Mathematica Aeterna, Vol. 4, 2014, no. 4, 389 - 401

Global controllability of a certain class of minimum time optimal control problems in 2-Banach spaces

Utpalendu Adak

Department of Mathematics
National Institute of Technology Durgapur
Durgapur-713 209
West Bengal, INDIA
uadak2010@gmail.com

Lakshmi Kanta Dey

Department of Mathematics
National Institute of Technology Durgapur
Durgapur-713 209
West Bengal, INDIA
lakshmikdey@yahoo.co.in

Hora Krishna Samanta

Department of Mathematics
Netaji Mahavidyalaya
Arambagh-712 601
West Bengal, INDIA

Abstract

In this paper, the global controllability of a certain class of minimum time control problems are considered in 2-Banach space. Necessary and sufficient conditions (N.A.S.C.) for global controllability of such problems are derived in 2-Banach space.

Mathematics Subject Classification: 34H05, 46N99, 49J15, 49J27, 49K27, 93B28

Keywords: Minimum time control, global controllability, 2-Banach spaces, Hahn-Banach theorem, supporting hyper-plane.

1 Introduction

Different types of optimization problems have been solved in normed linear spaces by many authors [[11]-[17]] during the last several decades. A minimum cost control problem was formulated and solved by Minamide and Nakamura [25] in Banach space. Burns [[11], [12]]; Choudhury and Mukherjee [[13]-[17]] developed a uniform theory of time optimal control problems for system which can be represented in terms of bounded, linear and onto transformation from a Banach space of control functions to another Banach space of control functions. Global controllability is an important concept in the field of control theory [22]. Mukherjee [27] solved the global controllability of a class of minimum time control problems in Banach space. Recently, important results of functional analysis in 2-Banach space were developed by different authors [[1], [18]-[21], [23], [24], [26], [30]]. They have developed a uniform theory in 2-Banach space. The concept of linear 2-normed spaces has been first introduced by Gähler [20] as an extension of the usual norm and as an interesting non-linear generalization of normed linear space which has been flourished extensively in different directions. He proved that if the space is a normed linear space of dimension greater than one, then it is possible to define a 2-norm on it. But, the converse is not true [19] i.e., every 2-normed linear space in not necessarily normable [[28], [29]]. In [5] authors have developed a certain class of minimum time optimal control problem in 2-Banach space. Here we define the global controllability of a certain class of generalized minimum time control problems in arbitrary 2-Banach space. We demonstrate how the solution of the original problem is obtained from that of the auxiliary problem of minimization of 2-norm for a terminal time given in advance, which is solved by generalized functional analytic techniques. More precisely we consider the following problem as follows:

Let B_t and D be 2-Banach spaces. Let $T_t : B_t \to D$ be a bounded linear transformation depending upon the parameter t. Let $U_e(y;t) = \{x \in B_t : N_1(x-y,e) \leq t\}$ for some non zero $y, e \in B_t$ be a ball in B_t and let $\xi \in D$. The problem is to determine $u \in U_e(y;t)$ such that $T_t u = \xi$ and t is minimum. Here B_t is an increasing function of t in the sense that $B_{t_1} \subset B_{t_2}$, whenever $t_1 \leq t_2$. Also $T_{t_1} : B_{t_1} \to D$ can be regarded as the restriction of $T_{t_2} : B_{t_2} \to D$. It is not tough to show that under the above condition $U_e(y;t_1) \subset U_e(y;t_2)$.

2 Technical Preliminaries

Throughout this article we consider, without any loss of generality, real 2-Banach space of any dimension. We present here some of the definitions and useful results for the organization of the paper.

Definition 2.1. *Let X be a real vector space of dimension d, $d \geq 2$. A 2-norm on X is a function $N(.,.) : X \times X \to \mathbb{R}$ which satisfies the following conditions:*

1. *$N(x, y) = 0$ iff x and y are linearly dependent (L.D.),*
2. *$N(x, y) = N(y, x)$, for all $x, y \in X$,*
3. *$N(\alpha x, y) = |\alpha| N(x, y), \alpha \in \mathbb{R}$ and for all $x, y \in X$,*
4. *$N(x, y + z) \leq N(x, y) + N(x, z)$ for all $x, y, z \in X$.*

The pair $(X, N(.,.))$ is then called a linear 2-normed space.

We observe that $N(.,.)$ is non-negative. A 2-normed space $(X, N(.,.))$ is called a 2-Banach space if every Cauchy sequence is convergent. Also if X and Y are 2-Banach spaces over the field of real numbers, it can be easily verified that $X \times Y$ is also 2-Banach space with respect to the 2-norm $N_3(.,.)$ where $N_3((x_i, y_i), (x_j, y_j)) = min\{N_1(x_i, x_j), N_2(y_i, y_j)\}$, i.e. $N_3(.,.) = min\{N_1(.,.), N_2(.,.)\}$, $N_1(.,.)$ and $N_2(.,.)$ are 2-norm functions defined on X and Y respectively and $N_3((x_i, y_i), (x_j, y_j)) = 0$ if either x_i, x_j are L.D. in X or y_i, y_j are L.D. in Y. Let $N_1^{'}, N_2^{'}, N_3^{'}$ are then 2-norm functions defined on the spaces $X^{'}, Y^{'}, (X \times Y)^{'}$ respectively, where $N_3^{'}(.,.) = min\{N_1^{'}(.,.), N_2^{'}(.,.)\}$; $X^{'}, Y^{'}, (X \times Y)^{'}$ denote the conjugate of $X, Y, (X \times Y)$ respectively.

Example 2.2. *Consider $(\mathbb{R}^2, N(.,.))$ with 2-norm defined by $N(a, b) = |a_1 b_2 - a_2 b_1|$ where $a = (a_1, a_2)$ and $b = (b_1, b_2) \in \mathbb{R}^2$ and we call this 2-norm is as standard 2-norm on $\mathbb{R}^2$. Geometrically this represents the area of the parallelogram determined by the vectors a and b as the adjacent sides. For $X = \mathbb{R}^3$. If we take*

$N_1(x, y) = \max\{|x_1 y_2 - x_2 y_1| + |x_1 y_3 - x_3 y_1|, |x_1 y_2 - x_2 y_1| + |x_2 y_3 - x_3 y_2|\}$, *where*

$x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3) \in \mathbb{R}^3$. Then $N_1(.,.)$ is a 2-norm on $\mathbb{R}^3$.

Remark 2.3. *Let $(X, \|.\|)$ be a normed linear space of dimension > 1, then we can always define a 2-norm $N(.,.)$ on it. For $x, y \in X$,*

$$N(x, y) = \sup_{f,g \in X^*, \|f\| = \|g\| = 1} |f(x)g(y) - g(x)f(y)|.$$

On the other hand, there are examples of 2-normed linear spaces X, where we can not define any induced norm [20, 29]. Evidently, 2-normed linear space can be considered as a non linear generalization of normed linear space. But it should be noted that given a 2-norm $N(.,.)$ on a finite dimensional 2-normed *linear space X, the 2-norm induces a derived norm $N_\infty(.)$ on X as follows: Let $\mathbb{B} = \{\beta_1, \beta_2, ..., \beta_d\}$ be a basis for X. Then for $x \in X$,*

$$N_\infty(x) := \max\{N(x, \beta_i) : i = 1, 2, ..., d\}.$$

For finite dimensional 2-normed linear space, all these norms are equivalents. Also for infinite dimensional 2-normed linear space; it may be true, provided the space is separable inner product space [21].

Example 2.4. *We construct an example of a 2-normed linear space $(\mathbb{R}, N(.,.))$ where*

$$N(a, b) = \frac{1}{2} \sup_{f,g \in F_\mathbb{R}} abs \left(\begin{vmatrix} f(a) & g(a) \\ f(b) & g(b) \end{vmatrix} \right)$$

and $F_\mathbb{R}$ is the set of all bounded functionals on domain $\mathbb{R}$ and with the norm less or equal to 1. It is to be noted that here we take $\mathbb{R}$ as a Banach space over the field of rationals $\mathbb{Q}$. It is not tough to prove that $(\mathbb{R}, N(.,.))$ is a 2-Banach space with the 2-norm $N(.,.)$.

Example 2.5. *The n-dimensional Euclidean 2-norm $N(.,.)$ defined on $\mathbb{R}^n (n \geq 2)$ is of the form*

$$N(a, b) = \sqrt{\sum_{i<j} (\alpha_i \beta_j - \alpha_j \beta_i)^2}$$

for $a = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $b = (\beta_1, \beta_2, ..., \beta_n) \in \mathbb{R}^n$.

Definition 2.6. *A 2-functional is a real valued mapping with domain $A \times B$, where A and B are linear manifolds of a 2-normed linear space X. Let $f : A \times B \to \mathbb{R}$ be a 2-functional on a 2-normed linear space X then f is called a linear 2-functional if*
(i) $f(a + b, c + d) = f(a, b) + f(a, d) + f(c, b) + f(c, d)$
(ii) $f(\alpha a, \beta b) = \alpha\beta f(a, b)$ for $\alpha, \beta \in \mathbb{R}$.

For other interesting examples one is referred to [[2]-[10]]. We refer [5] for necessary definitions and proofs of the following Theorems and Corollaries.

Definition 2.7. *The set of all points $\xi \in D$ such that $T_t u = \xi$ for some $u \in U_e(y; t)$ and for some non zero $y, e \in B_t$ is called the reachable set with respect to T_t and is denoted by $C(t)$.*

Theorem 2.8. *The reachable region $C(t)$ is bounded and a convex body, symmetrical with respect to the origin of D.*

Corollary 2.9. *The reachable region $C(t)$ is closed when B_t is either a reflexive space or it can be considered as a conjugate of some other 2-Banach space.*

Theorem 2.10. *An admissible control which will be optimal must satisfy* $\{N_1(u, u_1) : u \in U_e(y; t)\} = 1$ *for some* $u_1 \in U_t$.

Theorem 2.11. *Let* $\xi \in \delta C(t)$ *and* $\phi \in D^*$ *denotes a supporting hyper-plane to* $C(t)$ *at* ξ. *Then* $\langle \xi, \phi \rangle = \{N_1'(T_t^*\phi, f) : T_t^*\phi, f \in B_t^*\}$, *where* D^* *is the conjugate space to* D *and* T^* *is the transformation adjoint to* T.

Theorem 2.12. *Let* $\xi \in \delta C(t)$ *where* t *is the given terminal time and* $\phi \in D^*$ *denotes a supporting hyper-plane at* ξ. *Let* u_ϕ *be the optimal control to reach at* ξ *in the above sense. Then* u_ϕ *maximizes* $\langle u, T_t^*\phi \rangle$ *where* T_t^* *and* D^* *denote the adjoint transformation and adjoint space to* T_t *and* D *respectively and*

$$\langle u_\phi, T_t^*\phi \rangle = \min_{\{N_1(u,u_1): u \in U_e(y;t)\}=1} \langle u, T_t^*\phi \rangle = \{N_1'(T_t^*\phi, f) : T_t^*\phi, f \in B_t^*\}$$

for some $u_1 \in U_e(y; t)$ *and* $\{N_1(u_\phi, v_\phi) : u_\phi, v_\phi \in U_e(y; t)\} = 1$.

Theorem 2.13. *Let* K *be a weakly compact, convex set in a* 2*-Banach space* D *and let* ϕ *be any element* $\in D^*$, *the conjugate space of* D. *Then there exists a point* $\eta_0 \in K$, *such that* ϕ *denotes a supporting hyper-plane to* K *at* $\eta_0 \in \delta K$.

3 Main Results

In this section we study the existence of the optimal control of the following problem in an arbitrary 2-Banach space.

Auxiliary Problem: Let $\xi \in \delta C(t)$ where $\delta C(t)$ denotes the boundary of the reachable region $C(t)$ for some given time t. Then we have to determine $u \in U_e(y; t)$ such that $T_t u = \xi$ and $\{N_1(u, y) : u \in U_e(y; t)\}$ for some $y \in U_e(y; t)$ is minimum. We call this as minimum 2-norm problem. The corresponding control is regarded as the optimal control.

Now we find the form of the optimal control and also the shape of the reachable region $C(t)$ with respect to the minimum time t.

Theorem 3.1. *If* $\langle \xi, \phi \rangle = N_1'(T_t^*\phi, f)$ *where* $T_t^*\phi, f \in B_t^*$, *for some* $\xi \in C(t)$ *and some* $\phi \in D^*$, *then* $\xi \in \delta C(t)$ *and* ϕ *denotes a supporting hyper-plane to* $C(t)$ *at* ξ, *where* B_t *is either reflexive 2-Banach space or it can be considered as the conjugate of some other* 2*-Banach space.*

Proof. By the hypothesis made on B_t, we can say $C(t)$ is weakly compact. Also $C(t)$ is convex. Since $\phi \in D^*$ hence by Theorem 2.13, we can show that there exists a point $\eta \in \delta C(t)$ such that ϕ denotes a supporting hyper-plane to $C(t)$ at η. Consequently by Theorem 2.11, $\langle \eta, \phi \rangle = N_1'(T_t^*\phi, f)$ where $T_t^*\phi, f \in B_t^*$. But by hypothesis $\langle \xi, \phi \rangle = N_1'(T_t^*\phi, f)$ where $T_t^*\phi, f \in B_t^*$ for some $\xi \in C(t)$ and some $\phi \in D^*$. If $\xi \notin \delta C(t)$, then $\langle \xi, \phi \rangle < \langle \eta, \phi \rangle$. Therefore, $\langle \xi, \phi \rangle = N_1^{'}(T_t^* \phi, f)$ where $T_t^* \phi, f \in B_t^*$ which contradicts the hypothesis. Hence ξ must be in $\delta C(t)$. Also since $\langle \eta^{'}, \phi \rangle \leqslant \langle \eta, \phi \rangle = \langle \xi, \phi \rangle$, for $\eta^{'} \in C(t)$. Consequently, ϕ defines a supporting hyper-plane at ξ. □

Theorem 3.2. *The N.A.S.C. for the point $\xi \in C(t)$ to be in $\delta C(t)$ at the time $t = t_f$ is that*

$$\max_{\phi \in D^* such\ that\ N_1^{'}(T_t^* \phi, f) \neq 0} \frac{\langle \xi, \phi \rangle}{N_1^{'}(T_t^* \phi, f)} = 1,$$

where $T_t^* \phi, f \in B_t^*$ and B_t is either reflexive 2-Banach space or it can be considered as the conjugate of some other 2-Banach space.

Proof. Sufficiency: Suppose

$$\max_{\phi \in D^* such\ that\ N_1^{'}(T_t^* \phi, f) \neq 0} \frac{\langle \xi, \phi \rangle}{N_1^{'}(T_t^* \phi, f)} = 1,$$

Let the maximum be attained for some $\phi = \phi_\xi \in D^*$. Then $\langle \xi, \phi_\xi \rangle = N_1^{'}(T_{t_f}^* \phi_\xi, f)$ where $T_{t_f}^* \phi_\xi, f \in B_t^*$ for some $\xi \in C(t)$. Consequently, by Theorem 3.1, $\xi \in \delta C(t_f)$ and ϕ_ξ denotes a supporting hyper-plane to $C(t_f)$ at ξ.
Necessity: Let $\xi \in \delta C(t_f)$. Then by the Theorem 2.11, $\langle \xi, \phi_\xi \rangle = N_1^{'}(T_{t_f}^* \phi_\xi, f)$ where $T_{t_f}^* \phi_\xi, f \in B_t^*$ where ϕ_ξ is a supporting hyper-plane to $C(t_f)$ at ξ. Therefore

$$\max_{\phi \in D^* such\ that\ N_1^{'}(T_t^* \phi, f) \neq 0} \frac{\langle \xi, \phi \rangle}{N_1^{'}(T_t^* \phi, f)} = 1. \tag{1}$$

Now, we have to show that (1) gives the maximum value of the left hand side for all $\phi \in D^*$. Let $\psi \in D^*$ be any other 2-functional. If ψ is a supporting hyper-plane to $C(t_f)$ at ξ, then (1) holds. So, let us assume that $\psi \in D^*$ is not a supporting hyper-plane to $C(t_f)$ at ξ. Now by Theorem 2.8 and its Corollary 2.9 it can be shown that $C(t_f)$ is convex, weakly compact, closed and bounded set. Consequently, by Theorem 2.13, corresponding to $\psi \in D^*$ there exists a $\eta_0 \in C(t_f) \cap \delta C(t_f)$ such that ψ is a supporting hyper-plane to η_0. Hence we have $\langle \xi, \psi \rangle \leq \langle \eta_0, \psi \rangle = N_1^{'}(T_{t_f}^* \psi, f)$ where $T_{t_f}^* \psi, f \in B_t^*$. Therefore $\frac{\langle \xi, \psi \rangle}{N_1^{'}(T_{t_f}^* \psi, f)} \leq 1$.

This proves that $\displaystyle\max_{\psi \in D^* such\ that\ N_1^{'}(T_{t_f}^* \psi, f) \neq 0} \frac{\langle \xi, \psi \rangle}{N_1^{'}(T_{t_f}^* \psi, f)} = 1.$ □

The proof of the Theorems 3.3, 3.5 and Corollary 3.4 can be found in [5].

Theorem 3.3. *Let $\xi \in C(t_f) \cap \delta C(t_f)$ where $C(t_f)$ is the reachable region. Then $\displaystyle\max_{\psi \in D^* such\ that\ N_1^{'}(T_t^* \psi, f) \neq 0} \frac{\langle \xi, \psi \rangle}{N_1^{'}(T_t^* \psi, f)}$ is ≤ 1 or ≥ 1 according as $t \geq t_f$ or*

$t \leq t_f$ where $T_t^*\psi, f \in B_t^*$. Moreover the max is attained at a point $\psi \in D^*$, where ψ is the supporting hyper-plane to $\delta C(t)$ at the intersection with the ray through ξ.

To prove this we require the following Corollary.

Corollary 3.4. *Let $\xi \in C(t_f), \eta = l\xi \in \delta C(t)$ and $\psi \in D^*$ define the supporting hyper-plane at η, then $\langle \xi, \psi \rangle > 0$.*

Theorem 3.5. *Let $t_1 < t_2$ and $T_{t_1} : B_{t_1} \to D, T_{t_2} : B_{t_2} \to D$ be bounded linear onto transformations. Then $C(t_1) \subseteq C(t_2)$ and $\delta C(t_1) \cap \delta C(t_2) = \Phi$ iff $N_1'(T_{t_2}^*\phi, f_2) > N_1'(T_{t_1}^*\phi, f_1)$ where $T_{t_1}^*\phi, T_{t_2}^*\phi, f_1, f_2 \in B_t^*$, for some $\phi \in D^*$ and Φ denotes the null set.*

Theorem 3.6. *Let $\xi \in C(t_f) \cap \delta C(t_f)$ and $t \geq t_f$. Then $\max\limits_{\phi \in D^* such\ that\ N_1'(T_t^*\phi, f) \neq 0} \dfrac{\langle \xi, \phi \rangle}{N_1'(T_t^*\phi, f)}$ is a non-increasing function of $t, t \geq t_f$ where $T_t^*\phi, f \in B_t^*$, and for some $\phi \in D^*$.*

Proof. Let $t_f < t_1 < t_2$. Then from Theorem 3.3

$$\max_{\phi \in D^* such\ that\ N_1'(T_{t_1}^*\phi, f_1) \neq 0} \frac{\langle \xi, \phi \rangle}{N_1'(T_{t_1}^*\phi, f_1)} = \frac{\langle \xi, \phi_1 \rangle}{N_1'(T_{t_1}^*\phi_1, f_1)} \tag{2}$$

for $T_{t_1}^*\phi, f_1 \in B_t^*$ and for some $\phi_1 \in D^*, T_{t_1}^*\phi \in B_t^*$ where $\phi_1 \in D^*$, denotes a supporting hyper-plane to the point of intersection of the ray through ξ with $\delta C(t_1)$. Denote this point by $\xi_1 = l_1\xi$ for some $l_1 > 1$. Let $u_{t_1} \in U_e(y; t_1)$ be the optimal control to reach $\xi_1 = T_{t_1}u_{t_1}$ where $U_e(y; t_1)$ is a ball in B_{t_1}. Since T_{t_1} is the restriction of T_{t_2} on $U_e(y; t_1)$, we have

$$\xi_1 = T_{t_1}u_{t_1} = T_{t_2}u_{t_1}. \tag{3}$$

By Theorem 2.11, we also have, $\langle \xi_1, \phi_1 \rangle = N_1'(T_{t_1}^*\phi_1, f_1)$ where $T_{t_1}^*\phi_1, f_1 \in B_t^*$.

Thus from (2) we get, for $T_{t_1}^*\phi, T_{t_1}^*\phi_1, f_1 \in B_t^*$, and for some $\phi_1 \in D^*$,

$$\max_{\phi \in D^* such\ that\ N_1'(T_{t_1}^*\phi, f_1) \neq 0} \frac{\langle \xi, \phi \rangle}{N_1'(T_{t_1}^*\phi, f_1)} = \frac{\langle \xi, \phi_1 \rangle}{N_1'(T_{t_1}^*\phi_1, f_1)} = \frac{\langle \xi, \phi_1 \rangle}{\langle \xi_1, \phi_1 \rangle} = \frac{\langle \xi, \phi_1 \rangle}{\langle T_{t_1}u_{t_1}, \phi_1 \rangle} = \frac{\langle \xi, \phi_1 \rangle}{\langle T_{t_2}u_{t_1}, \phi_1 \rangle}. \tag{4}$$

Again, let

$$\max_{\phi \in D^* such\ that\ N_1'(T_{t_2}^*\phi, f_2) \neq 0} \frac{\langle \xi, \phi \rangle}{N_1'(T_{t_2}^*\phi, f_2)} = \frac{\langle \xi, \phi_2 \rangle}{N_1'(T_{t_2}^*\phi_2, f_2)}$$

where $T_{t_2}^*\phi, T_{t_2}^*\phi_2, f_2 \in B_t^*$ and some $\phi_2 \in D^*$, defines a supporting hyper-plane to $\xi_2 = l_2\xi_1$ for some $l_2 \geq l_1$ and $\xi_2 \in \delta C(t_2)$.
Then we obtain similarity as before

$$\max_{\phi \in D^* such\ that\ N_1'(T_{t_2}^*\phi, f_2) \neq 0} \frac{\langle \xi, \phi \rangle}{N_1'(T_{t_2}^*\phi, f_2)} = \frac{\langle \xi, \phi_2 \rangle}{N_1'(T_{t_2}^*\phi_2, f_2)} = \frac{\langle \xi, \phi_2 \rangle}{\langle T_{t_2}u_{t_2}, \phi_2 \rangle} \tag{5}$$

where $u_{t_2} \in U_{t_2}(y) \subset B_{t_2}$ is the optimal control to reach at ξ_2.

Now, $\xi_1 \in C(t_1) = T_{t_1}U_{t_1}(y) = T_{t_2}U_{t_1}(y) \subset T_{t_2}U_{t_2}(y) = C(t_2)$. Since ϕ_2 is a supporting hyper-plane to $C(t_2)$ at ξ_2, therefore we have $\langle \xi_1, \phi_2\rangle \leq \langle \xi_2, \phi_2\rangle = \langle T_{t_2}u_{t_2}, \phi_2\rangle$. Thus by (3), we have

$$\langle T_{t_2}u_{t_1}, \phi_2\rangle \leq \langle T_{t_2}u_{t_2}, \phi_2\rangle. \tag{6}$$

But

$$\langle T_{t_2}u_{t_1}, \phi_2\rangle = \langle \xi_1, \phi_2\rangle = l_1\langle \xi, \phi_2\rangle = \frac{1}{l_2}\langle \xi_2, \phi_2\rangle > 0. \tag{7}$$

Since $\theta \in Int\ C(t_2)$.

This also follows from the fact that $\langle \xi_2, \phi_2\rangle = N_1^{'}(T_{t_2}^{*}\phi_2, f_2)$ from Theorem 2.11, where $T_{t_2}^{*}\phi_2, f_2 \in B_t^{*}$ and for some $\xi_2 \in \delta C(t_2)$ and for some $\phi_2 \in D^{*}$.

Also

$$\langle \xi, \phi_2\rangle = \frac{1}{l_1 l_2}\langle \xi_2, \phi_2\rangle > 0. \tag{8}$$

Hence from (6),(7), (8) we have

$$\frac{\langle \xi, \phi_2\rangle}{\langle T_{t_2}u_{t_1}, \phi_2\rangle} \geq \frac{\langle \xi, \phi_2\rangle}{\langle T_{t_2}u_{t_2}, \phi_2\rangle}. \tag{9}$$

Since max is attained at ϕ_1, then from (4) and (9),

$$\max_{\phi \in D^{*} such\ that\ N_1^{'}(T_{t_1}^{*}\phi, f_1) \neq 0} \frac{\langle \xi, \phi\rangle}{N_1^{'}(T_{t_1}^{*}\phi, f_1)} = \frac{\langle \xi, \phi_1\rangle}{\langle T_{t_2}u_{t_1}, \phi_1\rangle} \geq \frac{\langle \xi, \phi_2\rangle}{\langle T_{t_2}u_{t_1}, \phi_2\rangle} \tag{10}$$

where $T_{t_1}^{*}\phi, f_1 \in B_t^{*}$ and for some $\xi \in \delta C(t)$ and for some $\phi_1, \phi_2 \in D^{*}$.

Now using (9) and (10), we have

$$\max_{\phi \in D^{*} such\ that\ N_1^{'}(T_{t_1}^{*}\phi, f) \neq 0} \frac{\langle \xi, \phi\rangle}{N_1^{'}(T_{t_1}^{*}\phi, f)} \geq \frac{\langle \xi, \phi_2\rangle}{\langle T_{t_2}u_{t_1}, \phi_2\rangle} \geq \frac{\langle \xi, \phi_2\rangle}{\langle T_{t_2}u_{t_2}, \phi_2\rangle} =$$

$\max_{\phi \in D^{*} such\ that\ N_1^{'}(T_{t_2}^{*}\phi, f) \neq 0} \frac{\langle \xi, \phi\rangle}{N_1^{'}(T_{t_2}^{*}\phi, f)}$. This proves the theorem. □

Theorem 3.7. *[5] Let X be a 2-normed linear space and X^{*} be its conjugate. Then $\exists$ a real bounded 2-linear functional $F \in X^{*}$, defined on X, such that $F(x_i, x_j) = N_1(x_i, x_j)$ where $x_i, x_j \in X$ and $\sup_{x_i, x_j\ are\ not\ L.D.} \frac{|F(x_i,x_j)|}{N_1(x_i,x_j)} = 1$. Such F will be called an extremal of x.*

Corollary 3.8. $\max_{\phi \in D^{*} such\ that\ N_1^{'}(T_t^{*}\phi, f) \neq 0} \frac{\langle \xi, \phi\rangle}{N_1^{'}(T_t^{*}\phi, f)}$ *is a non-increasing function of t, for $t \geq 0$. where $T_t^{*}\phi, f \in B_t^{*}$, and for some $\phi \in D^{*}$.*

4 Global Controllability

We consider the question of global controllability of the system. In this section we study the necessary and sufficient conditions for global controllability of a system defined in our previous paper [5]. We like to investigate the possibility of reaching any point $\eta \in D$ by applying a control $u \in U_e(y;t)$ where $U_e(y;t)$ is a ball in B_t, such that t is the minimum time taken. To resolve this question, let us first consider the reachable region $C(t)$ by applying all $u \in U_e(y;t) \subset B_t$ i.e. $T_t U_e(y;t) = C(t)$, where T_t is a linear bounded onto transformation from B_t onto D. Now, let $\eta \notin C(t)$ and let $\xi \in \delta C(t)$ be on the ray through η i.e. $\xi = l\eta$ where $0 < l < 1$ and t^* be the minimum time to reach ξ. Hence by Theorem 3.2, we can write

$$\max_{\phi \in D^* such\ that\ N_1'(T_{t^*}^*\phi, f) \neq 0} \frac{\langle \xi, \phi \rangle}{N_1'(T_{t^*}^*\phi, f)} = 1 \tag{11}$$

where $T_{t^*}^*\phi, f \in B_t^*$, and for some $\phi \in D^*$ for some $\xi \in \delta C(t) \cap \delta C(t^*)$. Suppose maximum is attained at $\phi = \phi_1$. Then $\langle \xi, \phi_1 \rangle = N_1'(T_{t^*}^*\phi_1, f) > 0$, by Corollary 3.4. Now, $\langle \xi, \phi \rangle$ is a continuous function of ϕ, and since $\langle \xi, \phi_1 \rangle > 0$ there exist a neighborhood of ϕ_1, such that $\langle \xi, \phi \rangle > 0$ for all ϕ in the neighborhood of ϕ_1. Put $\langle \xi, \phi \rangle = k_\phi > 0$ in this neighborhood. Thus $\langle \xi, \frac{\phi}{k_\phi} \rangle = 1$. Put $\psi = \frac{\phi}{k_\phi}$ in (11). Then from (11) we have

$$\frac{1}{\min_{\psi \in D^* such\ that\ N_1'(T_{t^*}^*\psi, f) \neq 0} N_1'(T_{t^*}^*\psi, f)} = 1$$

under the constraint $\langle \xi, \psi \rangle = 1$. Then the minimum root of the equation

$$\min_{\psi \in D^* such\ that\ N_1'(T_{t^*}^*\psi, f) \neq 0} N_1'(T_{t^*}^*\psi, f) = 1 \tag{12}$$

where $\langle \xi, \psi \rangle = 1$ will give the minimum time to reach at ξ. Now, $\xi \in \delta C(t^*)$. Here t^* is taken as the minimum root of (12). Obviously t^* is the minimum time to reach at ξ. Let $u_{t^*} \in U_e(y;t^*) \subset B_{t^*}$ be the optimal control to reach at $\xi \in \delta C(t^*)$. Thus $\xi = T_{t^*}u_{t^*}$. Hence $l\eta = T_{t^*}u_{t^*}$.

So in order to reach η in the time t^* we shall have to apply the control $\frac{u_{t^*}}{l} = v_{t^*}$ where $N_1(v_{t^*}, v_{t_1^*}) = \frac{1}{l} > 1$ where $v_{t^*}, v_{t_1^*} \in B_{t^*}$. Obviously $v_{t^*} \notin U_e(y;t^*)$. Now let t^{**} be the minimum time to reach η by applying an admissible control, if such a control exists. Then t^{**} will be greater than t^*, as found above. For if possible, let $t^{**} \leq t^*$. Obviously $t^* \neq t^{**}$ as in that case $\eta \in \delta C(t^*)$, which is not true. So, let $t^{**} < t^*$. Then by Theorem 3.3, we can write

$$\max_{\phi \in D^* such\ that\ N_1'(T_{t^{**}}^*\phi, f) \neq 0} \frac{\langle \xi, \phi \rangle}{N_1'(T_{t^{**}}^*\phi, f)} > 1 \tag{13}$$

where $T^*_{t^{**}}\phi, f \in B^*_t, \xi \in \delta C(t^*)$.

But by Theorem 3.2, $\max\limits_{\phi\in D^* such\ that\ N_1^{'}(T^*_{t^{**}}\phi,f)\neq 0} \dfrac{\langle \xi, \phi\rangle}{N_1^{'}(T^*_{t^{**}}\phi, f)} = \frac{\langle \xi,\phi_1\rangle}{N_1^{'}(T^*_{t^{**}}\phi_1,f)} =$

$$\frac{\langle l\eta,\phi_1\rangle}{N_1^{'}(T^*_{t^{**}}\phi_1,f)} = \frac{l\langle \eta,\phi_1\rangle}{N_1^{'}(T^*_{t^{**}}\phi_1,f)} = l < 1$$

which contradicts (13). Consequently, $t^{**} \nleq t^*$ and our assertion that $t^{**} > t^*$ is correct. Hence we have the following Theorem.

Theorem 4.1. *Let $T_tU_e(y;t) = C(t)$ for any given t, and let $\eta \notin C(t)$. Let $\xi \in \delta C(t^*)$ be the point on the ray through η and t^* be the minimum time to reach at ξ. If there exists an optimal control $u_t \in U_e(y;t)$ to reach η in minimum time t^{**}, then $t^{**} > t^*$.*

Proof. Again by applying Theorem 3.3, we have for $t = t^*$,

$$\max_{\phi\in D^* such\ that\ N_1^{'}(T^*_{t^*}\phi,f)\neq 0} \frac{\langle \xi,\phi\rangle}{N_1^{'}(T^*_{t^*}\phi,f)} = 1,$$

where $T^*_{t^*}\phi, f \in B^*_t$, for some $\xi \in \delta C(t^*)$ and for some $\phi \in D^*$,

i.e. $\max\limits_{\phi\in D^* such\ that\ N_1^{'}(T^*_{t^*}\phi,f)\neq 0} \dfrac{\langle l\eta, \phi\rangle}{N_1^{'}(T^*_{t^*}\phi, f)} = \dfrac{1}{l} > 1$, where $0 < l < 1$.

Evidently, $\max\limits_{\phi\in D^* such\ that\ N_1^{'}(T^*_{t^*}\phi,f)\neq 0} \dfrac{\langle \eta, \phi\rangle}{N_1^{'}(T^*_{t^*}\phi, f)}$ is also a non-increasing function of t. Now if there exists a time $t = t^{'}$, such that $\max\limits_{\phi\in D^* such\ that\ N_1^{'}(T^*_{t^{'}}\phi,f)\neq 0} \dfrac{\langle \eta, \phi\rangle}{N_1^{'}(T^*_{t^{'}}\phi, f)} < 1$ and also if $\max\limits_{\phi\in D^* such\ that\ N_1^{'}(T^*_{t^{'}}\phi,f)\neq 0} \dfrac{\langle \eta, \phi\rangle}{N_1^{'}(T^*_{t^{'}}\phi, f)}$ is a continuous function of t, then by the intermediate value property we can assert that there exists a time $t = t^{**}$, such that

$$\max_{\phi\in D^* such\ that\ N_1^{'}(T^*_{t^{**}}\phi,f)\neq 0} \frac{\langle \eta, \phi\rangle}{N_1^{'}(T^*_{t^{**}}\phi, f)} = 1, \tag{14}$$

Now $\eta \in C(t^{**})$. For if, $\eta \notin C(t^{**})$, let $\eta^{'} \in \delta C(t^{**})$ be the point on the ray through η so that $\eta^{'} = l\eta$ for some $l < 1$. Hence from (14), we get $\max\limits_{\phi\in D^* such\ that\ N_1^{'}(T^*_{t^{**}}\phi,f)\neq 0} \dfrac{\langle \eta^{'}, \phi\rangle}{N_1^{'}(T^*_{t^{**}}\phi, f)} = l < 1$,
which contradicts

$$\max_{\phi\in D^* such\ that\ N_1^{'}(T^*_{t^{**}}\phi,f)\neq 0} \frac{\langle \eta^{'}, \phi\rangle}{N_1^{'}(T^*_{t^{**}}\phi, f)} = 1\ (Theorem\ 3.2).$$

Hence again from Theorem 3.1 $\eta \in \delta C(t^{**})$, and the maximum in (14) will be attained at ψ which defines the supporting hyper-plane to $C(t^{**})$ at η.
Therefore we have $\langle \eta, \psi \rangle = N_1^{'}(T_{t^{**}}^{*}\psi, f)$ where $T_{t^{**}}^{*}\psi, f \in B_t^{*}$, for some $\eta \in \delta C(t^{**})$ and for some $\psi \in D^{*}$. Since $\eta \in \delta C(t^{**})$ there exists a $u_\eta \in U_e(y; t^{**})$ such that $\eta = T_{t^{**}} u_\eta$. So
$\langle T_{t^{**}} u_\eta, \psi \rangle = N_1^{'}(T_{t^{**}}^{*}\psi, f)$, or $\langle u_\eta, T_{t^{**}}^{*}\psi \rangle = N_1^{'}(T_{t^{**}}^{*}\psi, f)$.

Hence by Hahn Banach Theorem 3.7, u_η can be chosen to be $\overline{T_{t^{**}}^{*}\psi}$ with

$$N_1^{'}(\overline{T_{t^{**}}^{*}\psi}, f) = 1, \ where \ \overline{T_{t^{**}}^{*}\psi}, f \in B_t^{*}.$$

Similarly, it can be verified for $\eta \in Int\ C(t)$.

□

Theorem 4.2. *The sufficient conditions for the existence of minimum time control for η as in Theorem 4.1 are that*
(a) there exists a time t_1, such that $\max\limits_{\phi \in D^{*} such\ that\ N_1^{'}(T_{t_1}^{*}\phi, f) \neq 0} \dfrac{\langle \eta, \phi \rangle}{N_1^{'}(T_{t_1}^{*}\phi, f)} < 1$ *and*
(b) $\max\limits_{\phi \in D^{*} such\ that\ N_1^{'}(T_t^{*}\phi, f) \neq 0} \dfrac{\langle \eta, \phi \rangle}{N_1^{'}(T_t^{*}\phi, f)}$ *is a continuous function of t, where $T_t^{*}\phi, f \in B_t^{*}$.*

Theorem 4.3. *Necessary condition for existence of admissible optimal control is that* $\min\limits_{\psi \in D^{*} such\ that\ N_1^{'}(T_t^{*}\psi, f) \neq 0} N_1^{'}(T_t^{*}\psi, f) = 1$ *where $T_t^{*}\psi, f \in B_t^{*}$ under the constraint $\langle \eta, \psi \rangle = 1$ will have atleast one positive root.*

Proof. See Theorem 4.1. □

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Received: April, 2014