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# 3-Lie algebras and triangular matrices

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**Abstract**

In this paper we study the Rota-Baxter operator $P$ on the algebra $t(n, F)$ of upper triangular matrices. By the associative algebra $t(n, F)$, the Rota-Baxter operator $P$ on $t(n, F)$, and linear function $f$, we obatin 3-Lie algebra $(t(n, F), [, , ]_{f,P})$.

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# 1 Introduction

Rota-Baxter Lie algebras [1] are closely related to pre-Lie algebras and post-Lie algebras. Rota-Baxter algebras have played an important role in the Hopf algebra approach of renormalization of perturbative quantum field theory of Connes and Kreimer (see [2-4]), as well as in the application of the renormalization method in solving divergent problems in number theory. In this paper we first discuss the Rota-Baxter operators on the algebra $t(n, F)$ of upper triangular matrices. By the associative algebra $t(n, F)$, the Rota-Baxter operator $P$ on $t(n, F)$, and linear function $f$, we construct 3-Lie algebras [5], and study their structures.

*An associative algebra* is a vector space $L$ over a field $F$ endowed with a binary-linear operation $\circ : L \times L \to L$ satisfying the associative law: for $x_1, x_2, x_3 \in L$

$$(x_1 \circ x_2) \circ x_3 = x_1 \circ (x_2 \circ x_3). \tag{1}$$

Let $(L,\circ)$ be an associative algebra, $P: L\times L\to L$ be a linear mapping. If $P$ satisfies for $x_1,x_2\in L$

$$P(x_1)\circ R(x_2)=P(P(x_1)\circ x_2+x_1\circ P(x_2)+\lambda x_1\circ x_2), \quad (2)$$

then $P$ is called *a Rota-Baxter operator* of weight $\lambda$, where $\lambda$ is some element of $F$, and $(L,\circ,P)$ is called *a Rota-Baxter associative algebra* of weight $\lambda$.

*A* 3*-Lie algebra* is a vector space $L$ over a field $F$ endowed with a 3-ary multi-linear skew-symmetric operation $[x_1,x_2,x_3]$ satisfying the 3-Jacobi identity

$$[[x_1,x_2,x_3],y_2,y_3]=\sum_{i=1}^{3}[x_1,\cdots,[x_i,y_2,y_3],\cdots,x_3],\ \forall x_1,x_2,x_3\in L. \quad (3)$$

**Lemma 1.1** [6] *Let $(L,[,])$ be a Lie algebra, $f\in L^*$ satisfying $f([x,y])=0$ for every $x,y\in L$. Then $L$ is a 3-Lie algebra in the multiplication*

$$[x,y,z]_f=f(x)[y,z]+f(y)[z,x]+f(z)[x,y],\quad \forall x,y,z\in L.$$

# 2 Rota-Baxter operators on $t(n,F)$

**Lemma 2.1** *Let $(A,\circ,P)$ be a Rota-Baxter associative algebra of weight $\lambda$. Then $(A,*)$ is a associative algebra, where*

$$x*y=P(x)\circ y+x\circ P(y)+\lambda x\circ y,\quad \forall x,y\in A. \quad (4)$$

**Proof** By Eq.(2) and Eq. (4)

$(x*y)*z=(P(x)\circ y+x\circ P(y)+\lambda x\circ y)*z$

$=P(P(x)\circ y+x\circ P(y)+\lambda x\circ y))\circ z+(P(x)\circ y+x\circ P(y)+\lambda x\circ y)\circ P(z)$

$+\lambda(P(x)\circ y+x\circ P(y)+\lambda x\circ y)\circ z$

$=(P(x)\circ P(y))\circ z+(P(x)\circ y+x\circ P(y)+\lambda x\circ y)\circ P(z)$

$+\lambda(P(x)\circ y+x\circ P(y)+\lambda x\circ y)\circ z$

$=P(x)\circ(P(y)\circ z+y\circ P(z)+\lambda y\circ z)+x\circ(P(y)\circ P(z)))$

$+\lambda x\circ(P(y)\circ z+y\circ P(z)+\lambda y\circ z)=x*(y*z)$.

The proof is completed.

**Lemma 2.2** *Let $(A,\circ,P)$ be a Rota-Baxter associative algebra of weight $\lambda$. Then $(A,[,],P)$ is a Rota-Baxter Lie algebra of weight $\lambda$, where*

$$[x,y]=P(x)\circ y+x\circ P(y)+\lambda x\circ y-P(y)\circ x-y\circ P(x)-\lambda y\circ x,\ \ \forall x,y\in A. \quad (5)$$

**Proof** By Lemma 2.1, $A$ is a Lie algebra in the multiplication
$[x,y] = x*y - y*x = P(x)\circ y + x\circ P(y) + \lambda x\circ y - P(y)\circ x + y\circ P(x) + \lambda y\circ x.$
Now we prove that $P$ is a Rota-Baxter operator of Lie algebra $(A,[,])$. By Lemma 2.1 and Eq. (5),
$[P(x),P(y)] = P(x)*P(y) - P(y)*P(x)$
$= P^2(x)\circ P(y) + P(x)\circ P^2(y) + \lambda P(x)\circ P(y) - P^2(y)\circ P(x)$
$-P(y)\circ P^2(x) - \lambda P(y)\circ P(x)$
$= P(P^2(x)\circ y + P(x)\circ P(y) + \lambda P(x)\circ y + P(x)\circ P(y) + x\circ P^2(y) + \lambda x\circ P(y))$
$-P(P^2(y)\circ x - P(y)\circ P(x) - \lambda P(y)\circ x - P(y)\circ P(x) - y\circ P^2(x) - \lambda y\circ P(x))$
$+\lambda P(P(x)\circ y + x\circ P(y) + \lambda x\circ y - P(y)\circ x - y\circ P(x) - \lambda y\circ x).$
$= P([P(x),y] + [x,P(y)] + \lambda[x,y])$. It follows the result.

Now let $t(n,F)$ be the associative algebra constructed by $(n\times n)$-order upper triangular matrices. Then $t(n,F)$ has a basis $\{e_{ij} | 1\le i\le j\le n\}$, and the multiplication of $t(n,F)$ is

$$e_{ij}e_{kl} = \delta_{jk}e_{il},\quad 1\le i\le j\le n, 1\le k\le l\le n.$$

For a linear mapping $P: t(n,F)\to t(n,F)$, suppose

$$P(e_{ij}) = \sum_{1\le k\le l\le n} a_{kl}^{ij}e_{kl},\quad a_{kl}^{ij}\in F, 1\le i\le j\le n. \tag{6}$$

**Theorem 2.3** *Let $t(n,F)$ be the associative algebra constructed by $(n\times n)$-order upper triangular matrices. A linear mapping $P: t(n,F)\to t(n,F)$ defined as Eq.(6) is a Rota-Baxter operator of weight $\lambda$ on $t(n,F)$ if and only if $P$ satisfies, for every $1\le i\le j\le n, 1\le s\le t\le n, 1\le x\le y\le n$,*

$$\sum_{x\le u\le y} a_{ij}^{xu}a_{st}^{uy} = \sum_{1\le u\le s} a_{ij}^{us}a_{ut}^{xy} + \sum_{j\le w\le n} a_{st}^{jw}a_{iw}^{xy} + \lambda\delta_{js}a_{ij}^{xy}. \tag{7}$$

**Proof** For arbitrary basic vectors $e_{ij}, e_{st}$, since
$P(e_{ij})P(e_{st}) = (\sum\limits_{1\le x\le u\le n} a_{ij}^{xu}e_{xu})(\sum\limits_{1\le r\le y\le n} a_{st}^{ry}e_{ry}) = \sum\limits_{1\le x\le u\le y\le n} a_{ij}^{xu}a_{st}^{uy}e_{xy},$

$P(P(e_{ij})e_{st} + e_{ij}P(e_{st}) + \lambda e_{ij}e_{st})$
$= P(\sum\limits_{1\le u\le v\le n} a_{ij}^{uv}e_{uv}e_{st} + e_{ij}\sum\limits_{1\le r\le w\le n} a_{st}^{rw}e_{rw} + \lambda\delta_{js}e_{it})$
$= p(\sum\limits_{1\le u\le s} a_{ij}^{us}e_{ut} + \sum\limits_{j\le w\le n} a_{st}^{jw}e_{iw}) + \lambda\delta_{js}P(e_{it})$
$= \sum\limits_{1\le x\le y\le n}(\sum\limits_{1\le u\le s} a_{ij}^{us}a_{ut}^{xy} + \sum\limits_{j\le w\le n} a_{st}^{jw}a_{iw}^{xy})e_{xy} + \lambda\delta_{js}\sum\limits_{1\le x\le y\le n} a_{ij}^{xy}e_{xy}$
$= \sum\limits_{1\le x\le y\le n}[\sum\limits_{1\le u\le s} a_{ij}^{us}a_{ut}^{xy} + \sum\limits_{j\le w\le n} a_{st}^{jw}a_{iw}^{xy} + \lambda\delta_{js}a_{ij}^{xy}]e_{xy}.$
Summarizing above discussion, we obtain Eq. (7). It follows the result.

**Theorem 2.4** *Let $t(n,F)$ be the set of all $(n\times n)$-upper triangular matrices over $F$, and $\{e_{ij}|1\le i\le j\le n\}$ be a basis of $t(n,F)$. A linear mapping $P: t(n,F)\to t(n,F)$, $P(e_{ij})=\sum\limits_{1\le x\le y\le n} a_{ij}^{xy}e_{xy}$ satisfies Eq.(7), then $(t(n,F),[,],P)$ is a Rota-Baxter Lie algebra of weight $\lambda$, where*

$$[e_{ij},e_{kl}]=\sum_{1\le x\le k} a_{ij}^{xk}e_{xl}+\sum_{j\le y\le n} a_{kl}^{jy}e_{iy}+\lambda(\delta_{jk}e_{il}-\delta_{ik}e_{lj})$$
$$-\sum_{1\le x\le i} a_{kl}^{xi}e_{xj}-\sum_{k\le y\le n} a_{ij}^{ky}e_{ky}, \forall 1\le i\le j\le n, 1\le k\le l\le n.$$

**Proof** By Lemma 2.2 and Theorem 2.3, $(t(n,F),[,],P)$ is a Rota Baxter-Lie algebra. Thanks to Eq.(5), for every $1\le i\le j\le n, 1\le k\le l\le n$,

$$[e_{ij},e_{kl}]=P(e_{ij})e_{kl}+e_{ij}P(e_{kl})+\lambda e_{ij}e_{kl}-P(e_{kl})e_{ij}-e_{kl}P(e_{ij})-\lambda e_{kl}e_{ij}$$

$$=\sum_{1\le x\le y\le n} a_{ij}^{xy}e_{xy}e_{kl}+e_{ij}\sum_{1\le x\le y\le n} a_{kl}^{xy}e_{xy}+\lambda(\delta_{jk}e_{il}-\delta_{ik}e_{lj})$$
$$-\sum_{1\le x\le y\le n} a_{kl}^{xy}e_{xy}e_{ij}-e_{kl}\sum_{1\le x\le y\le n} a_{ij}^{xy}e_{xy}$$
$$=\sum_{1\le x\le k} a_{ij}^{xk}e_{xl}+\sum_{j\le y\le n} a_{kl}^{jy}e_{iy}+\lambda(\delta_{jk}e_{il}-\delta_{ik}e_{lj})$$
$$-\sum_{1\le x\le i} a_{kl}^{xi}e_{xj}-\sum_{k\le y\le n} a_{ij}^{ky}e_{ky}.$$

It follows the result.

**Theorem 2.5** *Let $t(n,F)$ be the set of all $(n\times n)$-upper triangular matrices over $F$, and $\{e_{ij}|1\le i\le j\le n\}$ be a basis of $t(n,F)$. A linear mapping $P: t(n,F)\to t(n,F)$, $P(e_{ij})=\sum\limits_{1\le x\le y\le n} a_{ij}^{xy}e_{xy}$ satisfies Eq.(7). If linear function $f: t(n,F)\to F$, $f(e_{ij})=\beta_{ij}$ satisfies $\forall 1\le i\le j\le n, 1\le k\le l\le n$,*

$$\sum_{x=1}^{k} a_{ij}^{xk}\beta_{xl}+\sum_{y=j}^{n} a_{kl}^{jy}\beta_{iy}+\lambda(\delta_{jk}\beta_{il}-\delta_{ik}\beta_{lj})-\sum_{x=1}^{i} a_{kl}^{xi}\beta_{xj}-\sum_{y=k}^{n} a_{ij}^{ky}\beta_{ky}=0, \quad (8)$$

*then $(t(n,F),[,,]_{f,P})$ is a 3-Lie algebra, where for $\forall 1\le i\le j\le n, 1\le k\le l\le n, 1\le s\le t\le n$,*

$$[e_{ij},e_{kl},e_{st}]_{f,P}$$
$$=\beta_{st}[\sum_{x=1}^{k} a_{ij}^{xk}e_{xl}+\sum_{y=j}^{n} a_{kl}^{jy}e_{iy}+\lambda(\delta_{jk}e_{il}-\delta_{ik}e_{lj})-\sum_{x=1}^{i} a_{kl}^{xi}e_{xj}-\sum_{y=k}^{n} a_{ij}^{ky}e_{ky}]$$
$$+\beta_{ij}[\sum_{x=1}^{s} a_{kl}^{xs}e_{xt}+\sum_{y=1}^{n} a_{st}^{ly}e_{ky}+\lambda(\delta_{ls}e_{kt}-\delta_{ks}e_{tl})-\sum_{x=1}^{k} a_{st}^{xk}e_{xl}-\sum_{y=s}^{n} a_{kl}^{sy}e_{sy}]$$
$$+\beta_{kl}[\sum_{x=1}^{i} a_{st}^{xi}e_{xj}+\sum_{y=t}^{n} a_{ij}^{ty}e_{sy}+\lambda(\delta_{ti}e_{sj}-\delta_{si}e_{jt})-\sum_{x=1}^{s} a_{ij}^{xs}e_{xt}-\sum_{y=i}^{n} a_{st}^{iy}e_{iy}].$$

**Proof** By Theorem 2.4, for $f: t(n,F) \to F$, $f(e_{ij}) = \beta_{ij}$, $\forall 1 \le i \le j \le n, 1 \le k \le l \le n$, satisfies $f([e_{ij}, e_{kl}]) = 0$ if and only if $f$ satisfies

$$0 = f(\sum_{x=1}^{k} a_{ij}^{xk} e_{xl} + \sum_{y=j}^{n} a_{kl}^{jy} e_{iy} + \lambda(\delta_{jk} e_{il} - \delta_{ik} e_{lj}) - \sum_{x=1}^{i} a_{kl}^{xi} e_{xj} - \sum_{y=k}^{n} a_{ij}^{ky} e_{ky}) =$$

$$\sum_{x=1}^{k} a_{ij}^{xk} f(e_{xl}) + \sum_{y=j}^{n} a_{kl}^{jy} f(e_{iy}) + \lambda(\delta_{jk} e_{il} - \delta_{ik} f(e_{lj})) - \sum_{x=1}^{i} a_{kl}^{xi} f(e_{xj}) - \sum_{y=k}^{n} a_{ij}^{ky} f(e_{ky})$$

we obtain Eq.(8). Again by Lemma 1.1, $[,,]_{f,P}$ is a 3-ary Lie product on $t(n,F)$. The proof is completed.

For the case $\lambda = 0$, and $n = 2$. For any Rota-Baxter operator $P$ on $t(2,F)$, we have $\sum_{x \le u \le y} a_{ij}^{xu} a_{st}^{uy} = \sum_{1 \le u \le s} a_{ij}^{us} a_{ut}^{xy} + \sum_{j \le w \le 2} a_{st}^{jw} a_{iw}^{xy}$, for $1 \le i \le j \le 2$, $1 \le s \le t \le 2$, $1 \le x \le y \le 2$.

Then the linear mapping $P$ on $t(2,F)$ is a Rota-Baxter operator on the associative algebra $t(2,F)$ if and only if $P$ satisfies the following identities:

$$a_{11}^{22} a_{12}^{11} = 0,\ a_{12}^{22} a_{12}^{11} = 0,\ a_{11}^{22} a_{22}^{11} = 0,\ a_{12}^{22} a_{22}^{11} = 0, a_{11}^{11} a_{11}^{11} + a_{12}^{11} a_{12}^{11} = 0,$$

$$a_{11}^{11} a_{12}^{11} + a_{12}^{12} a_{12}^{11} = 0,\ a_{12}^{11} a_{11}^{22} + a_{12}^{12} a_{12}^{22} = 0, a_{12}^{12} a_{12}^{22} + a_{12}^{22} a_{22}^{22} = 0,$$

$$a_{11}^{12} a_{12}^{11} + a_{11}^{22} a_{22}^{11} + a_{22}^{12} a_{12}^{11} = 0,\quad a_{22}^{12} a_{12}^{22} + a_{22}^{22} a_{22}^{22} = 0,$$

$$a_{11}^{12} a_{11}^{22} = a_{11}^{11} a_{11}^{12} + a_{11}^{12} a_{12}^{12}, a_{11}^{22} a_{11}^{22} = 2a_{11}^{11} a_{11}^{22} + a_{12}^{22} a_{11}^{12},$$

$$a_{22}^{12} a_{11}^{22} - a_{22}^{22} a_{22}^{12} = 0, a_{11}^{11} a_{22}^{12} + a_{11}^{12} a_{22}^{22} = a_{11}^{12} a_{12}^{12} + a_{11}^{22} a_{22}^{12} + a_{22}^{11} a_{11}^{12} + a_{22}^{12} a_{12}^{12},$$

$$a_{11}^{12} a_{12}^{22} + a_{22}^{11} a_{11}^{22} + a_{22}^{12} a_{12}^{22} = 0,\ a_{11}^{12} a_{12}^{22} = a_{12}^{11} a_{11}^{12} + a_{12}^{12} a_{12}^{12},$$

$$a_{12}^{11} a_{22}^{11} = a_{12}^{11} a_{12}^{12} + a_{12}^{22} a_{22}^{11} + a_{22}^{22} a_{12}^{11},\ a_{12}^{11} a_{22}^{12} = a_{12}^{12} a_{12}^{12} + a_{12}^{22} a_{22}^{12},$$

$$a_{22}^{11} a_{22}^{11} = a_{22}^{12} a_{12}^{11} + 2a_{22}^{22} a_{22}^{11},\ a_{22}^{11} a_{22}^{12} = a_{22}^{12} a_{12}^{12} + a_{11}^{22} a_{22}^{12}.$$

Therefore, $P(e_{11}) = e_{12}$, $P(e_{12}) = 0, P(e_{22}) = e_{11}$ is a Rota-Baxter operator of the algebra $t(2,F)$. By Lemma 2.2, we obtain Lie algebra $(t(n,F),[,])$, where

$[e_{11}, e_{12}] = P(e_{11})e_{12} - P(e_{12})e_{11} = 0,$

$[e_{11}, e_{22}] = P(e_{11})e_{22} - P(e_{22})e_{11} = e_{12}e_{22} - e_{11}e_{11} = e_{12} - e_{11},$

$[e_{12}, e_{22}] = P(e_{12})e_{22} - P(e_{22})e_{12} = -e_{11}e_{12} = -e_{12}.$

For every $f: t(2,F) \to F$ satisfies $f([e,e'] = 0)$ for $e, e' \in t(n,F)$, by Theorem 2.5 the 3-Lie algebra $(t(2,F), [,,]_{f,P})$ is abelian.

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