Mathematica Aeterna, Vol. 4, 2014, no. 4, 307 - 317

L-join meet approximation operators with Galois connections

Yong Chan Kim

Department of Mathematics, Gangneung-Wonju National University,
Gangneung, Gangwondo 210-702, Korea
yck@gwnu.ac.kr

Abstract

In this paper, we introduce join meet approximation operators with Galois connection in complete residuated lattices. We investigate relations between their operations and Alexandrov L-topologies.

Mathematics Subject Classification: 03E72, 03G10, 06A15, 06F07

Keywords:
Complete residuated lattices, Join meet approximation operators, Alexandrov L-topologies

1 Introduction

Hájek [2] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Pawlak [7,8] introduced rough set theory as a formal tool to deal with imprecision and uncertainty in data analysis. Radzikowska [9] developed fuzzy rough sets in complete residuated lattice. Bělohlávek [1] investigated information systems and decision rules in complete residuated lattices. Lai [5,6] introduced Alexandrov L-topologies induced by fuzzy rough sets. Kim [3,4] investigated the properties of Alexandrov topologies in complete residuated lattices. Algebraic structures of fuzzy rough sets are developed in many directions [3,9,10]

In this paper, we introduce join meet approximation operators with Galois connection in complete residuated lattices. We investigate relations between their operations and Alexandrov L-topologies.

Definition 1.1 [1,2] An algebra $(L, \wedge, \vee, \odot, \to, \perp, \top)$ is called a complete residuated lattice if it satisfies the following conditions:

(C1) $L = (L, \leq, \vee, \wedge, \bot, \top)$ is a complete lattice with the greatest element $\top$ and the least element $\bot$;
(C2) $(L, \odot, \top)$ is a commutative monoid;
(C3) $x \odot y \leq z$ iff $x \leq y \to z$ for $x, y, z \in L$.

In this paper, we assume $(L, \wedge, \vee, \odot, \to, ^* \bot, \top)$ is a complete residuated lattice with the law of double negation;i.e. $x^{**} = x$. For $\alpha \in L, A, \top_x \in L^X$, $(\alpha \to A)(x) = \alpha \to A(x)$, $(\alpha \odot A)(x) = \alpha \odot A(x)$ and $\top_x(x) = \top, \top_x(x) = \bot$, otherwise.

Lemma 1.2 [1,2] For each $x, y, z, x_i, y_i \in L$, we have the following properties.
(1) If $y \leq z$, $(x \odot y) \leq (x \odot z)$, $x \to y \leq x \to z$ and $z \to x \leq y \to x$.
(2) $x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$.
(3) $(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$.
(4) $\bigwedge_{i\in\Gamma} y_i^* = (\bigvee_{i\in\Gamma} y_i)^*$ and $\bigvee_{i\in\Gamma} y_i^* = (\bigwedge_{i\in\Gamma} y_i)^*$.
(5) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$.
(6) $x \odot y = (x \to y^*)^*$.
(7) $x \odot (x \to y) \leq y$.
(8) $(x \to y) \odot (y \to z) \leq x \to z$.
(9) $(x \to y) \to (x \to z) \geq y \to z$ and $(x \to z) \to (y \to z) \geq y \to x$.
(10) $x \odot y \to x \odot z \geq y \to z$.

Definition 1.3 [3,4] (1) A map $\mathcal{H} : L^X \to L^X$ is called an *L-upper approximation operator* iff it satisfies the following conditions
(H1) $A \leq \mathcal{H}(A)$,
(H2) $\mathcal{H}(\alpha \odot A) = \alpha \odot \mathcal{H}(A)$ where $\alpha(x) = \alpha$ for all $x \in X$,
(H3) $\mathcal{H}(\bigvee_{i\in I} A_i) = \bigvee_{i\in I} \mathcal{H}(A_i)$.
(2) A map $\mathcal{J} : L^X \to L^X$ is called an *L-lower approximation operator* iff it satisfies the following conditions
(J1) $\mathcal{J}(A) \leq A$,
(J2) $\mathcal{J}(\alpha \to A) = \alpha \to \mathcal{J}(A)$,
(J3) $\mathcal{J}(\bigwedge_{i\in I} A_i) = \bigwedge_{i\in I} \mathcal{J}(A_i)$.
(3) A map $\mathcal{K} : L^X \to L^X$ is called an *L-join meet approximation operator* iff it satisfies the following conditions
(K1) $\mathcal{K}(A) \leq A^*$,
(K2) $\mathcal{K}(\alpha \odot A) = \alpha \to \mathcal{K}(A)$,
(K3) $\mathcal{K}(\bigvee_{i\in I} A_i) = \bigwedge_{i\in I} \mathcal{K}(A_i)$.
(4) A map $\mathcal{M} : L^X \to L^X$ is called an *L-meet join approximation operator* iff it satisfies the following conditions
(M1) $A^* \leq \mathcal{M}(A)$,
(M2) $\mathcal{M}(\alpha \to A) = \alpha \odot \mathcal{M}(A)$,
(M3) $\mathcal{M}(\bigwedge_{i\in I} A_i) = \bigvee_{i\in I} \mathcal{M}(A_i)$.

Definition 1.4 [4,5] A subset $\tau \subset L^X$ is called an *Alexandrov L-topology* if it satisfies:

(T1) $\bot_X, \top_X \in \tau$ where $\top_X(x) = \top$ and $\bot_X(x) = \bot$ for $x \in X$.

(T2) If $A_i \in \tau$ for $i \in \Gamma$, $\bigvee_{i\in\Gamma} A_i, \bigwedge_{i\in\Gamma} A_i \in \tau$.

(T3) $\alpha \odot A \in \tau$ for all $\alpha \in L$ and $A \in \tau$.

(T4) $\alpha \to A \in \tau$ for all $\alpha \in L$ and $A \in \tau$.

Theorem 1.5 [4] (1) *τ is an Alexandrov topology on X iff $\tau_* = \{A^* \in L^X \mid A \in \tau\}$ is an Alexandrov topology on X.*

(2) *If $\mathcal{H}$ is an L-upper approximation operator, then $\tau_{\mathcal{H}} = \{A \in L^X \mid \mathcal{H}(A) = A\}$ is an Alexandrov topology on X.*

(3) *If $\mathcal{J}$ is an L-lower approximation operator, then $\tau_{\mathcal{J}} = \{A \in L^X \mid \mathcal{J}(A) = A\}$ is an Alexandrov topology on X.*

(4) *If $\mathcal{K}$ is an L-join meet approximation operator, then $\tau_{\mathcal{K}} = \{A \in L^X \mid \mathcal{K}(A) = A^*\}$ is an Alexandrov topology on X.*

(5) *If $\mathcal{M}$ is an L-meet join operator, then $\tau_{\mathcal{M}} = \{A \in L^X \mid \mathcal{M}(A) = A^*\}$ is an Alexandrov topology on X.*

2 L-join meet approximation operators with Galois connections

Theorem 2.1 *Let $\mathcal{K} : L^X \to L^X$ be an L-join meet approximation operators. Then the following properties hold.*

(1) *For $A \in L^X$, $\mathcal{K}(A)(y) = \bigwedge_{x\in X}(A(x) \to \mathcal{K}(\top_x)(y))$.*

(2) *Define $\mathcal{K}_1(B) = \bigvee\{A \mid B \le \mathcal{K}(A)\}$. Then $\mathcal{K}_1 : L^X \to L^X$ with*

$$\mathcal{K}_1(B)(x) = \bigwedge_{y\in X} (B(y) \to \mathcal{K}(\top_x)(y))$$

is an L-join meet approximation operator such that $(\mathcal{K}, \mathcal{K}_1)$ is a Galois connection;i.e.,

$$A \le \mathcal{K}_1(B) \text{ iff } B \le \mathcal{K}(A).$$

Moreover, $\tau_{\mathcal{K}_1} = (\tau_{\mathcal{K}})_*$.

(3) *If $\mathcal{K}(\mathcal{K}^*(A)) = \mathcal{K}(A)$ for $A \in L^X$, then $\mathcal{K}_1(\mathcal{K}_1^*(A)) = \mathcal{K}_1(A)$ for $A \in L^X$ such that $\tau_{\mathcal{K}_1} = (\tau_{\mathcal{K}})_*$ with*

$$\tau_{\mathcal{K}} = \{\mathcal{K}^*(A) = \bigvee_{x\in X} (A(x) \odot \mathcal{K}^*(\top_x)) \mid A \in L^X\},$$

$$\tau_{\mathcal{K}_1} = \{\mathcal{K}_1^*(A)(y) = \bigvee_{x\in X} (A(x) \odot \mathcal{K}^*(\top_y)(x)) \mid A \in L^X\}.$$

(4) *If* $\mathcal{K}(\mathcal{K}(A)) = \mathcal{K}^*(A)$ *for* $A \in L^X$, *then* $\mathcal{K}(\mathcal{K}^*(A)) = \mathcal{K}(A)$ *such that*

$$\tau_{\mathcal{K}} = \{\mathcal{K}(A) = \bigwedge_{x\in X} (A(x) \to \mathcal{K}(\top_x)) \mid A \in L^X\} = (\tau_{\mathcal{K}})_*.$$

(5) *Define* $\mathcal{M}_K(A) = \mathcal{K}(A^*)^*$. *Then* $\mathcal{M}_K : L^X \to L^X$ *with*

$$\mathcal{M}_K(A)(y) = \bigvee_{x\in X} (A^*(x) \odot \mathcal{K}^*(\top_x)(y))$$

is an L-meet join approximation operator. Moreover, the pair $(\mathcal{M}_K, \mathcal{M}_{K_1})$ *is a dual Galois connection;i.e.,*

$$\mathcal{M}_K(A) \leq B, \text{ iff } \mathcal{M}_{K_1}(B) \leq A$$

such that $\tau_{\mathcal{K}_1} = \tau_{\mathcal{M}_K} = (\tau_{\mathcal{K}})_* = (\tau_{\mathcal{M}_{K_1}})_*$.

(6) *If* $\mathcal{K}(\mathcal{K}^*(A)) = \mathcal{K}(A)$ *for* $A \in L^X$, *then* $\mathcal{M}_K(\mathcal{M}_K^*(A)) = \mathcal{M}_K(A)$ *for* $A \in L^X$ *such that* $\tau_{\mathcal{K}_1} = \tau_{\mathcal{M}_K} = (\tau_{\mathcal{K}})_* = (\tau_{\mathcal{M}_{K_1}})_*$. *with*

$$\tau_{\mathcal{M}_K} = \{\mathcal{M}_K^*(A) = \bigwedge_{x\in X} (\mathcal{K}^*(\top_x) \to A(x)) \mid A \in L^X\},$$

$$\tau_{(\mathcal{M}_K)_1} = \{(\mathcal{M}_K)_1^*(A)(y) = \bigwedge_{x\in X} (\mathcal{K}^*(\top_y)(x) \to A(x)) \mid A \in L^X\}.$$

(7) *If* $\mathcal{K}(\mathcal{K}(A)) = \mathcal{K}^*(A)$ *for* $A \in L^X$, *then* $\mathcal{M}_K(\mathcal{M}_K(A)) = \mathcal{M}_K^*(A)$ *such that*

$$\tau_{\mathcal{M}_K} = \{\mathcal{M}_K(A) = \bigvee_{x\in X} (A^*(x) \odot \mathcal{K}^*(\top_x)) \mid A \in L^X\} = (\tau_{\mathcal{M}_K})_*.$$

(8) *Define* $\mathcal{J}_K(A) = \mathcal{K}(A^*)$. *Then* $\mathcal{J}_K : L^X \to L^X$ *with*

$$\mathcal{J}_K(A)(y) = \bigwedge_{x\in X} (A^*(x) \to \mathcal{K}(\top_x)(y)) = \bigwedge_{x\in X} (\mathcal{K}^*(\top_x)(y) \to A(x))).$$

is an L-lower approximation operator.

(9) *If* $\mathcal{K}(\mathcal{K}^*(A)) = \mathcal{K}(A)$ *for* $A \in L^X$, *then* $\mathcal{J}_K(\mathcal{J}_K(A)) = \mathcal{J}_K(A)$ *for* $A \in L^X$ *such that* $\tau_{\mathcal{J}_{K_1}} = (\tau_{\mathcal{J}_K})_*$ *with*

$$\tau_{\mathcal{J}_K} = \{\mathcal{J}_K(A) = \bigwedge_{x\in X} (\mathcal{K}^*(\top_x) \to A(x))) \mid A \in L^X\},$$

$$\tau_{\mathcal{J}_{K_1}} = \{\mathcal{J}_{K_1}(A)(x) = \bigwedge_{x\in X} (\mathcal{K}^*(\top_x)(y) \to A(y))) \mid A \in L^X\}.$$

(10) *If* $\mathcal{K}(\mathcal{K}(A)) = \mathcal{K}^*(A)$ *for* $A \in L^X$, *then* $\mathcal{J}_K(\mathcal{J}_K^*(A)) = \mathcal{J}_K^*(A)$ *such that*

$$\tau_{\mathcal{J}_K} = \{\mathcal{J}_K^*(A) = \bigvee_{x\in X}(\mathcal{K}^*(\top_x) \odot A^*(x))) \mid A \in L^X\} = (\tau_{\mathcal{J}_K})_*.$$

(11) *Define* $\mathcal{H}_K(A) = (\mathcal{K}(A))^*$. *Then* $\mathcal{H}_K : L^X \to L^X$ *with*

$$\mathcal{H}_K(A)(y) = \bigvee_{x\in X}(A(x) \odot \mathcal{K}^*(\top_x)(y))$$

is an L-upper approximation operator. Moreover, $\tau_{\mathcal{H}_K} = \tau_{\mathcal{K}}$.

(12) *If* $\mathcal{K}(\mathcal{K}^*(A)) = \mathcal{K}(A)$ *for* $A \in L^X$, *then* $\mathcal{H}_K(\mathcal{H}_K(A)) = \mathcal{H}_K(A)$ *for* $A \in L^X$ *such that* $\tau_{\mathcal{H}_{K_1}} = (\tau_{\mathcal{H}_K})_*$ *with*

$$\tau_{\mathcal{H}_K} = \{\mathcal{H}_K(A) = \bigvee_{x\in X}(A(x) \odot \mathcal{K}^*(\top_x)) \mid A \in L^X\},$$

$$\tau_{(\mathcal{H}_K)_1} = \{(\mathcal{H}_K)_1(A)(y) = \bigvee_{x\in X}(A(x) \odot \mathcal{K}^*(\top_y)(x)) \mid A \in L^X\}.$$

(13) *If* $\mathcal{K}(\mathcal{K}(A)) = \mathcal{K}^*(A)$ *for* $A \in L^X$, *then* $\mathcal{H}_K(\mathcal{H}_K(A)) = \mathcal{H}_K^*(A)$ *such that*

$$\tau_{\mathcal{H}_K} = \{\mathcal{H}_K^*(A) = \bigwedge_{x\in X}(A(x) \to \mathcal{K}(\top_x)) \mid A \in L^X\} = (\tau_{\mathcal{H}_K})_*.$$

(14) $(\mathcal{H}_{K_1}, \mathcal{J}_K)$ *and* $(\mathcal{H}_K, \mathcal{J}_{K_1})$ *are a residuated connetion;i.e,*

$$\mathcal{H}_{K_1}(A) \le B \text{ iff } A \le \mathcal{J}_K(B),$$

$$\mathcal{H}_K(A) \le B \text{ iff } A \le \mathcal{J}_{K_1}(B).$$

Moreover, $\tau_{\mathcal{J}_K} = \tau_{\mathcal{H}_{K_1}}$ *and* $\tau_{\mathcal{J}_{K_1}} = \tau_{\mathcal{H}_K}$.

Proof (1) For $A = \bigvee_{x\in X}(A(x) \odot \top_x) \in L^X$, $\mathcal{K}(A)(y) = \bigwedge_{x\in X}(A(x) \to \mathcal{K}(\top_x)(y))$.

(2) (K1) Since $B \le \mathcal{K}(\mathcal{K}_1(B)) \le \mathcal{K}_1^*(B)$, we have $\mathcal{K}_1(B) \le B^*$.

(K2) Since $\mathcal{K}_1(B) \le \mathcal{K}_1(B)$, then $B \le \mathcal{K}(\mathcal{K}_1(B))$. Thus,

$$\begin{aligned} & B \le \mathcal{K}(\mathcal{K}_1(B)) \le \mathcal{K}(a \odot (a \to \mathcal{K}_1(B))) = a \to \mathcal{K}(a \to \mathcal{K}_1(B)) \\ & \text{iff } a \odot B \le \mathcal{K}(a \to \mathcal{K}_1(B)) \\ & \text{iff } a \to \mathcal{K}_1(B) \le \mathcal{K}_1(a \odot B). \end{aligned}$$

$$\begin{aligned} & a \odot B \le \mathcal{K}(\mathcal{K}_1(a \odot B)) \\ & \text{iff } B \le a \to \mathcal{K}(\mathcal{K}_1(a \odot B)) = \mathcal{K}(a \odot \mathcal{K}_1(a \odot B)) \\ & \text{iff } a \odot \mathcal{K}_1(a \odot B) \le \mathcal{K}_1(B) \\ & \text{iff } \mathcal{K}_1(a \odot B) \le a \to \mathcal{K}_1(B). \end{aligned}$$

(K3) $\mathcal{K}_1(\bigvee_{i\in\Gamma} A_i) = \bigwedge_{i\in\Gamma}\mathcal{K}_1(A_i)$. By the definition of $\mathcal{K}_1$, since $\mathcal{K}_1(A) \leq \mathcal{K}_1(B)$ for $B \leq A$, we have

$$\mathcal{K}_1(\bigvee_{i\in\Gamma} A_i) \leq \bigwedge_{i\in\Gamma}\mathcal{K}_1(A_i).$$

Since $\mathcal{K}(\bigwedge_{i\in\Gamma}\mathcal{K}_1(A_i)) \geq \mathcal{K}(\mathcal{K}_1(A_i)) \geq A_i$, then $\mathcal{K}(\bigwedge_{i\in\Gamma}\mathcal{K}_1(A_i)) \geq \bigvee_{i\in\Gamma} A_i$. Thus

$$\mathcal{K}_1(\bigvee_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma}\mathcal{K}_1(A_i).$$

Thus $\mathcal{K}_1 : L^X \to L^X$ is an L-join meet approximation operator. By the definition of $\mathcal{K}_1$, we have

$$A \leq \mathcal{K}_1(B) \text{ iff } B \leq \mathcal{K}(A).$$

Since $A^* \leq \mathcal{K}_1(A)$ iff $A \leq \mathcal{K}(A^*)$, we have $\tau_{\mathcal{K}_1} = (\tau_{\mathcal{K}})_*$.

(3) Let $\mathcal{K}(\mathcal{K}^*(A)) = \mathcal{K}(A)$ for $A \in L^X$. Then

$\mathcal{K}_1^*(A) \leq \mathcal{K}(B)$ iff $\mathcal{K}_1(A) \geq \mathcal{K}^*(B)$ iff $\mathcal{K}(\mathcal{K}^*(B)) = \mathcal{K}(B) \geq A$

$$\begin{aligned}\mathcal{K}_1(\mathcal{K}_1^*(A)) &= \bigvee\{B \mid \mathcal{K}_1^*(A) \leq \mathcal{K}(B)\}\\ &= \bigvee\{B \mid A \leq \mathcal{K}(B)\}\\ &= \mathcal{K}_1(A).\end{aligned}$$

(4) Let $\mathcal{K}(A) \in \tau_{\mathcal{K}}$. Since $\mathcal{K}(\mathcal{K}(A)) = \mathcal{K}^*(A)$, $\mathcal{K}(\mathcal{K}^*(A)) = \mathcal{K}(\mathcal{K}(\mathcal{K}(A))) = (\mathcal{K}(\mathcal{K}(A)))^* = \mathcal{K}(A)$. Hence $\mathcal{K}^*(A) \in \tau_{\mathcal{K}}$; i.e. $\mathcal{K}(A) \in (\tau_{\mathcal{K}})_*$.

Let $A \in (\tau_{\mathcal{K}})_*$. Then $A = \mathcal{K}(A^*)$. Since $\mathcal{K}(A) = \mathcal{K}(\mathcal{K}(A^*)) = \mathcal{K}^*(A^*) = A^*$, then $A \in\in \tau_{\mathcal{K}}$. Thus, $(\tau_{\mathcal{K}})_* \subset \tau_{\mathcal{K}}$.

(5) (M1) Since $A \leq \mathcal{K}(A^*)$, $\mathcal{M}_K(A) = \mathcal{K}(A^*)^* \leq A^*$.

(M2)

$$\begin{aligned}\mathcal{M}_K(\alpha \to A) &= (\mathcal{K}((\alpha \to A)^*)^* = (\mathcal{K}(\alpha \odot A^*))^*\\ &= (\alpha \to \mathcal{K}(A^*))^* = \alpha \odot \mathcal{K}(A^*)^*\\ &= \alpha \odot \mathcal{M}_K(A).\end{aligned}$$

(M3)

$$\begin{aligned}\mathcal{M}_K(\bigwedge_{i\in\Gamma} A_i) &= (\mathcal{K}(\bigwedge_{i\in\Gamma} A_i)^*)^* = (\mathcal{K}(\bigvee_{i\in\Gamma} A_i^*))^*\\ &= (\bigwedge_{i\in\Gamma}\mathcal{K}(A_i^*))^* = \bigvee_{i\in\Gamma}(\mathcal{K}(A_i^*))^*\\ &= \bigvee_{i\in\Gamma}\mathcal{M}_K(A_i).\end{aligned}$$

Moreover, the pair $(\mathcal{M}_K, \mathcal{M}_{K_1})$ is a dual Galois connection from:

$$\mathcal{M}_K(A) \leq B \text{ iff } B^* \leq \mathcal{K}(A^*) \text{ iff } A^* \leq \mathcal{K}_1(B^*)$$

$$\mathcal{K}_1^*(B^*) \leq A \text{ iff } \mathcal{M}_{K_1}(B) \leq A.$$

We have $\tau_{\mathcal{K}_1} = \tau_{\mathcal{M}_K} = (\tau_{\mathcal{K}})_* = (\tau_{\mathcal{M}_{K_1}})_*$ from:

$$A^* \leq \mathcal{K}_1(A) \text{ iff } A \leq \mathcal{K}(A^*)$$

$$\mathcal{M}_K(A) \le A^* \text{ iff } \mathcal{M}_{K_1}(A^*) \le A.$$

(6) Let $\mathcal{K}(\mathcal{K}^*(A)) = \mathcal{K}(A)$ for $A \in L^X$. Then

$$\begin{aligned}\mathcal{M}_K(\mathcal{M}_K^*(A)) &= \mathcal{K}^*(\mathcal{M}_K(A)) = (\mathcal{K}(\mathcal{K}^*(A^*)))^* \\ &= \mathcal{K}^*(A^*) = \mathcal{M}_K(A).\end{aligned}$$

By (3), since $\mathcal{K}_1(\mathcal{K}_1^*(A)) = \mathcal{K}_1(A)$ for $A \in L^X$, $(\mathcal{M}_K)_1((\mathcal{M}_K)_1^*(A)) = (\mathcal{M}_K)_1(A)$ for $A \in L^X$. Thus,

$$\tau_{\mathcal{M}_K} = \{\mathcal{M}_K^*(A) \mid A \in L^X\},\ \tau_{(\mathcal{M}_K)_1} = \{(\mathcal{M}_K)_1^*(A) \mid A \in L^X\}.$$

(7) Let $\mathcal{K}(\mathcal{K}(A)) = \mathcal{K}^*(A)$ for $A \in L^X$. Then

$$\begin{aligned}\mathcal{M}_K(\mathcal{M}_K(A)) &= \mathcal{K}^*(\mathcal{M}_K^*(A)) = (\mathcal{K}(\mathcal{K}(A^*)))^* \\ &= (\mathcal{K}^*(A^*))^* = \mathcal{M}_K^*(A).\end{aligned}$$

By the similarly method in (4), $\mathcal{M}_K(\mathcal{M}_K^*(A)) = \mathcal{M}_K(A)$ for $A \in L^X$. Thus,

$$\tau_{\mathcal{M}_K} = \{\mathcal{M}_K(A) \mid A \in L^X\} = (\tau_{\mathcal{M}_K})_*.$$

(8) It is similarly proved as (5).

(9) If $\mathcal{K}(\mathcal{K}^*(A)) = \mathcal{K}(A)$ for $A \in L^X$, then $\mathcal{J}_K(\mathcal{J}_K(A)) = \mathcal{J}_K(A)$

$$\begin{aligned}\mathcal{J}_K(\mathcal{J}_K(A)) &= \mathcal{J}_K(\mathcal{K}(A^*)) = \mathcal{K}(\mathcal{K}^*(A^*)) \\ &= \mathcal{K}(A^*) = \mathcal{J}_K(A).\end{aligned}$$

Similarly, $\mathcal{J}_{K_1}(\mathcal{J}_{K_1}(A)) = \mathcal{J}_{K_1}(A)$. Thus, the results hold.

(10) If $\mathcal{K}(\mathcal{K}(A)) = \mathcal{K}^*(A)$ for $A \in L^X$, then $\mathcal{J}_K(\mathcal{J}_K^*(A)) = \mathcal{J}_K^*(A)$

$$\begin{aligned}\mathcal{J}_K(\mathcal{J}_K^*(A)) &= \mathcal{J}_K(\mathcal{K}^*(A^*)) = \mathcal{K}(\mathcal{K}(A^*)) \\ &= \mathcal{K}^*(A^*) = \mathcal{J}_K^*(A).\end{aligned}$$

Since $\mathcal{J}_K(\mathcal{J}_K^*(A)) = \mathcal{J}_K^*(A)$

$$\begin{aligned}\mathcal{J}_K(\mathcal{J}_K(A)) &= \mathcal{J}_K(\mathcal{J}_K^*(\mathcal{J}_K^*(A))) \\ &= \mathcal{J}_K^*(\mathcal{J}_K^*(A)) = \mathcal{J}_K(A).\end{aligned}$$

Hence $\tau_{\mathcal{J}_K} = \{\mathcal{J}_K^*(A) \mid A \in L^X\} = (\tau_{\mathcal{J}_K})_*$.

(11) and (12) are similarly proved as (5) and (6), respectively.

(13) If $\mathcal{K}(\mathcal{K}(A)) = \mathcal{K}^*(A)$ for $A \in L^X$, then $\mathcal{H}_K(\mathcal{H}_K^*(A)) = \mathcal{H}_K^*(A)$ from:

$$\begin{aligned}\mathcal{H}_K(\mathcal{H}_K^*(A)) &= \mathcal{H}_K(\mathcal{K}(A)) = (\mathcal{K}(\mathcal{K}(A)))^* \\ &= (\mathcal{K}^*(A))^* = \mathcal{H}_K^*(A).\end{aligned}$$

(14) $(\mathcal{H}_{K_1}, \mathcal{J}_K)$ is a residuated connection;i.e,

$$\mathcal{H}_{K_1}(A) \le B \text{ iff } \mathcal{K}_1(A) \ge B^*,$$

$$A \le \mathcal{K}(B^*) \text{ iff } A \le \mathcal{J}_K(B),$$

Similarly,$(\mathcal{H}_K, \mathcal{J}_{K_1})$ is a residuated connection.

Example 2.2 Let R be a reflexive L-fuzzy relation. Define $\mathcal{K}_{R^*}: L^X \to L^X$ as follows:

$$\mathcal{K}_{R^*}(A)(y) = \bigwedge_{x\in X}(A(x) \to R^*(x,y)).$$

(1) (K1) $\mathcal{K}_{R^*}(A)(y) \leq A(y) \to R^*(y,y)) = A^*(x)$.

(K2) $\mathcal{K}_{R^*}(a \odot A)(y) = \bigwedge_{x\in X}((a\odot A)(x) \to R^*(x,y)) = a \to \bigwedge_{x\in X}(A(x) \to R^*(x,y)) = a \to \mathcal{K}_{R^*}(A)(y)$.

(K3) $\mathcal{K}_{R^*}(\bigvee_{i\in\Gamma} A_i)(y) = \bigwedge_{x\in X}(\bigvee_{i\in\Gamma} A_i(x) \to R^*(x,y)) = \bigwedge_{x\in X}\bigwedge_{i\in\Gamma}(A_i(x) \to R^*(x,y)) = \bigwedge_{i\in\Gamma}\mathcal{K}_{R^*}(A_i)(y)$. Hence $\mathcal{K}_{R^*}$ is an L-join meet approximation operator.

(2) Define $(\mathcal{K}_{R^*})_1(B) = \bigvee\{A \mid B \leq \mathcal{K}_{R^*}(A)\}$. Since $B(y) \leq \mathcal{K}_{R^*}(B)(y)$ iff $B(y) \leq A(x) \to R^*(x,y)$ iff $A(x) \leq B(y) \to R^*(x,y))$, then

$$(\mathcal{K}_{R^*})_1(B)(x) = \mathcal{K}_{R^{-1*}}(B)(x) = \bigwedge_{y\in X}(B(y) \to R^{-1*}(y,x)).$$

Then $(\mathcal{K}_{R^*})_1 = \mathcal{K}_{R^{-1*}}$ with

$$\mathcal{K}_{R^{-1*}}(A)(y) = \bigwedge_{x\in X}(A(x) \to R^{-1*}(x,y))$$

is an L-join meet approximation operator such that $(\mathcal{K}_R, \mathcal{K}_{R^{-1*}})$ is a Galois connection;i.e.,

$$A \leq \mathcal{K}_{R^{-1*}}(B) \text{ iff } B \leq \mathcal{K}_{R^*}(A).$$

Moreover, $\tau_{\mathcal{K}_{R^{-1*}}} = (\tau_{\mathcal{K}_{R^*}})_*$.

(3) If R is an L-fuzzy preorder, then R^{-1} is an L-fuzzy preorder. Since $R(x,y)\odot R(y,z) \leq R(x,z)$ iff

$$\begin{array}{l} A(x)\odot R(x,y)\odot (A(x) \to R^*(x,z) \leq R(x,y)\odot R^*(x,z) \leq R^*(y,z) \\ \text{iff } A(x) \to R^*(x,z) \leq A(x)\odot R(x,y) \to R^*(y,z) \end{array}$$

$$\begin{aligned} \mathcal{K}_{R^*}(\mathcal{K}^*_{R^*}(A))(z) &= \textstyle\bigwedge_{y\in X}(\mathcal{K}^*_{R^*}(A)(y) \to R^*(y,z)) \\ &= \textstyle\bigwedge_{y\in X}(\bigvee_{x\in X}(A(x)\odot R(x,y) \to R^*(y,z)) \\ &= \textstyle\bigwedge_{x\in X}(A(x) \to R^*(x,z)) = \mathcal{K}_{R^*}(A)(z). \end{aligned}$$

Thus $\mathcal{K}_{R^*}(\mathcal{K}^*_{R^*}(A)) = \mathcal{K}_{R^*}(A)$ for $A \in L^X$. Similarly, $\mathcal{K}_{R^{-1*}}(\mathcal{K}^*_{R^{-1*}}(A)) = \mathcal{K}_{R^{-1*}}(A)$ for $A \in L^X$ such that $\tau_{\mathcal{K}_{R^{-1*}}} = (\tau_{\mathcal{K}_{R^*}})_*$ with

$$\tau_{\mathcal{K}_{R^*}} = \{\mathcal{K}^*_{R^*}(A) = \bigvee_{x\in X}(A(x)\odot R(x,-)) \mid A \in L^X\},$$

$$\tau_{\mathcal{K}_{R^{-1*}}} = \{\mathcal{K}^*_{R^{-1*}}(A) = \bigvee_{x\in X}(A(x)\odot R(-,x)) \mid A \in L^X\}.$$

(4) Let R be a reflexive and Euclidean L-fuzzy relation. Since $R(x.z) \odot R(y,z) \leq R(x,y)$ iff $R(y,z) \leq R(x,z) \to R(x,y)$ iff $R(x,z) \odot R^*(x,y) \leq R^*(y,z)$, then

$$\begin{array}{l} A(x) \odot R(x,z) \odot (A(x) \to R^*(x,y)) \leq R(x,z) \odot R^*(x,y) \leq R^*(y,z) \\ \text{iff } A(x) \odot R(x,z) \leq (A(x) \to R^*(x,y)) \to R^*(y,z). \end{array}$$

$$\begin{array}{ll} \mathcal{K}_{R^*}(\mathcal{K}_{R^*}(A))(z) & = \bigwedge_{y\in X}(\mathcal{K}_{R^*}(A)(y) \to R^*(y,z)) \\ & = \bigwedge_{y\in X}(\bigwedge_{x\in X}(A(x) \to R^*(x,y)) \to R^*(y,z)) \\ & \geq \bigvee_{x\in X}(A(x) \odot R(x,z)) = \mathcal{K}_{R^*}(A)(z). \end{array}$$

Thus, $\mathcal{K}_{R^*}(\mathcal{K}_{R^*}(A)) = \mathcal{K}^*_{R^*}(A)$ for $A \in L^X$ such that $\tau_{\mathcal{K}_{R^*}} = (\tau_{\mathcal{K}_{R^*}})_*$ with

$$\tau_{\mathcal{K}_{R^*}} = \{\mathcal{K}_{R^*}(A) = \bigwedge_{x\in X}(A(x) \to R(x,-)) \mid A \in L^X\}.$$

(5) Define $\mathcal{M}_{\mathcal{K}_{R^*}}(A) = \mathcal{K}_{R^*}(A^*)^*$. By Theorem 2.1 (5), $\mathcal{M}_{\mathcal{K}_{R^*}} = \mathcal{M}_R$ and $\mathcal{M}_{(\mathcal{K}_{R^*})_1} = \mathcal{M}_{\mathcal{K}_{R^{-1*}}} = \mathcal{M}_{R^{-1}}$ are L-meet join approximation operators such that

$$\mathcal{M}_{\mathcal{K}_{R^*}}(A)(y) = (\bigwedge_{x\in X}(A^*(x) \to R(x,y)))^* = \bigvee_{x\in X}(A^*(x) \odot R(x,y)),$$

$$\mathcal{M}_{\mathcal{K}_{R^{-1*}}}(A)(y) = (\bigwedge_{x\in X}(A^*(x) \to R^{-1}(x,y)))^* = \bigvee_{x\in X}(A^*(x) \odot R^{-1}(x,y)).$$

Moreover, the pair $(\mathcal{M}_R, \mathcal{M}_{R^{-1}})$ is a dual Galois connection such that $\tau_{\mathcal{K}_{R^{-1*}}} = \tau_{\mathcal{M}_R} = (\tau_{\mathcal{K}_{R^{-1*}}})_* = (\tau_{\mathcal{M}_{R^{-1}}})_*$.

(6) If R is an L-fuzzy preorder, by (3), $\mathcal{K}_{R^*}(\mathcal{K}^*_{R^*}(A)) = \mathcal{K}_{R^*}(A)$ and $\mathcal{K}_{R^{-1*}}(\mathcal{K}^*_{R^{-1*}}(A)) = \mathcal{K}_{R^{-1*}}(A)$ for $A \in L^X$. By Theorem 2.1(6), $\mathcal{M}_R(\mathcal{M}^*_R(A)) = \mathcal{M}_R(A)$ and $\mathcal{M}_{R^{-1}}(\mathcal{M}^*_{R^{-1}}(A)) = \mathcal{M}_{R^{-1}}(A)$ for $A \in L^X$ such that $\tau_{\mathcal{K}_{R^{-1*}}} = \tau_{\mathcal{M}_R} = (\tau_{\mathcal{K}_{R^{-1*}}})_* = (\tau_{\mathcal{M}_{R^{-1}}})_*$ with

$$\tau_{\mathcal{M}_R} = \{\mathcal{M}^*_R(A) = \bigwedge_{x\in X}(R(x,-) \to A(x)) \mid A \in L^X\},$$

$$\tau_{\mathcal{M}_{R^{-1}}} = \{\mathcal{M}^*_{R^{-1}}(A) = \bigwedge_{x\in X}(R(-,x) \to A(x)) \mid A \in L^X\}.$$

(7) If R is a reflexive and Euclidean L-fuzzy relation, by (4), $\mathcal{K}_{R^*}(\mathcal{K}_{R^*}(A)) = \mathcal{K}^*_{R^*}(A)$ for $A \in L^X$. By Theorem 2.1(7), then $\mathcal{M}_R(\mathcal{M}_R(A)) = \mathcal{M}^*_R(A)$ for $A \in L^X$ such that $\tau_{\mathcal{M}_R} = (\tau_{\mathcal{M}_R})_*$ with

$$\tau_{\mathcal{M}_R} = \{\mathcal{M}_R(A) = \bigvee_{x\in X}(A^*(x) \odot R(x,-)) \mid A \in L^X\}.$$

(8) Define $\mathcal{J}_{\mathcal{K}_{R^*}}(A) = \mathcal{K}_{R^*}(A^*)$. By Theorem 2.1(8), $\mathcal{J}_{\mathcal{K}_{R^*}} = \mathcal{J}_R$ and $\mathcal{J}_{\mathcal{K}_{R^{-1*}}} = \mathcal{J}_{R^{-1}}$ are L-lower approximation operators such that

$$\mathcal{J}_{\mathcal{K}_{R^*}}(A)(y) = \bigwedge_{x\in X}(A^*(x) \to R^*(x,y)) = \bigwedge_{x\in X}(R(x,y) \to A(x)),$$

$$\mathcal{J}_{\mathcal{K}_{R^{-1*}}}(A)(y) = \bigwedge_{x\in X}(A^*(x) \to R^{-1*}(x,y)) = \bigwedge_{x\in X}(R(y,x) \to A(x)).$$

Moreover, $\tau_{\mathcal{J}_R} = (\tau_{\mathcal{K}_{R^*}})_* = \tau_{\mathcal{K}_{R^{-1*}}}$ and $\tau_{\mathcal{J}_{R^{-1}}} = (\tau_{\mathcal{K}_{R^{-1*}}})_* = \tau_{\mathcal{K}_{R^*}}$.

(9) If R is an L-fuzzy preorder, by (3), $\mathcal{K}_{R^*}(\mathcal{K}^*_{R^*}(A)) = \mathcal{K}_{R^*}(A)$ and $\mathcal{K}_{R^{-1*}}(\mathcal{K}^*_{R^{-1*}}(A)) = \mathcal{K}_{R^{-1*}}(A)$ for $A \in L^X$. By Theorem 2.1(9), then $\mathcal{J}_R(\mathcal{J}_R(A)) = \mathcal{J}_R(A)$ and $\mathcal{J}_{R^{-1}}(\mathcal{J}_{R^{-1}}(A)) = \mathcal{J}_{R^{-1}}(A)$ for $A \in L^X$ such that $\tau_{\mathcal{J}_{R^{-1}}} = (\tau_{\mathcal{J}_R})_*$ with

$$\tau_{\mathcal{J}_R} = \{\mathcal{J}_R(A) = \bigwedge_{x\in X}(R(x,-) \to A(x)) \mid A \in L^X\},$$

$$\tau_{\mathcal{J}_{R^{-1}}} = \{\mathcal{J}_{R^{-1}}(A) = \bigwedge_{x\in X}(R(-,x) \to A(x)) \mid A \in L^X\}.$$

(10) If R is a reflexive and Euclidean L-fuzzy relation, by (4), $\mathcal{K}_{R^*}(\mathcal{K}_{R^*}(A)) = \mathcal{K}^*_{R^*}(A)$ for $A \in L^X$. By Theorem 2.1(10), $\mathcal{J}_R(\mathcal{J}^*_R(A)) = \mathcal{J}^*_R(A)$ for $A \in L^X$ such that $\tau_{\mathcal{J}_R} = (\tau_{\mathcal{J}_R})_*$ with

$$\tau_{\mathcal{J}_R} = \{\mathcal{J}^*_R(A) = \bigvee_{x\in X}(A^*(x) \odot R(x,-)) \mid A \in L^X\}.$$

(11) Define $\mathcal{H}_{\mathcal{K}_{R^*}}(A) = (\mathcal{K}_{R^*}(A))^*$. Then $\mathcal{H}_{\mathcal{K}_{R^*}} = \mathcal{H}_R$ is an L-upper approximation operator such that

$$\mathcal{H}_{\mathcal{K}_{R^*}}(A)(y) = \bigvee_{x\in X}(R(x,y) \odot A(x)).$$

Moreover, $\tau_{\mathcal{H}_R} = \tau_{\mathcal{K}_{R^*}}$.

(12) If R is an L-fuzzy preorder, by (3), $\mathcal{K}_{R^*}(\mathcal{K}^*_{R^*}(A)) = \mathcal{K}_{R^*}(A)$ and $\mathcal{K}_{R^{-1*}}(\mathcal{K}^*_{R^{-1*}}(A)) = \mathcal{K}_{R^{-1*}}(A)$ for $A \in L^X$. By Theorem 2.1(12), $\mathcal{H}_{\mathcal{K}_{R^*}}(\mathcal{H}_{\mathcal{K}_{R^*}}(A)) = \mathcal{H}_{\mathcal{K}_{R^*}}(A)$ and $\mathcal{H}_{\mathcal{K}_{R^{-1*}}}(\mathcal{H}_{\mathcal{K}_{R^{-1*}}}(A)) = \mathcal{H}_{\mathcal{K}_{R^{-1*}}}(A)$ for $A \in L^X$ such that $\tau_{\mathcal{H}_{R^{-1}}} = (\tau_{\mathcal{H}_R})_*$ with

$$\tau_{\mathcal{H}_R} = \{\mathcal{H}_R(A) = \bigvee_{x\in X}(R(x,-) \odot A(x)) \mid A \in L^X\},$$

$$\tau_{\mathcal{H}_{R^{-1}}} = \{\mathcal{H}_{R^{-1}}(A) = \bigvee_{x\in X}(R(-,x) \odot A(x)) \mid A \in L^X\}.$$

(13) If R is a reflexive and Euclidean L-fuzzy relation, by (4), $\mathcal{K}_{R^*}(\mathcal{K}_{R^*}(A)) = \mathcal{K}^*_{R^*}(A)$ for $A \in L^X$. By Theorem 2.1(13), $\mathcal{H}_R(\mathcal{H}^*_R(A)) = \mathcal{H}^*_R(A)$ for $A \in L^X$ such that $\tau_{\mathcal{H}_R} = (\tau_{\mathcal{H}_R})_*$ with

$$\tau_{\mathcal{H}_R} = \{\mathcal{H}^*_R(A) = \bigwedge_{x\in X}(A(x) \to R^*(x,-)) \mid A \in L^X\}.$$

(14) $(\mathcal{H}_{R^{-1}}, \mathcal{J}_R)$ is a residuated connection;i.e,

$$\mathcal{H}_{R^{-1}}(A) \le B \text{ iff } \mathcal{K}_{R^{-1*}}(A) \ge B^*,$$

$$A \le \mathcal{K}_{R^*}(B^*) \text{ iff } A \le \mathcal{J}_R(B).$$

Similarly, $(\mathcal{H}_R, \mathcal{J}_{R^{-1}})$ is a residuated connection. Moreover, $\tau_{\mathcal{J}_R} = \tau_{\mathcal{H}_{R^{-1}}}$ and $\tau_{\mathcal{J}_{R^{-1}}} = \tau_{\mathcal{H}_R}$.

References

[1] R. Bělohlávek, Fuzzy Relational Systems, *Kluwer Academic Publishers*, New York , 2002.

[2] P. Hájek, Metamathematices of Fuzzy Logic,*Kluwer Academic Publishers*, Dordrecht (1998).

[3] Y.C. Kim, L-approximatins and join preserving maps,*J. Math. Comput. Sci.,* **3** (5) (2013), 1193-1210.

[4] Y.C. Kim, Alexandrov L-topologies and L-join meet approximation operators *International Journal of Pure and Applied Mathematics,* **91**(1)(2014), 113-129.

[5] H. Lai, D. Zhang, Fuzzy preorder and fuzzy topology,*Fuzzy Sets and Systems,* **157** (2006), 1865-1885.

[6] H. Lai, D. Zhang, Concept lattices of fuzzy contexts: Formal concept analysis vs. rough set theory,*Int. J. Approx. Reasoning,* **50** (2009), 695-707.

[7] Z. Pawlak, Rough sets, *Int. J. Comput. Inf. Sci.*, **11** (1982), 341-356.

[8] Z. Pawlak, Rough probability, *Bull. Pol. Acad. Sci. Math.*, **32**(1984), 607-615.

[9] A. M. Radzikowska, E.E. Kerre, A comparative study of fuzy rough sets, *Fuzzy Sets and Systems*, **126**(2002), 137-155.

[10] Y.H. She, G.J. Wang, An axiomatic approach of fuzzy rough sets based on residuated lattices,*Computers and Mathematics with Applications*, **58**(2009), 189-201.

[11] Zhen Ming Ma, Bao Qing Hu, Topological and lattice structures of L-fuzzy rough set determined by lower and upper sets, *Information Sciences*, **218**(2013), 194-204.

Received: March, 2014