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Alexandrov fuzzy topologies and fuzzy preorders

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Abstract

In this paper, we investigate the properties of Alexandrov fuzzy topologies and upper approximation operators. We study fuzzy preorder, Alexandrov topologies upper approximation operators induced by Alexandrov fuzzy topologies. We give their examples.

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Complete residuated lattices, fuzzy preorder, upper approximation operators, Alexandrov (fuzzy) topologies

1 Introduction

Hájek [2] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Höhle [3] introduced L-fuzzy topologies and L-fuzzy interior operators on complete residuated lattices. Pawlak [8,9] introduced rough set theory as a formal tool to deal with imprecision and uncertainty in data analysis. Radzikowska [10] developed fuzzy rough sets in complete residuated lattice. Bělohlávek [1] investigated information systems and decision rules in complete residuated lattices. Zhang [6,7] introduced Alexandrov L-topologies induced by fuzzy rough sets. Kim [5] investigated the properties of Alexandrov topologies in complete residuated lattices.

In this paper, we investigate the properties of Alexandrov fuzzy topologies and upper approximation operators in a sense as Höhle [3]. We study fuzzy preorder, Alexandrov topologies upper approximation operators induced by Alexandrov fuzzy topologies. We give their examples.

2 Preliminaries

Definition 2.1. [1-3] A structure $(L,\vee,\wedge,\odot,\rightarrow,\bot,\top)$ is called a *complete residuated lattice* iff it satisfies the following properties:

(L1) $(L,\vee,\wedge,\bot,\top)$ is a complete lattice where $\bot$ is the bottom element and $\top$ is the top element;

(L2) $(L,\odot,\top)$ is a monoid;

(L3) It has an adjointness,i.e.

$$x\le y\rightarrow z \text{ iff } x\odot y\le z.$$

An operator $^*: L\rightarrow L$ defined by $a^*=a\rightarrow\bot$ is called *strong negations* if $a^{**}=a$.

$$\top_x(y)=\begin{cases}\top, & \text{if } y=x,\\ \bot, & \text{otherwise.}\end{cases}\quad \top_x^*(y)=\begin{cases}\bot, & \text{if } y=x,\\ \top, & \text{otherwise.}\end{cases}$$

In this paper, we assume that $(L,\vee,\wedge,\odot,\rightarrow,^*,\bot,\top)$ be a complete residuated lattice with a strong negation *.

Definition 2.2. [6,7] Let X be a set. A function $e_X: X\times X\rightarrow L$ is called a fuzzy preorder if it satisfies the following conditions

(E1) reflexive if $e_X(x,x)=1$ for all $x\in X$,

(E2) transitive if $e_X(x,y)\odot e_X(y,z)\le e_X(x,z)$, for all $x,y,z\in X$'

Example 2.3. (1) We define a function $e_L: L\times L\rightarrow L$ as $e_L(x,y)=x\rightarrow y$. Then e_L is a fuzzy preorder on L.

(2) We define a function $e_{L^X}: L^X\times L^X\rightarrow L$ as $e_{L^X}(A,B)=\bigwedge_{x\in X}(A(x)\rightarrow B(x))$. Then e_{L^X} is a fuzzy preorder from Lemma 2.4 (9).

Lemma 2.4. [1,2] Let $(L,\vee,\wedge,\odot,\rightarrow,^*,\bot,\top)$ be a complete residuated lattice with a strong negation *. For each $x,y,z,x_i,y_i\in L$, the following properties hold.

(1) If $y\le z$, then $x\odot y\le x\odot z$.

(2) If $y\le z$, then $x\rightarrow y\le x\rightarrow z$ and $z\rightarrow x\le y\rightarrow x$.

(3) $x\rightarrow y=\top$ iff $x\le y$.

(4) $x\rightarrow\top=\top$ and $\top\rightarrow x=x$.

(5) $x\odot y\le x\wedge y$.

(6) $x\odot(\bigvee_{i\in\Gamma}y_i)=\bigvee_{i\in\Gamma}(x\odot y_i)$ and $(\bigvee_{i\in\Gamma}x_i)\odot y=\bigvee_{i\in\Gamma}(x_i\odot y)$.

(7) $x\rightarrow(\bigwedge_{i\in\Gamma}y_i)=\bigwedge_{i\in\Gamma}(x\rightarrow y_i)$ and $(\bigvee_{i\in\Gamma}x_i)\rightarrow y=\bigwedge_{i\in\Gamma}(x_i\rightarrow y)$.

(8) $\bigvee_{i\in\Gamma}x_i\rightarrow\bigvee_{i\in\Gamma}y_i\ge\bigwedge_{i\in\Gamma}(x_i\rightarrow y_i)$ and $\bigwedge_{i\in\Gamma}x_i\rightarrow\bigwedge_{i\in\Gamma}y_i\ge\bigwedge_{i\in\Gamma}(x_i\rightarrow y_i)$.

(9) $(x\rightarrow y)\odot x\le y$ and $(y\rightarrow z)\odot(x\rightarrow y)\le(x\rightarrow z)$.

(10) $x\rightarrow y\le(y\rightarrow z)\rightarrow(x\rightarrow z)$ and $x\rightarrow y\le(z\rightarrow x)\rightarrow(z\rightarrow y)$.

(11) $\bigwedge_{i\in\Gamma} x_i^* = (\bigvee_{i\in\Gamma} x_i)^*$ and $\bigvee_{i\in\Gamma} x_i^* = (\bigwedge_{i\in\Gamma} x_i)^*$.
(12) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$ and $(x \odot y)^* = x \to y^*$.
(13) $x^* \to y^* = y \to x$ and $(x \to y)^* = x \odot y^*$.
(14) $y \to z \leq x \odot y \to x \odot z$.

Definition 2.5. [5] A map $\mathcal{H} : L^X \to L^Y$ is called an *upper approximation operator* if it satisfies the following conditions, for all $A, A_i \in L^X$, and $\alpha \in L$,
(H1) $\mathcal{H}(\alpha \odot A) = \alpha \odot \mathcal{H}(A)$,
(H2) $\mathcal{H}(\bigvee_{i\in I} A_i) = \bigvee_{i\in I} \mathcal{H}(A_i)$,
(H3) $A \leq \mathcal{H}(A)$,
(H4) $\mathcal{H}(\mathcal{H}(A)) \leq \mathcal{H}(A)$.

Definition 2.6. [4] An operator $\mathbf{T} : L^X \to L$ is called an *Alexandrov fuzzy topology* on X iff it satisfies the following conditions, for all $A, A_i \in L^X$, and $\alpha \in L$,
(T1) $\mathbf{T}(\alpha) = \top$,
(T2) $\mathbf{T}(\bigwedge_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma} \mathbf{T}(A_i)$ and $\mathbf{T}(\bigvee_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma} \mathbf{T}(A_i)$,
(T3) $\mathbf{T}(\alpha \odot A) \geq \mathbf{T}(A)$,
(T4) $\mathbf{T}(\alpha \to A) \geq \mathbf{T}(A)$.

Definition 2.1 [5] A subset $\tau \subset L^X$ is called an *Alexandrov topology* if it satisfies satisfies the following conditions.
(O1) $\perp_X, \top_X \in \tau$ where $\top_X(x) = \top$ and $\perp_X(x) = \perp$ for $x \in X$.
(O2) If $A_i \in \tau$ for $i \in \Gamma$, $\bigvee_{i\in\Gamma} A_i, \bigwedge_{i\in\Gamma} A_i \in \tau$.
(O3) $\alpha \odot A \in \tau$ for all $\alpha \in L$ and $A \in \tau$.
(O4) $\alpha \to A \in \tau$ for all $\alpha \in L$ and $A \in \tau$.

3 Alexandrov fuzzy topologies and fuzzy preorders

Theorem 3.1 (1) *If* $\mathbf{T} : L^X \to L$ *is an Alexander fuzzy topology. Define* $\mathbf{T}^*(A) = \mathbf{T}(A^*)$. *Then* $\mathbf{T}^*$ *is an Alexander fuzzy topology.*

(2) *If* $\mathbf{T}$ *be an Alexandrov fuzzy topology on* X, $\tau_T^r = \{A \in L^X \mid \mathbf{T}(A) \geq r\}$ *is an Alexandrov topology on* X *and* $\tau_T^r \subset \tau_T^s$ *for* $s \leq r \in L$.

(3) *If* $\mathcal{H}$ *is an* L*-upper approximation operator, then* $\tau_{\mathcal{H}} = \{A \in L^X \mid \mathcal{H}(A) = A\}$ *is an Alexandrov topology on* X.

Proof. (1) (T1) $\mathbf{T}^*(\alpha) = \mathbf{T}((\alpha^*) = \top$.

(T2) $\mathbf{T}^*(\bigvee_{i\in\Gamma} A_i) = \mathbf{T}(\bigwedge_{i\in\Gamma} A_i^*) \geq \bigwedge_{i\in\Gamma} \mathbf{T}(A_i^*) = \bigwedge_{i\in\Gamma} \mathbf{T}^*(A_i)$ and $\mathbf{T}^*(\bigwedge_{i\in\Gamma} A_i) = \mathbf{T}(\bigvee_{i\in\Gamma} A_i^*) \geq \bigwedge_{i\in\Gamma} \mathbf{T}(A_i^*) = \bigwedge_{i\in\Gamma} \mathbf{T}^*(A_i)$.

(T3)

$$\begin{aligned}\mathbf{T}^*(\alpha\odot A) &= \mathbf{T}((\alpha\odot A)^*) = \mathbf{T}(\alpha\to A^*)\\ &\geq \mathbf{T}(A^*) = \mathbf{T}^*(A).\end{aligned}$$

(T4)

$$\begin{aligned}\mathbf{T}^*(\alpha\to A) &= \mathbf{T}((\alpha\to A)^*) = \mathbf{T}((\alpha\odot A^*)^{**})\\ &= \mathbf{T}(\alpha\odot A^*) \geq \mathbf{T}(A^*) = \mathbf{T}^*(A).\end{aligned}$$

Hence $\mathbf{T}^*$ is an Alexander fuzzy topology.

(2) (O1) Since $\mathbf{T}(\top_X) = \mathbf{T}(\bot_X) = \top \geq r$, $\bot_X, \top_X \in \tau^r_T$.

(O2) For $A_i \in \tau^r_T$ for each $i \in \Gamma$, by (H3), $\mathbf{T}(\bigvee_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma} \mathbf{T}(A_i) \geq r$ and $\mathbf{T}(\bigwedge_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma} \mathbf{T}(A_i) \geq r$.

Thus, $\bigvee_{i\in\Gamma} A_i, \bigwedge_{i\in\Gamma} A_i \in \tau^r_T$.

(O3) and (O4), For $A \in \tau^r_T$, by (H2), $\mathbf{T}(\alpha\odot A) \geq \mathbf{T}(A)$ and $\mathbf{T}(\alpha\to A) \geq \mathbf{T}(A)$. So, $\alpha\odot A, \alpha\to A \in \tau^r_T$.

Hence τ^r_T is an Alexandrov topology on X. Moreover, for $A \in \tau^r_T$ with $s \leq r$, $\mathbf{T}(A) \geq r \geq s$. Then $A \in \tau^s_T$. Thus $\tau^r_T \subset \tau^s_T$.

(3) (O1) Since $\top_X \leq \mathcal{H}(\top_X)$ and $\mathcal{H}(\bot_X) = \mathcal{H}(\bot_X \odot A) = \bot_X \odot \mathcal{H}(A)$, $\top_X = \mathcal{H}(\top_X)$ and $\top_X = \mathcal{H}(\top_X)$. Then $\bot_X, \top_X \in \tau_{\mathcal{H}}$.

(O2) For $A_i \in \tau_{\mathcal{H}}$ for each $i \in \Gamma$, by (H3), $\bigvee_{i\in\Gamma} A_i \in \tau_{\mathcal{H}}$. Since $\bigwedge_{i\in\Gamma} A_i \leq \mathcal{H}(\bigwedge_{i\in\Gamma} A_i) \leq \bigwedge_{i\in\Gamma} \mathcal{H}(A_i) = \bigwedge_{i\in\Gamma} A_i$, Thus, $\bigwedge_{i\in\Gamma} A_i \in \tau_{\mathcal{H}}$.

(O3) For $A \in \tau_{\mathcal{H}}$, by (H2), $\alpha\odot A \in \tau_{\mathcal{H}}$.

(O4) For $A \in \tau_{\mathcal{H}}$, since $\alpha \odot \mathcal{H}(\alpha\to A) = \mathcal{H}(\alpha\odot(\alpha\to A)) \leq \mathcal{H}(A)$, $\mathcal{H}(\alpha\to A) \leq \alpha\to\mathcal{H}(A) = \alpha\to A$. Then $\alpha\to A \in \tau_{\mathcal{H}}$.

Theorem 3.2 *Let* $\mathbf{T}$ *be an Alexandorv fuzzy topology on* X*. Define*

$$R^r_T(x,y) = \bigwedge\{A(x)\to A(y) \mid \mathbf{T}(A) \geq r^*\}$$

$$R^{-r}_T(x,y) = \bigwedge\{B(y)\to B(x) \mid \mathbf{T}(B) \geq r^*\}.$$

(1) R^r_T *is a fuzzy preorder with* $R^r_T \leq R^s_T$ *for each* $s \leq r$.

(2) R^{-r}_T *is a fuzzy preorder with* $R^{-r}_T \leq R^{-s}_T$ *for each* $s \leq r$ *and*

$$R^{-r}_T(x,y) = R^r_{T^*}(x,y).$$

(3) *Define* $\mathcal{H}_{R^r_T} : L^X \to L^X$ *as follows*

$$\mathcal{H}_{R^r_T}(A)(y) = \bigvee_{x\in X} (A(x)\odot R^r_T(x,y)).$$

Then $\mathcal{H}_{R^r_T}$ *is an upper approximation operator on* X *with* $\mathcal{H}_{R^r_T} \leq \mathcal{H}_{R^s_T}$ *for each* $s \leq r$ *such that* $\tau^{r*}_T = \tau_{\mathcal{H}_{R^r_T}}$.

(4) $\mathcal{H}_{R_T^{-r}}$ *is an upper approximation operator on* X *such that*

$$\mathcal{H}_{R_T^{-r}}(A)(y) = \bigvee_{x\in X}(A(x)\odot R_T^{-r}(x,y)) = \bigvee_{x\in X}(A(x)\odot R_{T^*}(x,y)).$$

Moreover, $(\tau_T^{r*})^* = \tau_{\mathcal{H}_{R_T^{-r}}}$.

(5) $\mathcal{H}_{R_T^r}(A) = \bigwedge\{A_i \mid A \le A_i,\ \mathbf{T}(A_i) \ge r^*\}$ *for all* $A \in L^X$ *and* $r \in L$. *Moreover,* $R_T^r(x,y) = \mathcal{H}_{R_T^r}(\top_x)(y)$, *for each* $x, y \in X$.

(6) $\mathcal{H}_{R_T^r}(A) = \bigwedge\{A_i \mid A \le A_i,\ \mathbf{T}^*(A_i) \ge r^*\}$ *for all* $A \in L^X$ *and* $r \in L$. *Moreover,* $R_T^{-r}(x,y) = R_{T^*}^r(x,y) = \mathcal{H}_{R_{T^*}^r}(\top_x)(y)$, *for each* $x, y \in X$.

(7) *If* $\mathcal{H}_{R_T^{r_i}}(A) = B$ *for all* $i \in \Gamma \ne \emptyset$, *then* $\mathcal{H}_{R_T^s}(A) = B$ *with* $s = \bigwedge_{i\in\Gamma} r_i$.

(8) *If* $\mathcal{H}_{R_T^{-r_i}}(A) = B$ *for all* $i \in \Gamma \ne \emptyset$, *then* $\mathcal{H}_{R_T^{-s}}(A) = B$ *with* $s = \bigwedge_{i\in\Gamma} r_i$.

(9) *Define* $\mathbf{T}_{H_T} : L^X \to L$ *as*

$$\mathbf{T}_{H_T}(A) = \bigvee\{r_i^* \in L \mid \mathcal{H}_{R_T^{r_i}}(A) = A\}.$$

Then $\mathbf{T}_{H_T} = \mathbf{T}$ *is an Alexandrov fuzzy topology on* X.

(10) *Define* $\mathbf{T}_{H_{T^*}} : L^X \to L$ *as*

$$\mathbf{T}_{H_{T^*}}(A) = \bigvee\{r_i^* \in L \mid \mathcal{H}_{R_T^{-r_i}}(A) = A\}$$

Then $\mathbf{T}_{H_{T^*}} = \mathbf{T}^*$ *is an Alexandrov fuzzy topology on* X.

(11) *There exists an Alexandrov fuzzy topology* $\mathbf{T}^r$ *such that*

$$\mathbf{T}^r(A) = e_{L^X}(\mathcal{H}_{R_T^r}(A), A).$$

If $r \le s$, *then* $\mathbf{T}^r \le \mathbf{T}^s$ *for all* $A \in L^X$.

(12) *There exists an Alexandrov fuzzy topology* $\mathbf{T}^{*r}$ *such that*

$$\mathbf{T}^{*r}(A) = e_{L^X}(\mathcal{H}_{R_T^{-r}}(A), A).$$

If $r \le s$, *then* $\mathbf{T}^{*r} \le \mathbf{T}^{*s}$ *for all* $A \in L^X$.

(13) *Define* $\mathbf{T}_T : L^X \to L$ *as*

$$\mathbf{T}_T(A) = \bigvee\{r^* \in L \mid \mathbf{T}^r(A) = \top\}.$$

Then $\mathbf{T}_T = \mathbf{T} = \mathbf{T}_{H_T}$ *is an Alexandrov fuzzy topology on* X.

(14) *Define* $\mathbf{T}_{T^*} : L^X \to L$ *as*

$$\mathbf{T}_{T^*}(A) = \bigvee\{r^* \in L \mid \mathbf{T}^{*r}(A) = \top\}.$$

Then $\mathbf{T}_{T^*} = \mathbf{T}^* = \mathbf{T}_{H_{T^*}}$ *is an Alexandrov fuzzy topology on* X.

Proof (1) Since $\mathbf{T}(B) \geq r^*$ iff $B \in \tau_T^{r*}$, then $R_T^r(x,y) = \bigwedge_{B\in\tau_T^{r*}}(B(x) \to B(y))$. Since $R_T^r(x,x) = \bigwedge_{B\in\tau_T^{r*}}(B(x) \to B(x)) = \top$ and

$$\begin{array}{l} R_T^r(x,y) \odot R_T^r(y,z) = \bigwedge_{B\in\tau_T^{r*}}(B(x) \to B(y)) \odot \bigwedge_{B\in\tau_T^{r*}}(B(y) \to B(z)) \\ \leq \bigwedge_{B\in\tau_T^{r*}}(B(x) \to B(y)) \odot (B(y) \to B(z)) \\ \leq \bigwedge_{B\in\tau_T^{r*}}(B(x) \to B(z)) = R_T^r(x,y). \end{array}$$

Hence R_T^r is a fuzzy preorder.

For $s \leq r$, since $\mathbf{T}(B) \geq s^* \geq r^*$, we have $R_T^r \leq R_T^s$.

(2) By a similar method as (1), R_T^{-r} is a fuzzy preorder. Moreover,

$$\begin{array}{ll} R_T^{-r}(x,y) & = \bigwedge\{B(y) \to B(x) \mid \mathbf{T}(B) \geq r^*\} \\ & = \bigwedge\{B^*(x) \to B^*(y) \mid \mathbf{T}(B^*) = \mathbf{T}^*(B) \geq r^*\} \\ & = R_{T^*}^r(x,y). \end{array}$$

(3) (H1) and (H2) follows from the definition of $\mathcal{H}_{R_T^r}$.

(H3) $\mathcal{H}_{R_T^r}(A)(y) = \bigvee_{x\in X}(A(x) \odot R_T^r(x,y)) \geq A(y) \odot R_T^r(y,y) = A(y)$.

(H4)

$$\begin{array}{ll} \mathcal{H}_{R_T^r}(\mathcal{H}_{R_T^r}(A))(x) & = \bigvee_{y\in X}(\mathcal{H}_{R_T^r}(A)(y) \odot R_T^r(y,x)) \\ & = \bigvee_{y\in X}(\bigvee_{z\in X}(A(z) \odot R_T^r(z,y)) \odot R_T^r(y,x)) \\ & = \bigvee_{z\in X}(A(z) \odot \bigvee_{y\in X}(R_T^r(z,y) \odot R_T^r(y,x))) \\ & \leq \bigvee_{z\in X}(A(z) \odot R_T^r(z,x)) \\ & = \mathcal{H}_{R_T^r}(A)(x). \end{array}$$

For $s \leq r$, since $R_T^r \leq R_T^s$, then $\mathcal{H}_{R_T^r} \leq \mathcal{H}_{R_T^s}$.

Since $A \in \tau_T^{r*}$;i.e. $\mathbf{T}(A) \geq r^*$, $R_T^r(x,y) \odot A(x) = \bigwedge_{B\in\tau_T^{r*}}(B(x) \to B(y)) \odot A(x) \leq (A(x) \to A(y)) \odot A(x) \leq A(y)$, by H(3), $\mathcal{H}_{R_T^r}(A) = A \in \tau_{\mathcal{H}_{R_T^r}}$. Thus $\tau_T^{r*} \subset \tau_{\mathcal{H}_{R_T^r}}$. Let $\mathcal{H}_{R_T^r}(A) = A \in \tau_{\mathcal{H}_{R_T^r}}$. Then

$$\begin{array}{ll} A(x) & = \mathcal{H}_{R_T^r}(A)(x) = \bigvee_{y\in X}(A(y) \odot R_T^r(y,x)) \\ & = \bigvee_{y\in X}(A(y) \odot \bigwedge_{B\in\tau_T^{r*}}(B(y) \to B(x))) \end{array}$$

Since $\bigwedge_{B\in\tau_T^{r*}}(B(y) \to B) \in \tau_T^{r*}$ and $\bigvee_{y\in X}(A(y) \odot \bigwedge_{B\in\tau_T^{r*}}(B(y) \to B)) \in \tau_T^{r*}$, we have $A \in \tau_T^{r*}$. Hence $\tau_{\mathcal{H}_{R_T^r}} \subset \tau_T^{r*}$.

(4) Since $A \in \tau_T^{r*}$, $R_T^{-r}(x,y) \odot A^*(x) = \bigwedge_{B\in\tau_T^{r*}}(B^*(x) \to B^*(y)) \odot A^*(x) \leq (A^*(x) \to A^*(y)) \odot A^*(x) \leq A^*(y)$. Hence $\mathcal{H}_{R_T^{-r}}(A^*) = A^* \in \tau_{\mathcal{H}_{R_T^{-r}}}$.

Let $\mathcal{H}_{R_T^{-r}}(A) = A \in \tau_{\mathcal{H}_{R_T^{-r}}}$. Then

$$\begin{array}{ll} A(x) & = \mathcal{H}_{R_T^{-r}}(A)(x) = \bigvee_{y\in X}(A(y) \odot R_T^{-r}(y,x)) \\ & = \bigvee_{y\in X}(A(y) \odot \bigwedge_{B\in\tau_T^{r*}}(B^*(y) \to B^*(x))) \end{array}$$

Since $\bigwedge_{B\in\tau_T^{r*}}(B^*(y) \to B^*) \in (\tau_T^{r*})^*$ and $\bigvee_{y\in X}(A(y) \odot \bigwedge_{B\in\tau}(B^*(y) \to B^*)) \in (\tau_T^{r*})^*$, we have $A \in (\tau_T^{r*})^*$. Hence $(\tau_T^{r*})^* = \tau_{\mathcal{H}_{R_T^{-r}}}$.

(5) For each $A \in L^X$ with $A \leq A_i,\ \mathbf{T}(A_i) \geq r^*$, since $A_i \in \tau_T^{r*} = \tau_{\mathcal{H}_{R_T^r}}$ with $A \leq A_i,\ A_i \in \tau$, then

$$\bigwedge_i A_i \leq \mathcal{H}_{R_T^r}(\bigwedge_i A_i) \leq A_i = \mathcal{H}_{R_T^r}(A_i).$$

So, $\mathcal{H}_{R_T^r}(\bigwedge_i A_i) = \bigwedge_i A_i$. Thus

$$\mathcal{H}_{R_T^r}(A) \leq \mathcal{H}_{R_T^r}(\bigwedge_i A_i) = \bigwedge_i A_i = \bigwedge\{A_i \mid A \leq A_i,\ \mathbf{T}(A_i) \geq r^*\}.$$

Since $\mathcal{H}_{R_T^r}(\mathcal{H}_{R_T^r}(A)) = \mathcal{H}_{R_T^r}(A) \geq A$ and $\mathcal{H}_{R_T^r}(A) \in \tau_{\mathcal{H}_{R_T^r}} = \tau_T^{r*}$. So,, $\bigwedge\{A_i \mid A \leq A_i,\ \mathbf{T}(A_i) \geq r^*\} \leq \mathcal{H}_{R_T^r}(A)$. Hence $\bigwedge\{A_i \mid A \leq A_i,\ \mathbf{T}(A_i) \geq r^*\} = \mathcal{H}_{R_T^r}(A)$ for all $A \in L^X$ and $r \in L$.

(6) It is proved in a similar way as (5).

(7) Let $\mathcal{H}_{R_T^{r_i}}(A) = B$ for all $i \in \Gamma \neq \emptyset$. Since

$$\mathcal{H}_{R_T^{r_i}}(A) = \bigvee_{x\in X}(A(x) \odot R_T^{r_i}(x,-)) = \bigvee_{x\in X}(A(x) \odot \bigwedge_{D\in\tau_T^{r_i^*}}(D(x) \to D)) \in \tau_T^{r_i^*}$$

$$\begin{aligned}&\mathcal{H}_{R_T^{r_i}}(A) = \bigvee_{x\in X}(A(x) \odot R_T^{r_i}(x,-))\\ &= \bigvee_{x\in X}(A(x) \odot \bigwedge\nolimits_{D\in\tau_T^{r_i^*}}(D(x) \to D)) \in \tau_T^{r_i^*}\end{aligned}$$

$\mathbf{T}(B) = \mathbf{T}(\mathcal{H}_{R_T^{r_i}}(A)) \geq r_i^*$, then $\mathbf{T}(B) \geq \bigvee_{i\in\Gamma} r_i^* = (\bigwedge_{i\in\Gamma} r_i)^* = s^*$ where $s = \bigwedge_{i\in\Gamma} r_i$. Since $\mathcal{H}_{R_T^s}(B)(y) = \bigvee_{x\in X}(B(x) \odot R_T^s(x,y)) \leq \bigvee_{x\in X}(B(x) \odot (B(x) \to B(y))) \leq B(y)$,

$$B = \mathcal{H}_{R_T^s}(B) \geq \mathcal{H}_{R_T^s}(A).$$

Since $s \leq r_i$, $\mathcal{H}_{R_T^s}(A) \geq \mathcal{H}_{R_T^{r_i}}(A) = B$. Thus $\mathcal{H}_{R_T^s}(A) = B$.

(9) Since $\mathbf{T}(A) = \mathbf{T}(\mathcal{H}_{R_T^{r_i}}(A)) \geq r_i^*$, for $\mathcal{H}_{R_T^{r_i}}(A) = A$,

$$\mathbf{T}_{H_T}(A) = \bigvee\{r_i^* \in L \mid \mathcal{H}_{R_T^{r_i}}(A) = A\} \leq \mathbf{T}(A).$$

Since $\mathbf{T}(A) \geq (\mathbf{T}^*(A))^*$, then $\mathcal{H}_{R_T^s}(A)$ where $\mathbf{T}^*(A) = s$. Thus

$$\mathbf{T}_{H_T}(A) = \bigvee\{r_i^* \in L \mid \mathcal{H}_{R_T^{r_i}}(A) = A\} \geq \mathbf{T}(A).$$

Hence $\mathbf{T}_{H_T} = \mathbf{T}$.

(11) (T1) $\mathbf{T}^r(\top_X) = \mathbf{T}^r(\bot_X) = \top$.

(T2)

$$\begin{aligned}&\mathbf{T}^r(\bigvee\nolimits_{i\in\Gamma} A_i) = e_{L^X}(\mathcal{H}_{R_T^r}(\bigvee\nolimits_{i\in\Gamma} A_i), \bigvee\nolimits_{i\in\Gamma} A_i)\\ &= e_{L^X}(\bigvee\nolimits_{i\in\Gamma} \mathcal{H}_{R_T^r}(A_i), \bigvee\nolimits_{i\in\Gamma} A_i)\\ &\geq \bigwedge\nolimits_{i\in\Gamma} e_{L^X}(\mathcal{H}_{R_T^r}(A_i), A_i) = \bigwedge\nolimits_{i\in\Gamma} \mathbf{T}^r(A_i)\end{aligned}$$

Since $\mathcal{H}_{R^r_T}(\bigwedge_{i\in\Gamma} A_i) \leq \bigwedge_{i\in\Gamma} \mathcal{H}_{R^r_T}(A_i)$, we have

$$\begin{array}{l}\mathbf{T}^r(\bigwedge_{i\in\Gamma} A_i) = e_{L^X}(\mathcal{H}_{R^r_T}(\bigwedge_{i\in\Gamma} A_i), \bigwedge_{i\in\Gamma} A_i)\\ \geq e_{L^X}(\bigwedge_{i\in\Gamma} \mathcal{H}_{R^r_T}(A_i), \bigwedge_{i\in\Gamma} A_i)\\ \geq \bigwedge_{i\in\Gamma} e_{L^X}(\mathcal{H}_{R^r_T}(A_i), A_i) = \bigwedge_{i\in\Gamma} \mathbf{T}^r(A_i)\end{array}$$

(T3)

$$\begin{array}{l}\mathbf{T}^r(\alpha\odot A) = e_{L^X}(\mathcal{H}_{R^r_T}(\alpha\odot A), \alpha\odot A)\\ = e_{L^X}(\alpha\odot\mathcal{H}_{R^r_T}(A), \alpha\odot A)\\ \geq e_{L^X}(\mathcal{H}_{R^r_T}(A), A) = \mathbf{T}^r(A)\end{array}$$

(T4) Since $\alpha\odot\mathcal{H}_{R^r_T}(\alpha\to A) = \mathcal{H}_{R^r_T}(\alpha\odot(\alpha\to A)) \leq \mathcal{H}_{R^r_T}(A)$, then $\mathcal{H}_{R^r_T}(\alpha\to A) \leq \alpha\to\mathcal{H}_{R^r_T}(A)$. Thus

$$\begin{array}{l}\mathbf{T}^r(\alpha\to A) = e_{L^X}(\mathcal{H}_{R^r_T}(\alpha\to A), \alpha\to A)\\ = e_{L^X}(\alpha\to\mathcal{H}_{R^r_T}(A), \alpha\to A)\\ \geq e_{L^X}(\mathcal{H}_{R^r_T}(A), A) = \mathbf{T}^r(A)\end{array}$$

Hence $\mathbf{T}^r$ is an Alexandrov fuzzy topology. Since $\mathcal{H}_{R^s_T} \leq \mathcal{H}_{R^r_T}$ for $r \leq s$, $\mathbf{T}^s(A) = e_{L^X}(\mathcal{H}_{R^s_T}, A) \geq e_{L^X}(\mathcal{H}_{R^r_T}, A) = \mathbf{T}^r(A)$.

(13) Since $\mathbf{T}^r(A) = e_{L^X}(\mathcal{H}_{R^r_T}(A), A) = \top$ iff $A = \mathcal{H}_{R^r_T}(A)$, by (9),

$$\begin{array}{ll}\mathbf{T}_T(A) & = \bigvee\{r^*\in L \mid \mathbf{T}^r(A) = \top\}\\ & = \bigvee\{r^*\in L \mid \mathcal{H}_{R^r_T}(A) = A\}\\ & = \mathbf{T}_{H_T}(A) = \mathbf{T}(A).\end{array}$$

(8), (10), (12) and (14) are similarly proved.

Example 3.3. Let $(L = [0,1], \odot, \to, ^*)$ be a complete residuated lattice with a strong negation.

(1) Let $X = \{x, y, z\}$ be a set. Define a map $\mathbf{T} : [0,1]^X \to [0,1]$ as

$$\mathbf{T}(A) = A(x) \to A(z).$$

Trivially, $\mathbf{T}(\alpha_X) = 1$

Since $\alpha\odot A(x) \to \alpha\odot A(z) \geq A(x) \to A(z)$ from Lemma 2.4 (14), $\mathbf{T}(\alpha\odot A) \geq \mathbf{T}(A)$. Since $(\alpha\to A(x)) \to (\alpha\to A(z)) \geq A(x) \to A(z)$ from Lemma 2.4 (10), $\mathbf{T}(\alpha\to A) \geq \mathbf{T}(A)$. By Lemma 2.4 (8), $\mathbf{T}(\bigvee_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma}\mathbf{T}(A_i)$ and $\mathbf{T}(\bigwedge_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma}\mathbf{T}(A_i)$. Hence $\mathbf{T}$ is an Alexandrov fuzzy topology.

Since $\mathbf{T}(A) = A(x) \to A(z) \geq r^*$, then $A(z) \geq A(x)\odot r^*$. Put $A(x) = 1, A(y) = 0$. So, $R^r_T(x,y) = \bigwedge\{A(x)\to A(y) \mid \mathbf{T}(A) \geq r^*\} = 0$ and $R^r_T(x,z) = \bigwedge\{A(x)\to A(z) \mid \mathbf{T}(A) \geq r^*\} = r^*$

$$\begin{pmatrix} R^r_T(x,x) = 1 & R^r_T(x,y) = 0 & R^r_T(x,z) = r^*\\ R^r_T(y,x) = 0 & R^r_T(y,y) = 1 & R^r_T(y,z) = 0\\ R^r_T(z,x) = 0 & R^r_T(z,y) = 0 & R^r_T(z,z) = 1\end{pmatrix}$$

By Theorem 3.1(3), we obtain $\mathcal{H}_{R^r_T}(A)(y) = \bigvee_{x\in X}(A(x)\odot R^r_T(x,y))$ such that

$$\mathcal{H}_{R^r_T}(A) = (A(x), A(y), A(z)\vee(A(x)\odot r^*))$$

If $A(x)\odot r^* \leq A(z)$, then $\mathcal{H}_{R^r_T}(A) = A$. Thus $A \in \tau_{\mathcal{H}_{R^r_T}}$. Moreover, since $\mathbf{T}(A) = A(x)\to A(z) \geq r^*$ iff $A(z) \geq A(x)\odot r^*$, $A \in \tau_T^{r^*}$ iff $A \in \tau_{\mathcal{H}_{R^r_T}}$.

So, $\tau_T^{r^*} = \tau_{\mathcal{H}_{R^r_T}}$.

$$\begin{aligned}\mathbf{T}_{H_T}(A) &= \bigvee\{r^*\in L \mid \mathcal{H}_{R^r_T}(A) = A\}\\ &= A(x)\to A(z) = \mathbf{T}(A).\end{aligned}$$

Moreover, we obtain

$$\begin{aligned}\mathbf{T}^r(A) &= \bigwedge_{x\in X}(\mathcal{H}_{R^r_T}(A)(x)\to A(x))\\ &= (A(x)\odot r^*)\to A(z) = r^*\to(A(x)\to A(z)).\end{aligned}$$

$$\begin{aligned}\mathbf{T}_T(A) &= \bigvee\{r^*\in L \mid \mathbf{T}^r(A) = 1\}\\ &= A(x)\to A(z)\end{aligned}$$

Hence $\mathbf{T}_T = \mathbf{T}_{H_T} = \mathbf{T}$.

$$\mathcal{H}_{R^r_T}(1_x)(z) = \bigwedge\{B(z) \mid B \geq 1_x,\ \mathbf{T}(B)\geq r^*\}$$

Since $B(x) = 1$ and $\mathbf{T}(B) = 1\to B(z) = B(z)\geq r^*$, then $\mathcal{H}_{R^r_T}(1_x)(z) = r^*$.

$$\mathcal{H}_{R^r_T}(1_x)(x) = \bigwedge\{B(x) \mid B \geq 1_x,\ \mathbf{T}(B)\geq r^*\} = 1$$

$$\mathcal{H}_{R^r_T}(1_x)(y) = \bigwedge\{B(y) \mid B \geq 1_x,\ \mathbf{T}(B)\geq r^*\} = 0$$

$$\mathcal{H}_{R^r_T}(1_z)(x) = \bigwedge\{B(x) \mid B \geq 1_z,\ \mathbf{T}(B)\geq r^*\}$$

Since $B(z) = 1$ and $\mathbf{T}(B) = B(x)\to 1 = 1$, then $\mathcal{H}_{R^r_T}(1_z)(x) = 0$.

$$\begin{pmatrix} \mathcal{H}_{R^r_T}(1_x)(x) = 1 & \mathcal{H}_{R^r_T}(1_x)(y) = 0 & \mathcal{H}_{R^r_T}(1_x)(z) = r^*\\ \mathcal{H}_{R^r_T}(1_y)(x) = 0 & \mathcal{H}_{R^r_T}(1_y)(y) = 1 & \mathcal{H}_{R^r_T}(1_y)(z) = 0\\ \mathcal{H}_{R^r_T}(1_z)(x) = 0 & \mathcal{H}_{R^r_T}(1_z)(y) = 0 & \mathcal{H}_{R^r_T}(1_z)(z) = 1 \end{pmatrix}$$

Then $R^r_T(x,y) = \mathcal{H}_{R^r_T}(1_x)(y)$.

(2) By (1), we obtain a map $\mathbf{T}^* : [0,1]^Y \to [0,1]$ as

$$\mathbf{T}^*(A) = A^*(x)\to A^*(z) = A(z)\to A(x).$$

Since $\mathbf{T}^*(A) = A(z)\to A(x)\geq r^*$, then $A(x)\geq A(z)\odot r^*$. Put $A(z) = 1, A(y) = 0$. So, $R^r_{T^*}(z,y) = \bigwedge\{A(z)\to A(y) \mid \mathbf{T}^*(A)\geq r^*\} = 0$ and $R^r_{T^*}(z,x) = \bigwedge\{A(z)\to A(x) \mid \mathbf{T}(A)\geq r^*\} = r^*$

$$\begin{pmatrix} R^r_{T^*}(x,x) = 1 & R^r_{T^*}(x,y) = 0 & R^r_{T^*}(x,z) = 0\\ R^r_{T^*}(y,x) = 0 & R^r_{T^*}(y,y) = 1 & R^r_{T^*}(y,z) = 0\\ R^r_{T^*}(z,x) = r^* & R^r_{T^*}(z,y) = 0 & R^r_{T^*}(z,z) = 1 \end{pmatrix}$$

Moreover, $R^r_{T^*}(x,y) = R^{-r}_T(x,y) = R^r_T(y,x)$ for all $x, y \in X$.

$$\mathcal{H}_{R^r_{T^*}}(A)(y) = \bigvee_{x\in X} (A(x) \odot R^r_{T*}(x,y)).$$

$$\mathcal{H}_{R^r_{T^*}}(A) = (A(x) \vee (A(z) \odot r^*), A(y), A(z))$$

If $A(z) \odot r^* \leq A(x)$, then $\mathcal{H}_{R^r_{T^*}}(A) = A$. If $A(x) \odot r^* \leq A(z)$, then $\mathcal{H}_{R^r_T}(A) = A$. Thus $A \in \tau_{\mathcal{H}_{R^r_{T^*}}}$. Moreover, since $\mathbf{T}^*(A) = A(z) \to A(x) \geq r^*$ iff $A(x) \geq A(z) \odot r^*$, $A \in \tau^{r^*}_{T^*}$ iff $A \in \tau_{\mathcal{H}_{R^r_{T^*}}}$. Thus

$$\begin{aligned}\mathbf{T}_{H_{T^*}}(A) &= \bigvee\{r^* \in L \mid \mathcal{H}_{R^r_{T^*}}(A) = A\} \\ &= A(z) \to A(x) = \mathbf{T}^*(A) = A^*(x) \to A^*(z) = \mathbf{T}(A^*).\end{aligned}$$

Moreover, we obtain

$$\begin{aligned}\mathbf{T}^{*r}(A) &= \bigwedge_{x\in X}(\mathcal{H}_{R^r_{T^*}}(A)(x) \to A(x)) \\ &= (A(z) \odot r^*) \to A(x) = r^* \to (A(z) \to A(x)).\end{aligned}$$

$$\begin{aligned}\mathbf{T}_{T^*}(A) &= \bigvee\{r^* \in L \mid \mathbf{T}^{*r}(A) = 1\} \\ &= A(z) \to A(x) = \mathbf{T}^*\end{aligned}$$

Hence $\mathbf{T}_{T^*} = \mathbf{T}_{H_{T^*}} = \mathbf{T}^*$.

$$\mathcal{H}_{R^r_{T^*}}(1_x)(z) = \bigwedge\{B(z) \mid B \geq 1_x,\ \mathbf{T}^*(B) \geq r^*\}$$

Since $B(x) = 1$ and $\mathbf{T}^*(B) = B(z) \to 1 = 1$, then $\mathcal{H}_{R^r_{T^*}}(1_x)(z) = 0$.

$$\mathcal{H}_{R^r_{T^*}}(1_z)(y) = \bigwedge\{B(y) \in L^X \mid B \geq 1_z,\ \mathbf{T}^*(B) \geq r^*\} = 0$$

$$\mathcal{H}_{R^r_{T^*}}(1_y)(y) = \bigwedge\{B(y) \in L^X \mid B \geq 1_y,\ \mathbf{T}^*(B) \geq r^*\} = 1$$

$$\mathcal{H}_{R^r_{T^*}}(1_z)(x) = \bigwedge\{B(x) \in L^X \mid B \geq 1_z,\ \mathbf{T}^*(B) \geq r^*\}$$

Since $B(z) = 1$ and $\mathbf{T}^*(B) = 1 \to B(x) = B(x) \geq r^*$, then $B(x) \geq r^*$. We have $\mathcal{H}_{R^r_{T^*}}(1_z)(x) = r^*$.

$$\begin{pmatrix} \mathcal{H}_{R^r_{T^*}}(1_x)(x) = 1 & \mathcal{H}_{R^r_{T^*}}(1_x)(y) = 0 & \mathcal{H}_{R^r_{T^*}}(1_x)(z) = 0 \\ \mathcal{H}_{R^r_{T^*}}(1_y)(x) = 0 & \mathcal{H}_{R^r_{T^*}}(1_y)(y) = 1 & \mathcal{H}_{R^r_{T^*}}(1_y)(z) = 0 \\ \mathcal{H}_{R^r_{T^*}}(1_z)(x) = r^* & \mathcal{H}_{R^r_{T^*}}(1_z)(y) = 0 & \mathcal{H}_{R^r_{T^*}}(1_z)(z) = 1 \end{pmatrix}$$

Then $R^r_{T^*}(x,y) = \mathcal{H}_{R^r_{T^*}}(1_x)(y)$.

(3) Let $(L = [0,1], \odot, \to, ^*)$ be a complete residuated lattice with a strong negation defined by, for each $n \in N$,

$$x \odot y = ((x^n + y^n - 1) \vee 0)^{\frac{1}{n}},\ \ x \to y = (1 - x^n + y^n)^{\frac{1}{n}} \wedge 1,\ x^* = (1 - x^n)^{\frac{1}{n}}.$$

By (1) and (2), we obtain

$$\mathbf{T}(A) = (1 - A(x)^n + A(z)^n)^{\frac{1}{n}} \wedge 1, \quad \mathbf{T}^*(A) = (1 - A(z)^n + A(x)^n)^{\frac{1}{n}} \wedge 1.$$

$$R^r_T = \begin{pmatrix} 1 & 0 & (1-r^n)^{\frac{1}{n}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad R^r_{T^*} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ (1-r^n)^{\frac{1}{n}} & 0 & 1 \end{pmatrix}$$

$$\mathcal{H}_{R^r_T}(A) = (A(x), A(y), A(z) \vee ((A(x)^n + 1 - r^n) \vee 0)^{\frac{1}{n}})$$

$$\mathcal{H}_{R^r_{T^*}}(A) = (A(x) \vee ((A(z)^n + 1 - r^n) \vee 0)^{\frac{1}{n}}, A(y), A(z))$$

Since $\mathbf{T}(A) = (1 - A(x)^n + A(z)^n)^{\frac{1}{n}} \wedge 1 \geq (1 - x^n)^{\frac{1}{n}}$, we have

$$\begin{array}{ll} \tau^{r^*}_T = \tau_{R^r_T} & = \{A \in L^X \mid A^n(x) - A^n(z) \leq r^n\} \\ \tau^{r^*}_{T^*} = \tau_{R^r_{T^*}} & = \{A \in L^X \mid A^n(z) - A^n(x) \leq r^n\}. \end{array}$$

$$\begin{array}{ll} \mathbf{T}^r(A) & = (A(x) \odot r^*) \to A(z) = (r^n - A(x)^n + A(z)^n)^{\frac{1}{n}} \wedge 1 \\ \mathbf{T}^{*r}(A) & = (A(z) \odot r^*) \to A(x) = (r^n - A(z)^n + A(x)^n)^{\frac{1}{n}} \wedge 1. \end{array}$$

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