Abstract

In this paper, we introduce join meet approximation operators with Galois connection in complete residuated lattices. We investigate relations between their operations and Alexandrov $L$-topologies.

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1 Introduction


In this paper, we introduce join meet approximation operators with Galois connection in complete residuated lattices. We investigate relations between their operations and Alexandrov $L$-topologies.

Definition 1.1 [1,2] An algebra $(L, \land, \lor, \circ, \rightarrow, \bot, \top)$ is called a complete residuated lattice if it satisfies the following conditions:
(C1) $L = (L, \leq, \lor, \land, \bot, \top)$ is a complete lattice with the greatest element $\top$ and the least element $\bot$;
(C2) $(L, \odot, \top)$ is a commutative monoid;
(C3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ for $x, y, z \in L$.

In this paper, we assume $(L, \land, \lor, \odot, \rightarrow, \ast, \bot, \top)$ is a complete residuated lattice with the law of double negation; i.e. $x^{**} = x$. For $\alpha \in L, A, \top_x \in L^X$, $(\alpha \rightarrow A)(x) = \alpha \rightarrow A(x)$, $(\alpha \odot A)(x) = \alpha \odot A(x)$ and $\top_x(x) = \top, \top_x(x) = \bot$, otherwise.

Lemma 1.2 [1,2] For each $x, y, z, x_i, y_i \in L$, we have the following properties.

1. If $y \leq z$, $(x \odot y) \leq (x \odot z)$, $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$.
2. $x \rightarrow (\land_{i \in I} y_i) = \land_{i \in I} (x \rightarrow y_i)$.
3. $(\lor_{i \in I} x_i) \rightarrow y = \land_{i \in I} (x_i \rightarrow y)$.
4. $\land_{i \in I} y_i = (\lor_{i \in I} y_i)$ and $\lor_{i \in I} y_i = (\land_{i \in I} y_i)$.
5. $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.
6. $(x \odot y) = (x \rightarrow y^*)^*$.
7. $(x \odot (x \rightarrow y)) \leq y$.
8. $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$.
9. $(x \rightarrow y) \rightarrow (x \rightarrow z) \geq y \rightarrow z$ and $(x \rightarrow z) \rightarrow (y \rightarrow z) \geq y \rightarrow x$.
10. $(x \odot y) \rightarrow x \odot z \geq y \rightarrow z$.

Definition 1.3 [3,4] (1) A map $H : L^X \rightarrow L^X$ is called an $L$-upper approximation operator iff it satisfies the following conditions

(H1) $A \leq H(A)$,
(H2) $H(\alpha \odot A) = \alpha \odot H(A)$ where $\alpha(x) = \alpha$ for all $x \in X$,
(H3) $H(\lor_{i \in I} A_i) = \land_{i \in I} H(A_i)$.

(2) A map $J : L^X \rightarrow L^X$ is called an $L$-lower approximation operator iff it satisfies the following conditions

(J1) $J(A) \leq A$,
(J2) $J(\alpha \rightarrow A) = \alpha \rightarrow J(A)$,
(J3) $J(\land_{i \in I} A_i) = \land_{i \in I} J(A_i)$.

(3) A map $K : L^X \rightarrow L^X$ is called an $L$-join meet approximation operator iff it satisfies the following conditions

(K1) $K(A) \leq A^*$,
(K2) $K(\alpha \odot A) = \alpha \rightarrow K(A)$,
(K3) $K(\lor_{i \in I} A_i) = \land_{i \in I} K(A_i)$.

(4) A map $M : L^X \rightarrow L^X$ is called an $L$-meet join approximation operator iff it satisfies the following conditions

(M1) $A^* \leq M(A)$,
(M2) $M(\alpha \rightarrow A) = \alpha \odot M(A)$,
(M3) $M(\land_{i \in I} A_i) = \lor_{i \in I} M(A_i)$.
**Definition 1.4** [4,5] A subset $\tau \subseteq L^X$ is called an **Alexandrov L-topology** if it satisfies:

1. (T1) $\bot_X, \top_X \in \tau$ where $\top_X(x) = \top$ and $\bot_X(x) = \bot$ for $x \in X$.
2. (T2) If $A_i \in \tau$ for $i \in \Gamma$, $\bigvee_{i \in \Gamma} A_i, \bigwedge_{i \in \Gamma} A_i \in \tau$.
3. (T3) $\alpha \odot A \in \tau$ for all $\alpha \in L$ and $A \in \tau$.
4. (T4) $\alpha \rightarrow A \in \tau$ for all $\alpha \in L$ and $A \in \tau$.

**Theorem 1.5** [4] (1) $\tau$ is an Alexandrov topology on $X$ iff $\tau^* = \{A^* \in L^X \mid A \in \tau\}$ is an Alexandrov topology on $X$.

2. If $\mathcal{H}$ is an $L$-upper approximation operator, then $\tau_\mathcal{H} = \{A \in L^X \mid \mathcal{H}(A) = A\}$ is an Alexandrov topology on $X$.
3. If $\mathcal{J}$ is an $L$-lower approximation operator, then $\tau_\mathcal{J} = \{A \in L^X \mid \mathcal{J}(A) = A\}$ is an Alexandrov topology on $X$.
4. If $\mathcal{K}$ is an $L$-join meet approximation operator, then $\tau_\mathcal{K} = \{A \in L^X \mid \mathcal{K}(A) = A^*\}$ is an Alexandrov topology on $X$.
5. If $\mathcal{M}$ is an $L$-meet join operator, then $\tau_\mathcal{M} = \{A \in L^X \mid \mathcal{M}(A) = A^*\}$ is an Alexandrov topology on $X$.

## 2 L-join meet approximation operators with Galois connections

**Theorem 2.1** Let $\mathcal{K} : L^X \rightarrow L^X$ be an $L$-join meet approximation operators. Then the following properties hold:

1. For $A \in L^X$, $\mathcal{K}(A)(y) = \bigwedge_{x \in X} (A(x) \rightarrow \mathcal{K}(\top_x)(y))$.
2. Define $\mathcal{K}_1(B) = \bigvee\{A \mid B \leq \mathcal{K}(A)\}$. Then $\mathcal{K}_1 : L^X \rightarrow L^X$ with
   $$\mathcal{K}_1(B)(x) = \bigwedge_{y \in X} (B(y) \rightarrow \mathcal{K}(\top_x)(y))$$
   is an $L$-join meet approximation operator such that $(\mathcal{K}, \mathcal{K}_1)$ is a Galois connection, i.e.,
   $$A \leq \mathcal{K}_1(B) \text{ iff } B \leq \mathcal{K}(A).$$

Moreover, $\tau_{\mathcal{K}_1} = (\tau_{\mathcal{K}})_*$.

3. If $\mathcal{K}(\mathcal{K}^*(A)) = \mathcal{K}(A)$ for $A \in L^X$, then $\mathcal{K}_1(\mathcal{K}^*_1(A)) = \mathcal{K}_1(A)$ for $A \in L^X$ such that $\tau_{\mathcal{K}_1} = (\tau_{\mathcal{K}})_*$ with
   $$\tau_{\mathcal{K}} = \{\mathcal{K}^*(A) = \bigvee_{x \in X} (A(x) \odot \mathcal{K}^*(\top_x)) \mid A \in L^X\},$$
\[ \tau_{K_1} = \{ K_1^*(A)(y) = \bigvee_{x \in X} (A(x) \cap K_1^*(T_y)(x)) \mid A \in L^X \}. \]

(4) If \( K(K(A)) = K^*(A) \) for \( A \in L^X \), then \( K(K^*(A)) = K(A) \) such that
\[ \tau_K = \{ K(A) = \bigwedge_{x \in X} (A(x) \rightarrow K(T_x)) \mid A \in L^X \} = (\tau_K)_*. \]

(5) Define \( M_K(A) = K(A^*)^* \). Then \( M_K : L^X \rightarrow L^X \) with
\[ M_K(A)(y) = \bigvee_{x \in X} (A^*(x) \cap K^*(T_x)(y)) \]
is an \( L \)-meet join approximation operator. Moreover, the pair \((M_K, M_{K_1})\) is a dual Galois connection; i.e.,
\[
M_K(A) \leq B, \text{ iff } M_{K_1}(B) \leq A
\]
such that \( \tau_{K_1} = \tau_{M_K} = (\tau_K)_* = (\tau_{M_{K_1}})_* \).

(6) If \( K(K^*(A)) = K(A) \) for \( A \in L^X \), then \( M_K(M_{K_1}(A)) = M_K(A) \) for \( A \in L^X \) such that \( \tau_{K_1} = \tau_{M_K} = (\tau_K)_* = (\tau_{M_{K_1}})_* \) with
\[ \tau_{M_K} = \{ M_K^*(A) = \bigwedge_{x \in X} (K^*(T_x) \rightarrow A(x)) \mid A \in L^X \}, \]
\[ \tau_{M_{K_1}} = \{ (M_{K_1})^*_1(A)(y) = \bigwedge_{x \in X} (K^*(T_y)(x) \rightarrow A(x)) \mid A \in L^X \}. \]

(7) If \( K(K(A)) = K^*(A) \) for \( A \in L^X \), then \( M_K(M_K(A)) = M_K(A) \) such that
\[ \tau_{M_K} = \{ M_K(A) = \bigvee_{x \in X} (A^*(x) \cap K^*(T_x)) \mid A \in L^X \} = (\tau_{M_K})_. \]

(8) Define \( J_K(A) = K(A^*) \). Then \( J_K : L^X \rightarrow L^X \) with
\[ J_K(A)(y) = \bigwedge_{x \in X} (A^*(x) \rightarrow K(T_x)(y)) = \bigwedge_{x \in X} (K^*(T_x)(y) \rightarrow A(x))). \]
is an \( L \)-lower approximation operator.

(9) If \( K(K^*(A)) = K(A) \) for \( A \in L^X \), then \( J_K(J_K(A)) = J_K(A) \) for \( A \in L^X \) such that \( \tau_{J_{K_1}} = (\tau_{J_K})_* \) with
\[ \tau_{J_K} = \{ J_K(A) = \bigwedge_{x \in X} (K^*(T_x) \rightarrow A(x))) \mid A \in L^X \}, \]
\[ \tau_{J_{K_1}} = \{ J_{K_1}(A)(x) = \bigwedge_{x \in X} (K^*(T_x)(y) \rightarrow A(y))) \mid A \in L^X \}. \]
(10) If $K(K(A)) = K^*(A)$ for $A \in L^X$, then $J_K(J_K^*(A)) = J_K^*(A)$ such that
\[ \tau_{J_K} = \{ J_K^*(A) = \bigvee_{x \in X} (K^*(\top_x) \odot A^*(x)) \mid A \in L^X \} = (\tau_{J_K})^* . \]

(11) Define $H_K(A) = (K(A))^*$. Then $H_K : L^X \to L^X$ with
\[ H_K(A)(y) = \bigvee_{x \in X} (A(x) \odot K^*(\top_x)(y)) \]
is an $L$-upper approximation operator. Moreover, $\tau_{H_K} = \tau_K$. 

(12) If $K(K^*(A)) = K^*(A)$ for $A \in L^X$, then $H_K(H_K(A)) = H_K(A)$ for $A \in L^X$ such that $\tau_{H_{K_1}} = (\tau_{H_K})^*$ with
\[ \tau_{H_K} = \{ H_K(A) = \bigvee_{x \in X} (A(x) \odot K^*(\top_x)) \mid A \in L^X \} , \]
\[ \tau_{H_{K_1}} = \{ (H_K)_1(A)(y) = \bigvee_{x \in X} (A(x) \odot K^*(\top_y)(x)) \mid A \in L^X \} . \]

(13) If $K(K(A)) = K^*(A)$ for $A \in L^X$, then $H_K(H_K(A)) = H_K^*(A)$ such that
\[ \tau_{H_K} = \{ H_K^*(A) = \bigwedge_{x \in X} (A(x) \rightarrow K(\top_x)) \mid A \in L^X \} = (\tau_{H_K})^* . \]

(14) $(H_K, J_K)$ and $(H_K, J_{K_1})$ are a residuated connection; i.e,
\[ H_{K_1}(A) \leq B \iff A \leq J_K(B), \]
\[ H_K(A) \leq B \iff A \leq J_{K_1}(B). \]
Moreover, $\tau_{J_K} = \tau_{H_{K_1}}$ and $\tau_{J_{K_1}} = \tau_{H_K}$.

**Proof** (1) For $A = \bigvee_{x \in X} (A(x) \odot \top_x) \in L^X$, $K(A)(y) = \bigwedge_{x \in X} (A(x) \rightarrow K(\top_x)(y))$.

(2) (K1) Since $B \leq K(K_1(B)) \leq K_1^*(B)$, we have $K_1(B) \leq B^*$. (K2) Since $K_1(B) \leq K_1(B)$, then $B \leq K(K_1(B))$. Thus,
\[ B \leq K(K_1(B)) \leq K(a \odot (a \rightarrow K_1(B))) = a \rightarrow K(a \rightarrow K_1(B)) \]
iff $a \circ B \leq K(a \rightarrow K_1(B))$
iff $a \rightarrow K_1(B) \leq K(a \circ B)$.
\[ a \circ B \leq K(K_1(a \circ B)) \]
iff $B \leq a \rightarrow K(K_1(a \circ B)) = K(a \rightarrow K_1(a \circ B))$
iff $a \circ K_1(a \circ B) \leq K_1(B)$
iff $K_1(a \circ B) \leq a \rightarrow K_1(B)$.
(K3) $K_1(\bigvee_{i \in \Gamma} A_i) = \bigwedge_{i \in \Gamma} K_1(A_i)$. By the definition of $K_1$, since $K_1(A) \leq K_1(B)$ for $B \leq A$, we have
\[
K_1(\bigvee_{i \in \Gamma} A_i) \leq \bigwedge_{i \in \Gamma} K_1(A_i).
\]
Since $K(\bigwedge_{i \in \Gamma} K_1(A_i)) \geq K(K_1(A_i)) \geq A$, then $K(\bigwedge_{i \in \Gamma} K_1(A_i)) \geq \bigvee_{i \in \Gamma} A_i$.
Thus
\[
K_1(\bigvee_{i \in \Gamma} A_i) \geq \bigwedge_{i \in \Gamma} K_1(A_i).
\]
Thus $K_1 : L^X \to L^X$ is an $L$-join meet approximation operator. By the definition of $K_1$, we have
\[
A \leq K_1(B) \iff B \leq K(A).
\]
Since $A^* \leq K_1(A)$ iff $A \leq K(A^*)$, we have $\tau_{K_1} = (\tau_K)_*$.
(3) Let $\mathcal{K}(\mathcal{K}^*(A)) = \mathcal{K}(A)$ for $A \in L^X$. Then
\[
K_1^*(A) \leq K(B) \iff K_1(A) \geq K^*(B) \iff K(K^*(B)) = K(B) \geq A
\]
\[
K_1(K_1^*(A)) = \bigvee \{B \mid K_1^*(A) \leq K(B)\}
\]
\[
= \bigvee \{B \mid A \leq K(B)\}
\]
\[
= K_1(A).
\]
(4) Let $\mathcal{K}(A) \in \tau_{K}$. Since $\mathcal{K}(\mathcal{K}(A)) = \mathcal{K}^*(A), \mathcal{K}(\mathcal{K}^*(A)) = \mathcal{K}(\mathcal{K}(A)) = (\mathcal{K}(\mathcal{K}(A)))^* = \mathcal{K}(A)$. Hence $\mathcal{K}^*(A) \in \tau_{K}$; i.e. $\mathcal{K}(A) \in (\tau_{K})_*$. Let $A \in (\tau_{K})_*$. Then $A = \mathcal{K}(A^*)$. Since $\mathcal{K}(A) = \mathcal{K}(\mathcal{K}(A^*)) = \mathcal{K}^*(A^*) = A^*$, then $A \in \tau_{K}$. Thus, $(\tau_{K})_* \subset \tau_{K}$.
(5) (M1) Since $A \leq \mathcal{K}(A^*), \mathcal{M}_K(A) = \mathcal{K}(A^*)^* \leq A^*$.
(M2)
\[
\mathcal{M}_K(\alpha \to A) = (\mathcal{K}(\alpha \to A))^* = (\mathcal{K}(\alpha \circ A^*))^*
\]
\[
= (\alpha \to \mathcal{K}(A^*))^* = \alpha \circ \mathcal{M}_K(A).
\]
(M3)
\[
\mathcal{M}_K(\bigwedge_{i \in \Gamma} A_i) = (\mathcal{K}(\bigwedge_{i \in \Gamma} A_i))^* = (\mathcal{K}(\bigvee_{i \in \Gamma} A_i^*))^*
\]
\[
= (\bigwedge_{i \in \Gamma} \mathcal{K}(A_i^*))^* = \bigvee_{i \in \Gamma} (\mathcal{K}(A_i^*))^*
\]
\[
= \bigvee_{i \in \Gamma} \mathcal{M}_K(A_i).
\]
Moreover, the pair $(\mathcal{M}_K, \mathcal{M}_K)$ is a dual Galois connection from:
\[
\mathcal{M}_K(A) \leq B \iff B^* \leq \mathcal{K}(A^*) \iff A^* \leq K_1(B^*)
\]
\[
K_1^*(B^*) \leq A \iff \mathcal{M}_K(B) \leq A.
\]
We have $\tau_{K_1} = \tau_{\mathcal{M}_K} = (\tau_K)_* = (\tau_{K_1})_*$, from:
\[
A^* \leq K_1(A) \iff A \leq \mathcal{K}(A^*)
\]
\[ \mathcal{M}_K(A) \leq A^* \text{ iff } \mathcal{M}_{K_1}(A^*) \leq A. \]

(6) Let \( \mathcal{K}(\mathcal{K}^*(A)) = \mathcal{K}(A) \) for \( A \in L^X \). Then
\[
\mathcal{M}_K(\mathcal{M}_K^*(A)) = \mathcal{K}^*(\mathcal{M}_K(A)) = (\mathcal{K}(\mathcal{K}^*(A^*)))^* = \mathcal{K}^*(A^*) = \mathcal{M}_K(A).
\]

By (3), since \( \mathcal{K}_1(\mathcal{K}_1^*(A)) = \mathcal{K}_1(A) \) for \( A \in L^X \), \( (\mathcal{M}_K)_1((\mathcal{M}_K)_1^*(A)) = (\mathcal{M}_K)_1(A) \) for \( A \in L^X \). Thus,
\[
\tau_{\mathcal{M}_K} = \{ \mathcal{M}_K^*(A) \mid A \in L^X \}, \quad \tau_{(\mathcal{M}_K)_1} = \{ (\mathcal{M}_K)_1^*(A) \mid A \in L^X \}.
\]

(7) Let \( \mathcal{K}(\mathcal{K}(A)) = \mathcal{K}^*(A) \) for \( A \in L^X \). Then
\[
\mathcal{M}_K(\mathcal{M}_K(A)) = \mathcal{K}^*(\mathcal{M}_K(A)) = (\mathcal{K}(\mathcal{K}(A^*)))^* = (\mathcal{K}^*(A^*))^* = \mathcal{M}_K(A).
\]

By the similarly method in (4), \( \mathcal{M}_K(\mathcal{M}_K^*(A)) = \mathcal{M}_K(A) \) for \( A \in L^X \). Thus,
\[
\tau_{\mathcal{M}_K} = \{ \mathcal{M}_K(A) \mid A \in L^X \} = (\tau_{\mathcal{M}_K})^*.
\]

(8) It is similarly proved as (5).

(9) If \( \mathcal{K}(\mathcal{K}^*(A)) = \mathcal{K}(A) \) for \( A \in L^X \), then \( \mathcal{J}_K(\mathcal{J}_K(A)) = \mathcal{J}_K(A) \)
\[
\mathcal{J}_K(\mathcal{J}_K(A)) = \mathcal{J}_K(\mathcal{K}(A^*)) = \mathcal{K}(\mathcal{K}(A^*)) = \mathcal{K}(A^*) = \mathcal{J}_K(A).
\]

Similarly, \( \mathcal{J}_{K_1}(\mathcal{J}_{K_1}(A)) = \mathcal{J}_{K_1}(A) \). Thus, the results hold.

(10) If \( \mathcal{K}(\mathcal{K}(A)) = \mathcal{K}^*(A) \) for \( A \in L^X \), then \( \mathcal{J}_K(\mathcal{J}_K(A)) = \mathcal{J}_K(A) \)
\[
\mathcal{J}_K(\mathcal{J}_K(A)) = \mathcal{J}_K(\mathcal{K}^*(A^*)) = \mathcal{K}(\mathcal{K}(A^*)) = \mathcal{K}^*(A^*) = \mathcal{J}_K(A).
\]

Since \( \mathcal{J}_K(\mathcal{J}_K^*(A)) = \mathcal{J}_K^*(A) \)
\[
\mathcal{J}_K(\mathcal{J}_K(A)) = \mathcal{J}_K(\mathcal{J}_K^*(\mathcal{J}_K^*(A))) = \mathcal{J}_K(\mathcal{J}_K(A)) = \mathcal{J}_K(A).
\]

Hence \( \tau_{\mathcal{J}_K} = \{ \mathcal{J}_K^*(A) \mid A \in L^X \} = (\tau_{\mathcal{J}_K})^* \).

(11) and (12) are similarly proved as (5) and (6), respectively.

(13) If \( \mathcal{K}(\mathcal{K}(A)) = \mathcal{K}^*(A) \) for \( A \in L^X \), then \( \mathcal{H}_K(\mathcal{H}_K^*(A)) = \mathcal{H}_K^*(A) \) from:
\[
\mathcal{H}_K(\mathcal{H}_K^*(A)) = \mathcal{H}_K(\mathcal{K}(A)) = (\mathcal{K}(\mathcal{K}(A)))^* = (\mathcal{K}^*(A))^* = \mathcal{H}_K^*(A).
\]

(14) \( (\mathcal{H}_K, \mathcal{J}_K) \) is a residuated connection; i.e,
\[
\mathcal{H}_K(A) \leq B \text{ iff } \mathcal{K}_1(A) \geq B^*,
\]
\[
A \leq \mathcal{K}(B^*) \text{ iff } A \leq \mathcal{J}_K(B),
\]

Similarly, \( (\mathcal{H}_K, \mathcal{J}_K) \) is a residuated connection.
Example 2.2 Let $R$ be a reflexive $L$-fuzzy relation. Define $\mathcal{K}_{R'} : L^X \rightarrow L^X$ as follows:

$$\mathcal{K}_{R'}(A)(y) = \bigwedge_{x \in X} (A(x) \rightarrow R^*(x, y)).$$

1. (K1) $\mathcal{K}_{R'}(A)(y) \leq A(y) \rightarrow R^*(y, y)) = A^*(x)$.
2. (K2) $\mathcal{K}_{R'}(a \circ A)(y) = \Lambda_{x \in X} ((a \circ A)(x) \rightarrow R^*(x, y)) = a \rightarrow \Lambda_{x \in X} (A(x) \rightarrow R^*(x, y)) = a \rightarrow \mathcal{K}_{R'}(A)(y)$.
3. (K3) $\mathcal{K}_{R'}(\bigvee_{i \in I} A_i)(y) = \Lambda_{x \in X} (\bigvee_{i \in I} A_i(x) \rightarrow R^*(x, y)) = \Lambda_{x \in X} \bigwedge_{i \in I} (A_i(x) \rightarrow R^*(x, y)) = \bigwedge_{i \in I} \mathcal{K}_{R'}(A_i)(y)$. Hence $\mathcal{K}_{R'}$ is an $L$-join meet approximation operator.

(2) Define $(\mathcal{K}_{R'})_1(B) = \bigvee \{ A \mid B \leq \mathcal{K}_{R'}(A) \}$. Since $B(y) \leq \mathcal{K}_{R'}(B)(y)$ iff $B(y) \leq A(x) \rightarrow R^*(x, y)$ iff $A(x) \leq B(y) \rightarrow R^*(x, y)$, then

$$(\mathcal{K}_{R'})_1(B)(x) = \mathcal{K}_{R'}^{-1,*}(B)(x) = \bigwedge_{y \in X} (B(y) \rightarrow R^{-1}* (y, x)).$$

Then $(\mathcal{K}_{R'})_1 = \mathcal{K}_{R'}^{-1,*}$ with

$$\mathcal{K}_{R'}^{-1,*}(A)(y) = \bigwedge_{x \in X} (A(x) \rightarrow R^{-1}* (x, y))$$

is an $L$-join meet approximation operator such that $(\mathcal{K}_R, \mathcal{K}_{R'}^{-1,*})$ is a Galois connection; i.e.,

$$A \leq \mathcal{K}_{R'}^{-1,*}(B) \iff B \leq \mathcal{K}_{R'}(A).$$

Moreover, $\tau_{\mathcal{K}_{R'}^{-1,*}} = (\tau_{\mathcal{K}_R})_*^{-1}$.

(3) If $R$ is an $L$-fuzzy preorder, then $R^{-1}$ is an $L$-fuzzy preorder. Since $R(x, y) \circ R(y, z) \leq R(x, z)$ iff

$$A(x) \circ R(x, y) \circ (A(x) \rightarrow R^*(x, z)) \leq R(x, y) \circ R^*(x, z) \leq R^*(y, z) \iff A(x) \rightarrow R^*(x, z) \leq A(x) \circ R(x, y) \rightarrow R^*(y, z)$$

$$\mathcal{K}_{R'}(\mathcal{K}_{R'}(A))(z) = \bigwedge_{y \in X} (\mathcal{K}_{R'}(A)(y) \rightarrow R^*(y, z))$$

$$= \bigwedge_{y \in X} \bigvee_{x \in X} (A(x) \circ R(x, y) \rightarrow R^*(y, z))$$

$$= \bigwedge_{x \in X} (A(x) \rightarrow R^*(x, z)) = \mathcal{K}_{R'}(A)(z).$$

Thus $\mathcal{K}_{R'}(\mathcal{K}_{R'}(A)) = \mathcal{K}_{R'}(A)$ for $A \in L^X$. Similarly, $\mathcal{K}_{R'^{-1,*}}(\mathcal{K}_{R'^{-1,*}}(A)) = \mathcal{K}_{R'^{-1,*}}(A)$ for $A \in L^X$ such that $\tau_{\mathcal{K}_{R'^{-1,*}}} = (\tau_{\mathcal{K}_{R'}})^{-1}$ with

$$\tau_{\mathcal{K}_{R'}} = \{ \mathcal{K}_{R'}(A) = \bigvee_{x \in X} (A(x) \circ R(x, -)) \mid A \in L^X \},$$

$$\tau_{\mathcal{K}_{R'^{-1,*}}} = \{ \mathcal{K}_{R'^{-1,*}}(A) = \bigvee_{x \in X} (A(x) \circ R(-, x)) \mid A \in L^X \}.$$
(4) Let $R$ be a reflexive and Euclidean $L$-fuzzy relation. Since $R(x,z) \circ R(y,z) \leq R(x,y)$ iff $R(y,z) \leq R(x,z) \rightarrow R(x,y)$ iff $R(x,z) \circ R^*(x,y) \leq R^*(y,z)$, then
\[
A(x) \circ R(x,z) \circ (A(x) \rightarrow R^*(x,y)) \leq R(x,z) \circ R^*(x,y) \leq R^*(y,z)
\]
iff $A(x) \circ R(x,z) \leq (A(x) \rightarrow R^*(x,y)) \rightarrow R^*(y,z)$.
\[
\mathcal{K}_{R^*}(\mathcal{K}_{R^*}(A))(z) = \bigwedge_{y \in X} (\mathcal{K}_{R^*}(A)(y) \rightarrow R^*(y,z))
\]
\[
= \bigwedge_{y \in X} (\bigwedge_{x \in X} (A(x) \rightarrow R^*(x,y)) \rightarrow R^*(y,z))
\]
\[
\geq \bigvee_{x \in X} (A(x) \circ R(x,z)) = \mathcal{K}_{R^*}(A)(z).
\]
Thus, $\mathcal{K}_{R^*}(\mathcal{K}_{R^*}(A)) = \mathcal{K}_{R^*}(A)$ for $A \in L^X$ such that $\tau_{\mathcal{K}_{R^*}} = (\tau_{\mathcal{K}_{R^*}})_*$ with
\[
\tau_{\mathcal{K}_{R^*}} = \{ \mathcal{K}_{R^*}(A) = \bigwedge_{x \in X} (A(x) \rightarrow R(x,-)) \mid A \in L^X \}.
\]

(5) Define $\mathcal{M}_{\mathcal{K}_{R^*}}(A) = \mathcal{K}_{R^*}(A^*)^*$. By Theorem 2.1 (5), $\mathcal{M}_{\mathcal{K}_{R^*}} = \mathcal{M}_R$ and $\mathcal{M}_{(\mathcal{K}_{R^*})^*} = \mathcal{M}_{\mathcal{K}_{R^*}} = \mathcal{M}_{R-1}$ are $L$-meet join approximation operators such that
\[
\mathcal{M}_{\mathcal{K}_{R^*}}(A)(y) = (\bigwedge_{x \in X} (A^*(x) \rightarrow R(x,y)))^* = \bigvee_{x \in X} (A^*(x) \circ R(x,y))
\]
\[
\mathcal{M}_{\mathcal{K}_{R^{-1}}}(A)(y) = (\bigwedge_{x \in X} (A^*(x) \rightarrow R^{-1}(x,y)))^* = \bigvee_{x \in X} (A^*(x) \circ R^{-1}(x,y)).
\]
Moreover, the pair $(\mathcal{M}_R, \mathcal{M}_{R^{-1}})$ is a dual Galois connection such that $\tau_{\mathcal{M}_{R^{-1}}} = \tau_{\mathcal{M}_R} = (\tau_{\mathcal{M}_{R^{-1}}})_*= (\tau_{\mathcal{M}_R})_*$.

(6) If $R$ is an $L$-fuzzy preorder, by (3), $\mathcal{K}_{R^*}(\mathcal{K}_{R^*}(A)) = \mathcal{K}_{R^*}(A)$ and $\mathcal{K}_{R^{-1}}(\mathcal{K}_{R^{-1}}(A)) = \mathcal{K}_{R^{-1}}(A)$ for $A \in L^X$. By Theorem 2.1(6), $\mathcal{M}_R(\mathcal{M}_{R^*}(A)) = \mathcal{M}_R(A)$ and $\mathcal{M}_{R^{-1}}(\mathcal{M}_{R^{-1}}(A)) = \mathcal{M}_{R^{-1}}(A)$ for $A \in L^X$ such that $\tau_{\mathcal{M}_{R^{-1}}} = (\tau_{\mathcal{M}_R})_*$ with
\[
\tau_{\mathcal{M}_R} = \{ \mathcal{M}_R(A) = \bigwedge_{x \in X} (R(x,-) \rightarrow A(x)) \mid A \in L^X \},
\]
\[
\tau_{\mathcal{M}_{R^{-1}}} = \{ \mathcal{M}_{R^{-1}}(A) = \bigwedge_{x \in X} (R(-,x) \rightarrow A(x)) \mid A \in L^X \}.
\]

(7) If $R$ is a reflexive and Euclidean $L$-fuzzy relation, by (4), $\mathcal{K}_{R^*}(\mathcal{K}_{R^*}(A)) = \mathcal{K}_{R^*}(A)$ for $A \in L^X$. By Theorem 2.1(7), then $\mathcal{M}_R(\mathcal{M}_R(A)) = \mathcal{M}_R(A)$ for $A \in L^X$ such that $\tau_{\mathcal{M}_R} = (\tau_{\mathcal{M}_R})_*$ with
\[
\tau_{\mathcal{M}_R} = \{ \mathcal{M}_R(A) = \bigvee_{x \in X} (A^*(x) \circ R(x,-)) \mid A \in L^X \}.
\]

(8) Define $\mathcal{J}_{\mathcal{K}_{R^*}}(A) = \mathcal{K}_{R^*}(A^*)$. By Theorem 2.1(8), $\mathcal{J}_{\mathcal{K}_{R^*}} = \mathcal{J}_R$ and $\mathcal{J}_{\mathcal{K}_{R^{-1}}^*} = \mathcal{J}_{R^{-1}}$ are $L$-lower approximation operators such that
\[
\mathcal{J}_{\mathcal{K}_{R^*}}(A)(y) = \bigwedge_{x \in X} (A^*(x) \rightarrow R^*(x,y)) = \bigwedge_{x \in X} (R(x,y) \rightarrow A(x)).
\]
Moreover, \( \tau_{J_R} = (\tau_{K^*_R})_* = \tau_{K^*_R-1} \) and \( \tau_{J_R-1} = (\tau_{K^*_R-1})_* = \tau_{K^*_R} \).

(9) If \( R \) is an \( L \)-fuzzy preorder, by (3), \( K^*_R(\mathcal{K}^*_R(A)) = K^*_R(A) \) and \( K^*_R(\mathcal{K}^*_R(A)) = K^*_R(A) \) for \( A \in \mathbb{L}^X \). By Theorem 2.1(9), then \( J_R(\mathcal{J}_R(A)) = J_R(A) \) and \( J_{R-1}(\mathcal{J}_{R-1}(A)) = J_{R-1}(A) \) for \( A \in \mathbb{L}^X \) such that \( \tau_{J_{R-1}} = (\tau_{J_R})_* \) with

\[
\tau_{J_R} = \{ J_R(A) = \bigvee_{x \in \mathbb{X}} (R(x, \cdot) \rightarrow A(x)) \mid A \in \mathbb{L}^X \},
\]

\[
\tau_{J_{R-1}} = \{ J_{R-1}(A) = \bigvee_{x \in \mathbb{X}} (R(\cdot, x) \rightarrow A(x)) \mid A \in \mathbb{L}^X \}.
\]

(10) If \( R \) is a reflexive and Euclidean \( L \)-fuzzy relation, by (4), \( K^*_R(\mathcal{K}^*_R(A)) = K^*_R(A) \) for \( A \in \mathbb{L}^X \). By Theorem 2.1(10), \( J_R(\mathcal{J}_R(A)) = J_R(A) \) for \( A \in \mathbb{L}^X \) such that \( \tau_{J_R} = (\tau_{J_R})_* \) with

\[
\tau_{J_R} = \{ J_R(A) = \bigvee_{x \in \mathbb{X}} (A^*(x) \odot R(x, \cdot)) \mid A \in \mathbb{L}^X \}.
\]

(11) Define \( \mathcal{H}_{K^*_R}(A) = (\mathcal{K}_R(A))^* \). Then \( \mathcal{H}_{K^*_R} = \mathcal{H}_R \) is an \( L \)-upper approximation operator such that

\[
\mathcal{H}_{K^*_R}(A)(y) = \bigvee_{x \in \mathbb{X}} (R(x, y) \odot A(x)).
\]

Moreover, \( \tau_{\mathcal{H}_R} = \tau_{K^*_R} \).

(12) If \( R \) is an \( L \)-fuzzy preorder, by (3), \( K^*_R(\mathcal{K}^*_R(A)) = K^*_R(A) \) and \( K^*_R(\mathcal{K}^*_R(A)) = K^*_R(A) \) for \( A \in \mathbb{L}^X \). By Theorem 2.1(12), \( \mathcal{H}_{K^*_R}(\mathcal{H}_{K^*_R}(A)) = \mathcal{H}_{K^*_R}(A) \) and \( \mathcal{H}_{K^*_R}(\mathcal{H}_{K^*_R}(A)) = \mathcal{H}_{K^*_R}(A) \) for \( A \in \mathbb{L}^X \) such that \( \tau_{\mathcal{H}_{R-1}} = (\tau_{\mathcal{H}_R})_* \) with

\[
\tau_{\mathcal{H}_R} = \{ \mathcal{H}_R(A) = \bigvee_{x \in \mathbb{X}} (R(x, \cdot) \odot A(x)) \mid A \in \mathbb{L}^X \},
\]

\[
\tau_{\mathcal{H}_{R-1}} = \{ \mathcal{H}_{R-1}(A) = \bigvee_{x \in \mathbb{X}} (R(\cdot, x) \odot A(x)) \mid A \in \mathbb{L}^X \}.
\]

(13) If \( R \) is a reflexive and Euclidean \( L \)-fuzzy relation, by (4), \( K^*_R(\mathcal{K}^*_R(A)) = K^*_R(A) \) for \( A \in \mathbb{L}^X \). By Theorem 2.1(13), \( \mathcal{H}_R(\mathcal{H}_R(A)) = \mathcal{H}_R(A) \) for \( A \in \mathbb{L}^X \) such that \( \tau_{\mathcal{H}_R} = (\tau_{\mathcal{H}_R})_* \) with

\[
\tau_{\mathcal{H}_R} = \{ \mathcal{H}_R(A) = \bigvee_{x \in \mathbb{X}} (A(x) \rightarrow R^*(x, \cdot)) \mid A \in \mathbb{L}^X \}.
\]

(14) \( (\mathcal{H}_{R-1}, J_R) \) is a residuated connection; i.e,

\[
\mathcal{H}_{R-1}(A) \leq B \quad \text{iff} \quad K_{R-1}(A) \geq B^*,
\]

\[A \leq K_R(B^*) \quad \text{iff} \quad A \leq J_R(B).
\]

Similarly, \( (\mathcal{H}_R, J_{R-1}) \) is a residuated connection. Moreover, \( \tau_{J_R} = \tau_{\mathcal{H}_{R-1}} \) and \( \tau_{J_{R-1}} = \tau_{\mathcal{H}_R} \).
References


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