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Coefficient Bounds for Certain Subclasses of Analytic Function Involving Hadamard Products

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Abstract

In this present investigation authors obtained sharp bounds for $|a_3 - \mu a_2^2|$ for certain subclasses of analytic functions when $\mu \geq 1$.

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1 Introduction

Let $\mathcal{A}$ denote the family of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\Delta := \{z : |z| < 1\}$. Further, let $\mathcal{S}$ denote the class of functions which are univalent in Δ. Let the function be defined by (1). Also let the function $g(z)$ be defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n. \tag{2}$$

Then the Hadamard product (or convolution) of the functions $f(z)$ and $g(z)$ is defined by

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{3}$$

Fekete and Szegö [8] obtained sharp upper bounds for $|a_3 - \mu a_2^2|$ when μ is real. For various subclasses of $\mathcal{S}$, sharp upper bound for functional $|a_3 - \mu a_2^2|$ has been studied by many different authors including [1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. In this paper we obtain sharp upper bounds for $|a_3 - \mu a_2^2|$ when $f(z)$ belonging to the class of functions defined as follows:

Definition 1.1. *Let $0 < \alpha \leq 1$, $0 \leq \lambda \leq 1$, and $\beta > 0$, and let $f(z) \in \mathcal{A}$. Then $f(z) \in M(\Phi, \Psi, \lambda, \alpha, \beta)$ if and only if*

$$\left| arg \left[\frac{(1-\lambda z f'(z)}{g(z)} + \left(\frac{f'(z) + z f''(z)}{g'(z)} \right) \right] \right| < \frac{\pi \alpha}{2}, \tag{4}$$

with $g(z) \in \mathcal{A}$ and satisfies

$$\left| arg \left(\frac{g(z) * \Phi(z)}{g(z) * \Psi(z)} \right) \right| < \frac{\pi \beta}{2}, z \in \Delta \tag{5}$$

where $\Phi(z) = z + \sum_{n=2}^{\infty} \Upsilon_n z^n$ and
$\Psi(z) = z + \sum_{n=2}^{\infty} \gamma_n z^n$, analytic in Δ such that $g(z) * \Psi(z) \neq 0$, $\Upsilon_n \geq 0$, $\gamma_n \geq 0$ and $\Upsilon_n > \gamma_n$ $(n \geq 2)$.

Definition 1.2. *Let $0 < \alpha \leq 1$, $0 \leq \lambda \leq 1$, $\beta > 0$, and let $f(z) \in \mathcal{A}$. Then $f(z) \in N(\Phi, \Psi, \lambda, \alpha, \beta)$ if and only if*

$$\left| arg \left(\frac{z^{(1-\lambda)} f'(z)}{g(z)^{(1-\lambda)}} \right) \right| < \frac{\pi \alpha}{2}, z \in \Delta \tag{6}$$

with $g(z) \in \mathcal{A}$ and satisfies equation (5).

Lemma 1.3 ([18])**.** *Let $h(z) \in P$ that is $h(z)$ be analytic in Δ and be given by $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ and $Reh(z) > 0$ for $z \in \Delta$ then*

$$\left| |c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1|^2}{2}. \tag{7}$$

2 Coefficient Problem

Theorem 2.1. *Let $f(z)$ be given by (1). If $f(z) \in M(\Phi, \Psi, \lambda, \alpha, \beta)$ and $3\eta\mu \geq 2\delta^2 + 4\delta\gamma_2$, where $\delta = \Upsilon_2 - \gamma_2$, $\eta = \Upsilon_3 - \gamma_3$ and $\mu \geq 1$, then we have the sharp inquality*

$$|a_3 - \mu a_2^2| \leq \frac{\beta^2}{3\eta\delta^2}[3\mu\eta - 2\delta^2 - 4\delta\gamma_2) + \frac{\alpha[\alpha\delta(3\mu(1+2\lambda) - 2(1+\lambda^2)) + 2\beta(1+\lambda)(3\mu(1+2\lambda) - 2(1+3\lambda))]}{3\delta(1+\lambda)^2(1+2\lambda)} \tag{8}$$

Proof. Let $f(z) \in M(\Phi, \Psi, \lambda, \alpha, \beta)$. It follows from (4) that

$$(1-\lambda)zf'(z)g'(z) + \lambda[f'(z) + zf''(z)]g(z) = q^\alpha(z)g(z)g'(z) \tag{9}$$

for $z \in \Delta$, with $q(z) \in P$ given by $q(z) = 1 + q_1 z + q_2 z^2 + q_3 z^3 + \cdots$. Equating coefficients, we obtain

$$2a_2(1+\lambda) = b_2(1+\lambda) + \alpha q_1 \tag{10}$$

and

$$3a_3(1+2\lambda) = b_3(1+2\lambda) + 3\alpha q_1 b_2 - 2\alpha q_1 b_2 \frac{1}{1+\lambda} + \alpha q_2 + \frac{\alpha(\alpha-1)}{2} q_1^2 \tag{11}$$

Also, it follows from (5) that

$$g(z) * \Phi(z) = [g(z) * \Psi(z)]p^\beta(z), \tag{12}$$

where $p(z) \in P$ with $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ for $z \in \Delta$. Thus equating coefficients, we obtain

$$\delta b_2 = \beta p_1 \tag{13}$$

$$\eta b_3 = \beta\left[p_2 + \frac{\beta(\delta + 2\gamma_2) - \delta}{2\delta} p_1^2\right]. \tag{14}$$

From (10), (11), (13), (14) we have

$$a_3 - \mu a_2^2 = \frac{\alpha}{3(1+2\lambda)}\left[q_2 - \frac{q_1^2}{2}\right] + \frac{\beta}{3\eta}\left[p_2 - \frac{p_1^2}{2}\right] + \frac{\alpha^2 q_1^2[2(1+\lambda)^2 - 3\mu(1+2\lambda)]}{12(1+\lambda)^2(1+2\lambda)} + \frac{\alpha\beta p_1 q_1[2(1+3\lambda) - 3\mu(1+2\lambda1)]}{6\delta(1+\lambda)(1+2\lambda)} + \frac{\beta^2 p_1^2(2\delta^2 + 4\gamma_2\delta - 3\mu\eta)}{12\eta\delta^2}. \tag{15}$$

Assume that $a_3-\mu a_2^2$ positive. Thus we now estimate $Re(a_3-\mu a_2^2)$ by applying the same technique done by London [15], and so from (15) and by using lemma 1.3 and letting $p_1 = 2re^{i\theta}$, $q_1 = 2Re^{i\phi}$, $0 \le r \le 1$, $0 \le R \le 1$, $0 \le \theta \le 2\pi$ and $0 \le \phi \le 2\pi$, we obtain,

$$\begin{aligned}
Re[a_3-\mu a_2^2] =& \frac{\alpha}{3(1+2\lambda}Re\left(q_2-\frac{q_1^2}{2}\right)+\frac{\beta}{2\eta}Re\left(p_2-\frac{p_1^2}{2}\right)\\
&+\frac{\alpha^2[2(1+\lambda)^2-3\mu(1+2\lambda)]Re\ q_1^2}{12(1+\lambda)^2(1+2\lambda)}\\
&+\frac{\alpha\beta[2(1+3\lambda)-3\mu(1+2\lambda)]Re\ p_1q_1}{6\delta(1+\lambda)(1+2\lambda)}\\
&+\frac{\beta^2(2\delta^2+4\gamma_2\delta-3\mu\eta)Re\ p_1^2}{12\eta\delta^2}\\
\le& \frac{2\alpha}{3(1+2\lambda)}(1-R^2)+\frac{2\beta}{3\eta}(1-r^2)\\
&+\frac{\alpha^2[2(1+\lambda)^2-3\mu(1+2\lambda)]R^2cos2\phi}{3(1+\lambda)^2(1+2\lambda)}\\
&+\frac{2\alpha\beta[2(1+3\lambda)-3\mu(1+2\lambda)]rRcos(\theta+\phi)}{3\delta(1+\lambda)(1+2\lambda)}\\
&+\frac{\beta^2(2\delta^2+4\gamma_2\delta-3\mu\eta)r^2cos2\theta}{3\eta\delta^2}\\
\le& \frac{2\alpha}{3(1+2\lambda)}(1-R^2)+\frac{2\beta}{3\eta}(1-r^2)\\
&+\frac{\alpha^2[3\mu(1+2\lambda)-2(1+\lambda)^2]R^2}{3(1+\lambda)^2(1+2\lambda)}\\
&+\frac{2\alpha\beta[3\mu(1+2\lambda)-2(1+3\lambda)]rR}{3\delta(1+\lambda)(1+2\lambda)}\\
&+\frac{\beta^2(3\mu\eta-2\delta^2-4\gamma_2\delta)r^2}{3\eta\delta^2}=G(r,R)
\end{aligned}$$

Letting α, β and μ fixed and differentiating $G(r,R)$ partially when $0<\alpha\le 1$,

$\beta \geq 1$ and $\mu \geq 1$ we observe that

$$\begin{aligned} G_{rr}G_{RR} - (G_{rR})^2 &= \frac{16\alpha\beta}{9\eta(1+2\lambda)} \\ &+ \frac{4\alpha^2\beta^2(3\mu\eta - 2\delta^2 - 4\gamma_2\delta)[3\mu(1+2\lambda) - 2(1+\lambda)^2}{9\eta\delta^2(1+\lambda)^2(1+2\lambda)} \\ &- \frac{8\alpha^2\beta[3\mu(1+2\lambda) - 2(1+\lambda)^2]}{9\eta(1+\lambda)^2(1+2\lambda)} \\ &- \frac{8\alpha\beta^2[3\mu\eta - 2\delta^2 - 4\gamma_2\delta]}{9\eta\delta^2(1+2\lambda)} \\ &- \frac{4\alpha^2\beta^2[3\mu(1+2\lambda) - 2(1+3\lambda)]^2}{9\delta^2(1+\lambda)^2(1+2\lambda)^2} \\ &< 0. \end{aligned}$$

Therefore, the maximum of $G(r, R)$ occurs on boundaries. Thus the desired inequality follows by observing that

$$\begin{aligned} G(r, R) \leq G(1, 1) &= \frac{\beta^2(3\mu\eta - 2\delta^2 - 4\gamma_2\delta)}{3\eta\delta^2} \\ &+ \frac{\alpha[\alpha\delta(3\mu(1+2\lambda) - 2(1+\lambda)^2) + 2\beta(1+\lambda)(3\mu(1+2\lambda) - 2(1+3\lambda))]}{3\delta(1+\lambda)^2(1+2\lambda)} \end{aligned} \tag{16}$$

The equality is attained when choosing $p_1 = q_1 = 2i$ and $p_2 = q_2 = -2$ in (15). □

Theorem 2.2. *Let $f(z)$ given by (1). If $f(z) \in N(\Phi, \Psi, \lambda, \alpha, \beta)$ and $3\eta\mu \geq 2\delta^2 + 4\gamma_2\delta$ where $\delta = \Upsilon_2 - \gamma_2$, $\eta = \Upsilon_3 - \gamma_3$ and $\mu \geq 1$, then we have the sharp inequality*

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{\beta^2(1-\lambda)}{3\eta\delta^2}[3\mu\eta(1-\lambda) - 2\delta^2 - 4\gamma_2\delta] \\ &+ \frac{\alpha[\alpha\delta + 2\beta(1-\lambda)](3\mu - 2)}{3\delta} \\ &+ \frac{2\lambda(1-\lambda)\beta^2}{3\delta^2} \end{aligned} \tag{17}$$

Proof of Theorem 2.2 is similar to Theorem 2.1. So the details are omitted.

Remark 2.3. *Letting $\lambda = 0$ in Theorem 2.1, we have the result given by Darus and Thomas [7].*

Remark 2.4. *Letting $\Phi(z) = \frac{z}{(1-z)^2}$, $\Psi(z) = \frac{z}{1-z}$, $\lambda = 0$ and $\alpha = 1$ in Theorem 2.1, we have the result given by Jahangiri [11].*

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