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Hyers-Ulam Stability of a multivariate partial differential equation

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Abstract

We prove the Hyers-Ulam stability of a partial differential equation. That is, if u is an approximate solution of $\sum_{i=1}^{n} a_i u_{x_i}(x_1, x_2, \cdots, x_n) + g(x_j)u(x_1, x_2, \cdots, x_n) + h(x_j) = 0$, then there exists an exact solution of the differential equation near to u.

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1 Introduction

Assume that X is a normed space over a scalar field $\mathbb{K}$ and that I is an open interval. Let a_i be fixed elements of $\mathbb{K}$. Assume that for a fixed function $g : I \to X$ and for any n-times differentiable function $y : I \to X$ satisfying the inequality

$$||y^{(n)}(t) + \sum_{i=0}^{n-1} a_i y^{(i)}(t) + g(t)|| \leq \varepsilon$$

for all $t \in I$ and for a given $\varepsilon > 0$, there exists a function $y_0 : I \to X$ satisfying

$$y_0^{(n)}(t) + \sum_{i=0}^{n-1} a_i y_0^{(i)}(t) + g(t) = 0$$

and $||y(t) - y_0(t)|| \leq K(\varepsilon)$ for any $t \in I$, where $K(\varepsilon)$ is an expression for ε with $\lim_{\varepsilon \to 0} K(\varepsilon) = 0$. Then, we say that the above differential equation has Hyers-Ulam stability.

M. Obłoza may be the first one who to investigate the Hyers-Ulam stability of differential equations (see [3] and [4]). The work of Alsina and Ger [1] has been generalized by many mathematicians (see [2], [5], [6], [7], [8]).

Recently, some mathematicians investigated the Hyers-Ulam stability of partial differential equations (see [10], [11], [12]).

We should mention that Jung [9] proved that stability of two linear partial differential equations of first order in the field of $\mathbb{R}^+$:

$$au_x(x,y) + bu_y(x,y) + g(y)u(x,y) + h(y) = 0 \qquad (a \leq 0, b > 0)$$

and

$$au_x(x,y) + bu_y(x,y) + g(x)u(x,y) + h(x) = 0 \qquad (a > 0, b \leq 0)$$

In 2006, Jung (see [5]) proved the following theorem of the Hyers-Ulam stability of a linear differential equation:

Theorem 1.1. *Let X be a complex Banach space and let $I = (a,b)$ be an open interval,where $a, b \in \mathbb{R} \cup \{\pm\infty\}$ are arbitrarily given with $a < b$. Assume that $g : I \to \mathbb{C}$ and $h : I \to X$ are continuous functions such that $g(t)$ and $e^{\int_a^t g(u)}h(t)$ are integrable on (a,c) for each $c \in I$. Moreover, suppose $\varphi : I \to [0,\infty)$ is a function such that $\varphi(t)e^{\Re\int_a^t g(u)du}$ is integrable on I, where $\Re z$ denotes the real part of a complex number z. If a continuously differentiable function $y : I \to X$ satisfies the differential inequality*

$$||y'(t) + g(t)y(t) + h(t)|| \leq \varphi(t)$$

for all $t \in I$, then there exists a unique $x \in X$ such that

$$||y(t) - e^{-\int_a^t g(u)du}(x - \int_a^t e^{\int_a^v g(u)du}h(v)dv)|| \leq e^{-\Re\int_a^t g(u)du} \int_t^b \varphi(v)e^{\Re\int_a^v g(u)du}dv$$

for every $t \in I$.

This theorem will be useful in our proof of the main result of this paper.

Throughout this paper, we denote by $\Re(z)$ the real part of a complex number z. The purpose of this paper is to prove the Hyers-Ulam stability of first-order linear partial differential equations in n-dimensional space of the form

$$\sum_{i=1}^{n} a_i u_{x_i}(x_1, x_2, \cdots, x_n) + g(x_j)u(x_1, x_2, \cdots, x_n) + h(x_j) = 0 \tag{1.1}$$

where $a_i \in \mathbb{R}$ are arbitrarily given.

2 Main Results

The following theorem is our main result.

Theorem 2.1. *Let I be a real n-dimensional space, $u : I \to \mathbb{C}$ be a function which has continuous partial derivatives with respect to all variables. Moreover, assume that u satisfies the following inequality:*

$$||\sum_{i=1}^{n} a_i u_{x_i}(x_1, x_2, \cdots, x_n) + g(x_j)u(x_1, x_2, \cdots, x_n) + h(x_j)|| \leq \varepsilon \qquad (2.1)$$

for all $x_i \in \mathbb{R}$ and for some $\varepsilon \geq 0$, where $g, h : \mathbb{R} \to \mathbb{C}$ are continuous functions and $j(j \leq n)$ is an arbitrary positive integer. Furthermore, assume that g, h and u satisfy
(a) $\int_{-\infty}^{y} g(t)dt$ exists for all $y \in \mathbb{R} \cup \{\pm\infty\}$;
(b) $\int_{-\infty}^{y} e^{\frac{1}{a_j}\int_{-\infty}^{w} g(t)dt} h(w)dw$ exists for all $y \in \mathbb{R} \cup \{\pm\infty\}$;
(c) $\int_{-\infty}^{\infty} e^{\frac{1}{a_j}\Re(\int_{-\infty}^{w} g(t)dt)} dw$ exists;
(d) $\lim_{x_j \to \infty} u(x_1, x_2, \cdots, x_n)$ is differential with respect to every $x_i (i \neq j)$.
Then, there exists a function u_0 satisfies Eq.(2.1) with

$$||u(x_1, x_2, \cdots, x_n) - u_0(x_1, x_2, \cdots, x_n)|| \leq M\varepsilon,$$

where M is a constant.

Proof. Without loss of generality, suppose that $j = n$.

Firstly, we will introduce new coordinates $Y = (y_1, y_2, \cdots, y_n)$ by a suitable change of axes:

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & -\frac{a_1}{a_2} & 0 & \cdots & 0 \\ 0 & 1 & -\frac{a_2}{a_3} & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \frac{1}{a_n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = AX, \qquad (2.3)$$

where $A_{i(i+1)} = -\frac{a_i}{a_{i+1}}, A_{ii} = 1(i \in \mathbb{N}, 1 \leq i \leq n-1), A_{nn} = \frac{1}{a_n}$ and the other $A_{ij} = 0$, and we define $\tilde{u}(y_1, y_2, \cdots, y_n) = u(A^{-1}Y) = u(x_1, x_2, \cdots, x_n)$, then it follows from Eq.(2.3) that

$$\begin{pmatrix} u_{x_1} \\ u_{x_2} \\ \vdots \\ u_{x_n} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -\frac{a_1}{a_2} & 1 & 0 & \cdots & 0 \\ 0 & -\frac{a_2}{a_3} & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \frac{1}{a_n} \end{pmatrix} \begin{pmatrix} \tilde{u}_{y_1} \\ \tilde{u}_{y_2} \\ \vdots \\ \tilde{u}_{y_n} \end{pmatrix} = A^T \begin{pmatrix} \tilde{u}_{y_1} \\ \tilde{u}_{y_2} \\ \vdots \\ \tilde{u}_{y_n} \end{pmatrix}. \qquad (2.4)$$

Hence, we have

$$\sum_{i=1}^{n} a_i u_{x_i}(x_1, x_2, \cdots, x_n) = \tilde{u}_{y_n}(y_1, y_2, \cdots, y_n)$$

and if we apply this equality to Eq.(2.1), and define $\tilde{g}(y_n) = g(a_n y_n) = g(x_n)$ and $\tilde{h}(y_n) = h(a_n y_n) = h(x_n)$, we get

$$||\tilde{u}_{y_n}(y_1, y_2, \cdots, y_n) + \tilde{g}(y_n)\tilde{u}(y_1, y_2, \cdots, y_n) + \tilde{h}(y_n)|| \leq \varepsilon \tag{2.5}$$

for all $y_1, y_2, \cdots, y_n \in \mathbb{R}$.

We can know from (a) that,

$$\int_{-\infty}^{y} \tilde{g}(\mu)d\mu = \frac{1}{a_n}\int_{-\infty}^{a_n y} g(t)dt \tag{2.6}$$

exists for all $y \in \mathbb{R} \cup \{\pm\infty\}$. Then, we can know from Eq.(2.6) that

$$\int_{-\infty}^{y} e^{\int_{-\infty}^{v}\tilde{g}(\mu)d\mu}\tilde{h}(v)dv = \frac{1}{a_n}\int_{-\infty}^{a_n y} e^{\frac{1}{a_n}\int_{-\infty}^{w} g(t)dt} h(w)dw. \tag{2.7}$$

Hence, it follows from (b) that, for all $y \in \mathbb{R}$, the following integration exists:

$$\int_{-\infty}^{y} e^{\int_{-\infty}^{v}\tilde{g}(\mu)d\mu}\tilde{h}(v)dv. \tag{2.8}$$

Similarly, we can know from Eq.(2.6) and (c) that

$$\int_{-\infty}^{\infty} \varepsilon e^{\Re(\int_{-\infty}^{v}\tilde{g}(\mu)d\mu)}dv = \frac{\varepsilon}{a_n}\int_{-\infty}^{\infty} e^{\frac{1}{a_n}\Re(\int_{-\infty}^{w} g(t)dt)}dw \qquad \text{exists.} \tag{2.9}$$

By using inequality (2.5), the condition (2.6),(2.8)and(2.9), together with Theorem 1.1, imply that for each fixed $(y_1, y_2, \cdots, y_{n-1})$, there will exists a unique complex number $\tilde{\theta}(y_1, y_2, \cdots, y_{n-1})$ such that

$$||\tilde{u}(y_1, y_2, \cdots, y_n) - e^{-\int_{-\infty}^{y_n}\tilde{g}(\mu)d\mu}(\tilde{\theta}(y_1, y_2, \cdots, y_{n-1}) - \int_{-\infty}^{y_n} e^{\int_{-\infty}^{v}\tilde{g}(\mu)d\mu}\tilde{h}(v)dv)||$$
$$\leq \varepsilon e^{-\Re\int_{-\infty}^{y_n}\tilde{g}(\mu)d\mu}\int_{y_n}^{+\infty} e^{\Re\int_{-\infty}^{v}\tilde{g}(\mu)d\mu}dv \tag{2.10}$$

for all $y_n \in (-\infty, +\infty)$.

According to a formula in the proof of theorem 1.1 (see [5]), in the view of Eq.(2.6) and (a),(b)and (d), we can conclude that

$$\tilde{\theta}(y_1, y_2, \cdots, y_{n-1}) = \lim_{y_n \to +\infty}[e^{\int_{-\infty}^{y_n}\tilde{g}(\mu)d\mu}\tilde{u}(y_1, y_2, \cdots, y_n) + \int_{-\infty}^{y_n} e^{\int_{-\infty}^{v}\tilde{g}(\mu)d\mu}\tilde{h}(v)dv],$$

which is differential with respect to all variables. Then we define

$$\tilde{u}_0(y_1, y_2, \cdots, y_n) = e^{-\int_{-\infty}^{y_n} \tilde{g}(\mu)d\mu}(\tilde{\theta}(y_1, y_2, \cdots, y_{n-1}) - \int_{-\infty}^{y_n} e^{\int_{-\infty}^{v} \tilde{g}(\mu)d\mu}\tilde{h}(v)dv)$$

and $u_0(X) = \tilde{u}_0(AX) = \tilde{u}_0(Y)$. Thus, we can obtain that

$$||u(x_1, x_2, \cdots, x_n) - u_0(x_1, x_2, \cdots, x_n)|| \leq M\varepsilon,$$

where $M = \sup_{x_n \in \mathbb{R}} \frac{1}{a_n} e^{-\frac{1}{a_n}\Re\int_{-\infty}^{x_n} g(\mu)d\mu} \int_{x_n}^{+\infty} e^{\frac{1}{a_n}\Re\int_{-\infty}^{v} g(\mu)d\mu} dv.$

□

Remark 2.2. *It is easy to know that*

$$u_0(x_1, x_2, \cdots, x_n) = e^{-\frac{1}{a_n}\int_{-\infty}^{x_n} g(t)dt}[\theta(x_1, x_2, \cdots, x_n) - \frac{1}{a_n}\int_{-\infty}^{x_n} e^{\frac{1}{a_n}\int_{-\infty}^{w} g(t)dt} h(w)dw]$$

is a solution of the partial differential equation(1.1), where $\theta(x_1, x_2, \cdots, x_n) = \tilde{\theta}(x_1 - \frac{a_1}{a_2}x_2, x_2 - \frac{a_2}{a_3}x_3, \cdots, x_{n-1} - \frac{a_{n-1}}{a_n}x_n)$.

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