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On oid-semigroups and universal semigroups "at infinity"

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Abstract

In this paper, we present an important new results for study of oid-semigroup and universal semigroup "at infinity". Principal results are theorem 3.4 and theorem 4.4.

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1 Introduction

Let S be a semigroup and topological space. S is called topological semigroup if the multiplication $(s,t) \to st : S \times S \to S$ is jointly continuous. Civin and Yood [5] shows that the Stone-Cech compactification of a discrete semigroup S could be given a semigroup structure. Indeed the operation on S extends uniquely to βS, so that S contained in it's topological center. Pym [4] introduced the concept of an oid. Oids are important because nearly all semigroups contains them and all oids are oid-isomorphic [6]. Through out this paper we will let T be a commutative oid with a discrete topology. Then the compact space βT produces a compact right topological semigroup T^∞. Our aim of the present paper is to introduce oid-semigroup and universal semigroup at infinity.

2 Definitions and preliminaries

Definition 2.1. *Let* $x = (x(n))_{n\in\mathbb{N}}$ *be any sequence consisting of* $1s$ *and* ∞s. *We define* $\operatorname{supp}(x(n))_{n\in\mathbb{N}} = \{n \in \mathbb{N} : x(n) = \infty\}$ *and write*

$$T = \{(x(n))_{n\in\mathbb{N}} : \operatorname{supp}(x(n))_{n\in\mathbb{N}} \text{ is finite and non-empty}\}.$$

A commutative standard oid is the set T *together with the product* xy *defined in* T *if and only if* $(\operatorname{supp} x) \cap (\operatorname{supp} y) = \emptyset$ *to be* $(x(n)y(n))$ *where* $x(n)y(n)$ *is ordinary multiplication* $(1 \cdot 1 = 1, 1 \cdot \infty = \infty \cdot 1 = \infty)$.

Any commutative standard oid T can be considered as $\bigoplus_{n=1}^{\infty}\{1,\infty\}\backslash\{(1,1,\ldots,1)\}$ so that T is a countable set. Obviously $\operatorname{supp}(xy) = (\operatorname{supp} x) \cup (\operatorname{supp} y)$ whenever xy is defined in T. A more detailed analysis of oids can be found in [4]. For $x, y \in T$, $\operatorname{supp} x < \operatorname{supp} y$ means that $n < m$ if $n \in \operatorname{supp} x$ and $m \in \operatorname{supp} y$, and $\operatorname{supp} x_\alpha \to \infty$ for some net (x_α) in T will means that for arbitrary $k \in \mathbb{N}$ eventually $\min(\operatorname{supp} x_\alpha) > k$. Then for a fixed $t \in T$, eventually $\operatorname{supp} t < \operatorname{supp} y$ and so eventually tx_α is defined in T.

Remark 2.2. *Write* $u_n = (1, 1, \ldots, \infty, 1, 1, \ldots)$ *(with* ∞ *in the nth place). Put* $U = \{u_n : n \in \mathbb{N}\}$. *Then* U *is countable subset of* T. *Moreover, any* $x \in T$ *can be written uniquely as a finite product* $x = u_{i_1}u_{i_2}\ldots u_{i_k}$ *with* $i_1 < i_2 < \ldots < i_k$, $\operatorname{supp} x = \{i_1, \ldots, i_k\}$.

The compact space βT produces a compact right topological semigroup at infinity T^∞ defined by

$$T^\infty = \{\mu \in \beta T : \ \mu = \lim_\alpha x_\alpha \text{ with } \operatorname{supp} x_\alpha \to \infty\}$$

with the multiplication $\mu\nu = \lim_\alpha \lim_\beta x_\alpha y_\beta$ if $\mu = \lim_\alpha x_\alpha$, $\nu = \lim_\beta y_\beta$.
Infact, the product $(\mu, \nu) \to \mu\nu : \beta T \times T^\infty \to T^\infty$ is defined and is right continuous, and left continuity holds when $\mu = t \in T$.
Let $\nu \in T^\infty$. The left operator determined by ν is the mapping $L_\nu : B(T) \to B(T)$ defined by $L_\nu f(t) = \lim_\beta f(y_\beta t)$ $(t \in T, f \in B(T))$ with $y_\beta \to \nu$, $\operatorname{supp} y_\beta \to \infty$. Since T is commutative then $L_\nu f(t) = (L_t f)^\beta(\nu)$. If $\mu \in T^\infty$, then $L_{\mu\nu} = L_\mu \circ L_\nu$, so that $(\mu, \nu) \to \mu\nu : T^\infty \times T^\infty \to T^\infty$ is a binary operation on T^∞ relative to which T^∞ is a compact right topological semigroups.

Definition 2.3. (*a*) *The cardinal function is the map* $c : T \to \mathbb{N}$ *given by* $c(x) = \operatorname{card}(\operatorname{supp} x)$. *If* $(\operatorname{supp} x) \cap (\operatorname{supp} y) = \emptyset$ *then* xy *is defined,* $c(xy) = c(x) + c(y)$. *It follows that* c *extends to homomorphism* c^β *from* T^∞ *into the one-point compactification* $\mathbb{N} \cup \{\infty\}$.
Notation: *We denoted* $\frac{1}{c(x)}$ *by* $k(x)$, *for* $x \in T$. *If* $A \subseteq T$ *then* 1_A *denoted the indicator function of* A. *That is the function whose value* 1 *on* A *and* 0 *on* $T \backslash A$.
(*b*) *Let* $z : T \to \mathbb{Z}^+$. *For* $x \in T$, $z(x)$ *be the largest number of consecutive* 1*'s between* $\min(\operatorname{supp} x)$ *and* $\max(\operatorname{supp} x)$, *then the function* k *defined on* T *by* $k(x) = \frac{1}{z(x)+1}$ *is bounded, so extends to a unique continuous function* k^β *from* βT *into* $\mathbb{Z}^+ \cup \{\infty\}$.
(*c*) *Let* T *be a standard oid, and let* $x = u_{i_1} u_{i_2} \ldots u_{i_k}$. *We define* $\ell : T \to \mathbb{N}$ *by* $\ell(x) = i_k - i_1 + 1$, $(x \in T)$. *Then obvious that, there is a unique function* $\ell^\beta : \beta T \to \mathbb{N} \cup \{\infty\}$, *and put* $r(x) = \frac{1}{\ell(x)}$, $(x \in T)$.

3 Oid-semigroup

Definition 3.1. *Let T' be any set with an operation "0" defined on T' and let T be a standard oid. We say that $\varphi : T \to T'$ is an oid-map if for any $x, y \in T$ which xy is defined in T, then $\varphi(xy) = \varphi(x) \circ \varphi(y)$.*

For example let T be a standard oid and let $\mathbb{N}$ be additive semigroup of positive integers. Then $x = u_{i_1}u_{i_2}\ldots u_{i_k}$, $i_1 < i_2 < \ldots < i_k$. Define $\varphi : T \to \mathbb{N}$ by $\varphi(x) = 2^{i_1} + 2^{i_2} + \ldots + 2^{i_k}$. Then it is easily seen that φ is an oid-map.

Definition 3.2. *Suppose that there is on T a multiplication $m : T \times T \to T$ which makes (T, m) a commutative semigroup, and which has the property that the identity map from an oid T onto (T, m) is an oid map. Then we say (T, m) is a commutative oid-semigroup "at infinity".*

Suppose T ba a commutative oid-map, and $f \in C(T)$, $s \in T$. The left (right) translate $l_s f(r_s f)$ of f by s is defined by $l_s f(t) = f(st)$ $(r_s f(t) = f(ts))$ $\forall\, t \in T$. A subspace X of $C(T)$ is called left (right) translation invariant if $l_s f \in X$, $(r_s f \in X)$, $\forall\, f \in X$, $s \in T$. As T is a commutative, it follows that $l_s f = r_s f$ for all $s \in T$. Left (right) translation invariant subspace are discussed in [1].
We recall that a function $f \in C(T)$ is said to be almost periodic if the set $\{r_s f, s \in T\}$ of right translation of f is relatively norm compact in $C(T)$ [2]. The set of all almost periodic functions on T is denoted by $\mathrm{AP}(T)$.

Lemma 3.3. *Let T be a commutative oid-semigroup and let $f \in \mathrm{AP}(T)$. Then $L_\nu f \in n - \mathrm{cl}\{r_s f, s \in T\}$ for all $\nu \in \beta T$.*

Proof. By definition when T is a semigroup, $L_\nu f(t) = \lim_\alpha f(tx_\alpha)$ where $x_\alpha \to \nu$ in βT and $t \in T$. Since $f(tx_\alpha) = r_{x_\alpha} f(t)$ and $(r_{x_\alpha} f)$ is a net in $\{r_s f, s \in T\}$ which is relatively norm compact in $C(T)$, then there exists a subnet (x_{α_β}) of

(x_α) and $g \in C(T)$ such that $\|r_{x_{\alpha_\beta}} f - g\| \to 0$, i.e., $|r_{x_{\alpha_\beta}} f(t) - g(t)| \to 0$ for all $t \in T$. It follows that

$$L_\nu f(t) = \lim_\alpha r_{x_\alpha} f(t) = \lim_\beta r_{x_{\alpha_\beta}} f(t) = g(t)$$

for all $t \in T$. Thus $L_\nu f = g \in n - \mathrm{cl}\{r_s f, s \in T\}$, as required. □

The next theorem get us the eqivalent condition for AP(T) when T is oid-semigroup.

Theorem 3.4. *Let T and βT be semigroups and $f \in C(T)$. Then $f \in$ AP(T) if and only if $(\mu, \nu) \to f^\beta(\mu\nu) : \beta T \times \beta T \to \mathbb{C}$ is jointly continuous.*

Proof. Necessity: Suppose $f \in$ AP(T), let $(\mu_\alpha), (\nu_\alpha)$ be nets in βT with $\mu_\alpha \to \mu$ and $\nu_\alpha \to \nu$ in βT. Then for each α, $L_{\nu_\alpha} f \in n - \mathrm{cl}\{r_s f : s \in T\}$. Since $n - \mathrm{cl}\{r_s f : s \in T\}$ is norm compact in $C(T)$, it follows that there exists a subnet (ν_{α_δ}) of (ν_α) and $g \in C(T)$ such that $\|L_{\nu_{\alpha_\delta}} f - g\| \to 0$. Now by a similar argument, we have that $f^\beta(\mu_\alpha \nu_\alpha) \to f^\beta(\mu\nu)$ i.e., $(\mu, \nu) \to f^\beta(\mu\nu) : \beta T \times \beta T \to \mathbb{C}$ is jointly continuous, as desired.
Sufficiency: Suppose $f \in C(T)$ and $\varphi : (\mu, \nu) \to f^\beta(\mu\nu) : \beta T \times \beta T \to \mathbb{C}$ is jointly continuous. Then $\varphi(0, T) \subseteq \varphi(0, \beta T)$, and for all $\mu \in \beta T$, $t \in T$ we have

$$\varphi(\mu, t) = f^\beta(\mu t) = f^\beta(t\mu) = (l_t f)^\beta(\mu).$$

Therefore $\varphi(0, T) = \{\varphi(0, t) : \ t \in T\} = \{(l_t f)^\beta : \ t \in T\}$. Now it is easy to check that the function φ satisfies all suitable conditions. It follows that $\nu \to \varphi(0, \nu) : \beta T \to C(\beta T)$ is norm continuous. This proves $\varphi(0, \beta T)$ is norm compact in $C(\beta T)$, and therefore $n - \mathrm{cl}\, \varphi(0, T)$ is norm compact in $C(\beta T)$. Since $C(T)$ and $C(\beta T)$ are isometrically isomorphic Banach spaces and T is a commutative semigroup, it follows that $n - \mathrm{cl}\{r_t f : \ t \in T\}$ is norm compact in $C(T)$. Thus $f \in$ AP(T) and the result now follows. □

The next theorem is a key result in the theory of oid-semigroup T.

Theorem 3.5. *Let T be an oid-semigroup and $f \in C(T)$, $t \in T$. Then $l_t f$ is jointly continuous on $\beta T \times T^\infty$.*

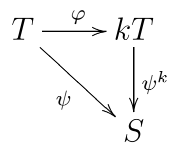
Proof. It is enough to show that $(\mu, \nu) \to (l_t f)^\beta(\mu\nu) : \beta T \times T^\infty \to \mathbb{C}$ is continuous. Since left continuity holds at each point of T by definition 3.2, then map $\mu \to t\mu$ is continuous from βT into βT. Therefore the composite map $(\mu, \nu) \to (t\mu, \nu) \to f^\beta(t\mu\nu) = (l_t f)^\beta(\mu\nu)$ is continuous from $\beta T \times T^\infty$ to $\mathbb{C}$. □

Remark 3.6. *Let T be a commutative oid-semigroup (Definition 3.2). Then the product $\mu\nu = \mu \circ L_\nu$ can be defined whenever $\mu, \nu \in \beta T$. The product $\mu\nu \in \beta T$ and the formula $\mu\nu = \mu \circ L_\nu$ is a binary operation on βT relative to which βT is compact right topological semigroup and left continuity holds when $\mu \in T$. Moreover, $t\nu = \nu t$ for $t \in T$, $\nu \in \beta T$. Therefore βT contains T^∞ as a subsemigroup.*

4 Universal semigroups "at infinity"

In this section we prove that associated with each commutative standard oid T, there is a commutative semigroup, called the universal semigroup of the oid T by starting with the countable subset U of the oid T and producing a unique algebraic isomorphism between the universal semigroup of the oid and the free abelian semigroup generated by U which has a universal mapping property relative to U. We first give the definition of universal semigroup.

Definition 4.1. *Let T be a commutative standard oid. Then a universal semigroup of T is a pair (φ, kT) such that kT is a commutative semigroup, $\varphi : T \to kT$ is an oid-map (Definition 3.1) and if $\psi : T \to S$ is an oid-map of T into a commutative semigroup S, then there exists a unique algebraic* *homomorphism* $\psi^k : kT \to S$ *such that the diagram*



commutes.

Write $F_{u_i} = \{1, u_i, u_i^2, u_i^3, \ldots\}$ for each $u_i \in U$. Then $\bigoplus_{i=1}^{\infty} F_{u_i} \backslash \{(1, 1, \ldots, 1, \ldots)\}$ is called the free abelian semigroup generated by U and will be denoted by F_U. We usually write

$$(u_1^{n_1}, u_2^{n_2}, \ldots, u_r^{n_r}, \ldots) = u_{i_1}^{\alpha_1} u_{i_2}^{\alpha_2} \ldots u_{i_k}^{\alpha_k}$$

where $i_1 < i_2 < \ldots < i_k$ and $\alpha_1, \alpha_2, \ldots, \alpha_k \neq 0$, $\alpha_1 = n_{i_1}, \ldots, \alpha_k = n_{i_k}$.

Lemma 4.2. *Let* T *be a standard oid, define* $\theta : T \to F_U$ *by*

$$\theta(u_{i_1} u_{i_2} \ldots u_{i_k}) = u_{i_1} u_{i_2} \ldots u_{i_k}$$

where $i_1 < i_2 < \ldots < i_k$. *Then* θ *is an injective oid-map.*

Proof. Straightforward. □

Lemma 4.3. *Let* S *be any commutative semigroup. If* $\varphi_0 : T \to S$ *is any oid-map of an oid* T *into* S *than* φ_0 *can be extended in one and only one way to a homomorphism* φ *of* F_U *into* S.

Proof. Define $\varphi : F_U \to S$ by

$$\varphi(u_{i_1}^{\alpha_1} u_{i_2}^{\alpha_2} \ldots u_{i_k}^{\alpha_k}) = \varphi_0(u_{i_1})^{\alpha_1} \varphi_0(u_{i_2})^{\alpha_2} \ldots \varphi_0(u_{i_k})^{\alpha_k}.$$

Since S is commutative, it is straightforward to prove that φ is a homomorphism. Now, let $x = u_{i_1} u_{i_2} \ldots u_{i_k} \in T$, $i_1 < i_2 < \ldots < i_k$ and let $\varphi_0 : T \to S$ be an oid-map. Then

$$\varphi_0(x) = \varphi_0(u_{i_1} u_{i_2} \ldots u_{i_k}) = \varphi_0(u_{i_1}) \varphi_0(u_{i_2}) \ldots \varphi_0(u_{i_k}) = \varphi(u_{i_1} u_{i_2} \ldots u_{i_k}) = \varphi(x).$$

Which implies that $\varphi|_T = \varphi_0$. Moreover, φ is unique, for if $\psi : F_U \to S$ is to have the required properties then

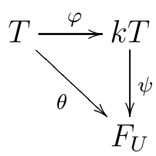
$$\begin{aligned}\psi(u_{i_1}^{\alpha_1} u_{i_2}^{\alpha_2} \dots u_{i_k}^{\alpha_k}) &= \psi(u_{i_1})^{\alpha_1} \psi(u_{i_2})^{\alpha_2} \dots \psi(u_{i_k})^{\alpha_k} \\ &= \varphi_0(u_{i_1})^{\alpha_1} \varphi_0(u_{i_2})^{\alpha_2} \dots \varphi_0(u_{i_k})^{\alpha_k} \\ &= \varphi(u_{i_1}^{\alpha_1} u_{i_2}^{\alpha_2} \dots u_{i_k}^{\alpha_k})\end{aligned}$$

and we have that $\psi = \varphi$, as desired. □

This lemma shows that F_U is a universal semigroup for T. We prove next that every universal semigroup is isomorphic to F_U.

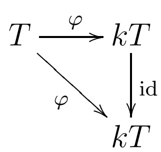
Theorem 4.4. *Let (φ, kT) be a universal semigroup of an oid T and let F_U be the free abelian semigroup on U. Then kT is algebraically isomorphic to F_U.*

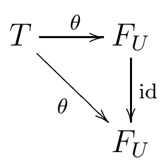
Proof. By lemma 4.2, $\theta : T \to F_U$ is an injective oid-map. Since (φ, kT) is a universal semigroup of the oid T and F_U is commutative, there exists a unique homomorphism $\psi : kT \to F_U$ such that the diagram



commutes.

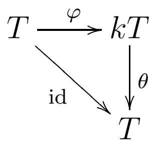
Now $\varphi : T \to kT$ is an oid-map, kT is a commutative semigroup, by lemma 4.3 there exists a unique homomorphism $\psi' : F_U \to kT$ such that $\psi'\theta = \varphi$. In view of the commuting diagrams:





and the uniqueness of $\psi' \circ \psi$, we see that $\psi' \circ \psi = \text{id}$ and similarly $\psi \circ \psi' = \text{id}$. We conclude that ψ is an isomorphism and the result follows. □

Remark 4.5. *Let T be a standard oid. Suppose in addition that T is a commutative oid-semigroup so that if the oid product xy of two elements $x, y \in T$ is defined, then it is the same as the semigroup product. Since* $\mathrm{id} : T \to T$ *is an oid-map then the diagram*



commutes.

Clearly, θ is a surmorphism. We denote by $R(\theta)$ the relation

$$\{(x, y) \in kT \times kT : \theta(x) = \theta(y)\}.$$

Then $R(\theta)$ is a congruence on kT. Moreover, $\frac{kT}{R(\theta)}$ is a quotient semigroup and so by the first isomorphism theorem ([3], chapter 1, Theorem 1.49*), the semigroup T is isomorphic to $\frac{kT}{R(\theta)}$.*

Any time a topology is used on kT without explicitly being described, it is assumed to be the discrete topology.

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