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Optimal Inequalities for Generalized Logarithmic and Seiffert Means

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Abstract

For $r \in \mathbf{R}$, the generalized logarithmic mean $L_r(a,b)$ and Seiffert mean $P(a,b)$ of two positive numbers a and b are defined by $L_r(a,b) = a, for\ a = b, L_r(a,b) = [(b^r - a^r)/r(b-a)]^{\frac{1}{r-1}}, for\ r \neq 1, r \neq 0, and\ a \neq b, L_r(a,b) = \frac{1}{e}(\frac{b^b}{a^a})^{\frac{1}{b-a}}, for\ r = 1\ and\ a \neq b, L_r(a,b) = (b-a)/(\ln b - \ln a),\ for\ r = 0\ and\ a \neq b, and\ P(a,b) = (a-b)/(4\arctan\sqrt{a/b} - \pi)$ respectively. In this paper, we find the greatest value α and the least value β such that the inequality

$$L_\alpha(a,b) < P(a,b)\ (or\ P(a,b) < L_\beta(a,b), resp.)$$

holds for all $a,\ b > 0$ with $a \neq b$.

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1 Introduction

For $r \in \mathbf{R}$, the generalized logarithmic mean $L_r(a,b)$ with parameter r of two positive numbers a and b is defined by

$$L_r(a,b) = \begin{cases} a, & a=b, \\ (\frac{b^r-a^r}{r(b-a)})^{\frac{1}{r-1}}, & r \neq 1, r \neq 0, \ a \neq b, \\ \frac{1}{e}(\frac{b^b}{a^a})^{\frac{1}{b-a}}, & r=1 \ , \ a \neq b, \\ (b-a)/(\ln b - \ln a), & r=0 \ , \ a \neq b. \end{cases}$$

It is well known that the generalized logarithmic mean is continuous and increasing with respect to $r \in \mathbf{R}$ for fixed a and b.

For $a,b>0$ with $a \neq b$ the Seiffert mean $P(a,b)$ was introduced by Seiffert [8] as follows:

$$P(a,b) = \frac{a-b}{4\arctan(\sqrt{a/b}) - \pi}.$$

Recently, both means have been the subject of intensive research [1-15] and therein.

Let $H(a,b) = 2ab/(a+b), A(a,b) = (a+b)/2, G(a,b) = \sqrt{ab}, I(a,b) = 1/e(b^b/a^a)^{1/(b-a)}$ and $L(a,b) = (b-a)/(\ln b - \ln a)$ be the harmonic , arithmetic, geometric , identric and logarithmic means of two positive real numbers a and b with $a \neq b$. Then

$$\begin{aligned} \min\{a,b\} < H(a,b) < G(a,b) = L_{-1}(a,b) < L(a,b) = L_0(a,b) \\ < I(a,b) = L_1(a,b) < A(a,b) = L_2(a,b) < \max\{a,b\}. \end{aligned}$$

The following bounds for the Seiffert mean $P(a,b)$ in terms of the power mean $M_r(a,b) = ((a^r+b^r)/2)^{1/r} (r \neq 0)$ were presented by Jagers in [13]:

$$M_{1/2} < P(a,b) < M_{2/3}(a,b)$$

for all $a,b>0$ with $a \neq b$.

Hästö [15] found the sharp lower bound for the Seiffert mean as follow:

$$M_{log2/log\pi}(a,b) < P(a,b)$$

for all $a,b>0$ with $a \neq b$.

In [9], Seiffert proved

$$P(a,b) > \frac{3A(a,b)G(a,b)}{A(a,b)+2G(a,b)} \quad \text{and} \quad P(a,b) > \frac{2}{\pi}A(a,b)$$

for all $a,b>0$ with $a \neq b$.

In [10], the authors found the greatest value α and the least value β such that the double inequality $\alpha A(a,b) + (1-\alpha)H(a,b) < P(a,b) < \beta A(a,b) + (1-\beta)H(a,b)$ holds for all $a, b > 0$ with $a \neq b$.

In [11], the author proved that

$$L_{3\alpha-2}(a,b) < \alpha A(a,b) + (1-\alpha)G(a,b), for\ \alpha \in (0, \frac{1}{2})$$
$$L_{3\alpha-2}(a,b) > \alpha A(a,b) + (1-\alpha)G(a,b), for\ \alpha \in (\frac{1}{2}, 1)$$

In [9], Seiffert proved

$$L(a,b) < P(a,b) < I(a,b) = L_1(a,b)$$

for all $a, b > 0$ with $a \neq b$.

The purpose of the present paper is to find the greatest value α such that the inequality

$$L_\alpha(a,b) < P(a,b)$$

holds for all $a,\ b > 0$ with $a \neq b$, at the same time we prove the parameter 1 in inequality $P(a,b) < I(a,b) = L_1(a,b)$ is optimal.

2 Main Results

Lemma 2.1 *Let* $g(t) = 4\arctan t - \pi + \frac{1}{1-r}(rt^{2r-2} - 1)(4\arctan t - \pi) - \frac{2(t^2-1)(1-t^{2r})}{t+t^3}$*, one has the following: if* r *is the solution of equation* $\frac{1}{r-1}\ln r = \ln \pi$*, then there exists* $\lambda \in (1, +\infty)$ *such that* $g(t) < 0$ *for* $t \in [1, \lambda)$ *and* $g(t) > 0$ *for* $t \in (\lambda, +\infty)$*.*

proof. Let $g_1(t) = \frac{1}{2}t^{3-2r}g'(t),\ g_2(t) = (1+t^2)^3 g_1'(t),\ g_3(t) = \frac{1}{2}t^{1+2r}g_2{}'(t),\ g_4(t) = \frac{1}{2}t^{-1}g_3{}'(t),\ g_5(t) = \frac{1}{2}t^{-1}g_4{}'(t),\ g_6(t) = \frac{1}{2}t^{-1}g_5{}'(t),\ g_7(t) = \frac{t^{5-2r}}{2r(r+1)}g_6'(t),\ g_8(t) = \frac{1}{4(r+2)t}g_7'(t)$, then simple computations lead to

$$\lim_{t\to 1^+} g(t) = 0, \tag{2.1}$$

$$\lim_{t\to +\infty} g(t) = +\infty, \tag{2.2}$$

$$\begin{aligned} g_1(t) = \ & -r(4\arctan t - \pi) + \tfrac{2}{(1-r)(1+t^2)}(r^2 t - t^{3-2r}) + \tfrac{2(1+r)t^3}{1+t^2} \\ & + \tfrac{(t^2-1)(1-t^{2r})}{(1+t^2)^2}(t^{1-2r} + 3t^{3-2r}), \end{aligned} \tag{2.3}$$

$$\lim_{t\to 1^+} g_1(t) = 0, \tag{2.4}$$

$$\lim_{t\to +\infty} g_1(t) = +\infty, \tag{2.5}$$

$$g_2(t) = (3-6r)t^{6-2r} + (9-2r)t^{4-2r} + (6r-3)t^{2-2r} - (1-2r)t^{-2r} + (2r-1)t^6 \\ +(4r-9)t^4 + (9-2r)t^2 + 1 - 4r + \tfrac{2}{1-r}[(2r-1)t^{6-2r} - (3-2r)t^{2-2r} - r^2t^4 + r^2], \tag{2.6}$$

$$\lim_{t\to 1^+} g_2(t) = 0, \tag{2.7}$$

$$\lim_{t\to +\infty} g_2(t) = +\infty, \tag{2.8}$$

$$\begin{aligned} g_3(t) = \;& (3-6r)(3-r)t^6 + (9-2r)(2-r)t^4 + (-6r^2+13r-9)t^2 \\ & +r(1-2r) + 3(2r-1)t^{6+2r} + 2(4r-9)t^{4+2r} + (9-2r)t^{2+2r} \\ & +\tfrac{1}{1-r}[(2r-1)(6-2r)t^6 - 4r^2t^{4+2r}], \end{aligned} \tag{2.9}$$

$$\lim_{t\to 1^+} g_3(t) = 0, \tag{2.10}$$

$$\lim_{t\to +\infty} g_3(t) = +\infty, \tag{2.11}$$

$$\begin{aligned} g_4(t) = \;& 3(3-6r)(3-r)t^4 + 2(9-2r)(2-r)t^2 + (-6r^2+13r-9) \\ & +3(2r-1)(3+r)t^{4+2r} + (4r-9)(4+2r)t^{2+2r} + (9-2r)(1+r)t^{2r} \\ & +\tfrac{1}{1-r}[3(2r-1)(6-2r)t^4 - 2r^2(4+2r)t^{2+2r}], \end{aligned} \tag{2.12}$$

$$\lim_{t\to 1^+} g_4(t) = 32r(r-1) < 0, \tag{2.13}$$

$$\lim_{t\to +\infty} g_4(t) = +\infty, \tag{2.14}$$

$$\begin{aligned} g_5(t) = \;& 6(3-6r)(3-r)t^2 + 2(9-2r)(2-r) + 3(2r-1)(3+r)(2+r)t^{2+2r} \\ & +(4r-9)(4+2r)(1+r)t^{2r} + r(9-2r)(1+r)t^{2r-2} \\ & +\tfrac{1}{1-r}[6(2r-1)(6-2r)t^2 - r^2(4+2r)(2+2r)t^{2r}], \end{aligned} \tag{2.15}$$

$$\lim_{t\to 1^+} g_5(t) = 16r(r-1)(r+7) < 0, \tag{2.16}$$

$$\lim_{t\to +\infty} g_5(t) = +\infty, \tag{2.17}$$

$$\begin{aligned} g_6(t) = \;& 6(3-6r)(3-r) + 3(2r-1)(3+r)(2+r)(1+r)t^{2r} \\ & +2r(4r-9)(2+r)(1+r)t^{2r-2} + r(9-2r)(1+r)(r-1)t^{2r-4} \\ & +\tfrac{1}{1-r}[12(2r-1)(3-r) - 4r^3(2+r)(+r)t^{2r-2}], \end{aligned} \tag{2.18}$$

$$\lim_{t\to 1^+} g_6(t) = 8r(2r^3+8r^2+9r-15), \tag{2.19}$$

if r is the solution of equation $\frac{1}{r-1}\ln r = \ln \pi$, we can get $\frac{3}{4} < r < \frac{13}{16}$, from simple computations we get

$$\lim_{t\to 1^+} g_6(t) = 8r(2r^3+8r^2+9r-15) < 0, \tag{2.20}$$

$$\lim_{t\to +\infty} g_6(t) = +\infty, \tag{2.21}$$

$$g_7(t) = 3(2r-1)(3+r)(2+r)t^4+(12r^3-2r^2-34r+36)t^2+(9-2r)(r-2)(r-1), \tag{2.22}$$

$$\lim_{t\to 1^+} g_7(t) = 4(4r^3 + 10r^2 - 11r + 9), \tag{2.23}$$

$4r^3 + 10r^2 - 11r + 9 > 0$ for $\frac{3}{4} < r < \frac{13}{16}$, so we have

$$\lim_{t\to 1^+} g_7(t) > 0, \tag{2.24}$$

$$g_8(t) = 3(2r-1)(3+r)t^2 + 6r^2 - 13r + 9, \tag{2.25}$$

$$\lim_{t\to 1^+} g_8(t) = 2r(6r+1) > 0, \tag{2.26}$$

$$g_8'(t) = 6(2r-1)(3+r)t > 0,$$

and $g_8(t)$ is strictly increasing in $[1, +\infty)$. From (2.26) and and the monotonicity of $g_8(t)$ we clearly see that $g_8(t) > 0$ for $t > 1$, hence $g_7(t)$ is strictly increasing in $[1, +\infty)$.

The monotonicity of $g_7(t)$ and (2.24) implies that $g_7(t) > 0$ for $t > 1$, then we conclude that $g_6(t)$ is strictly increasing in $[1, +\infty)$.

It follows from (2.20)and (2.21) together with the monotonicity of $g_6(t)$ that there exists $t_1 > 1$ such that $g_6(t) < 0$ for $t \in [1, t_1)$ and $g_6(t) > 0$ for $t \in (t_1, +\infty)$, hence we know that $g_5(t)$ is strictly decreasing in $[1, t_1]$ and strictly increasing in $[t_1, +\infty)$.

the monotonicity of $g_5(t)$ in $[1, t_1]$ and in $[t_1, +\infty)$ together with (2.16) and (2.17) imply that there exists $t_2 > t_1$ such that $g_5(t) < 0$ for $t \in [1, t_2)$ and $g_5(t) > 0$ for $t \in (t_2, +\infty)$, hence $g_4(t)$ is strictly decreasing in $[1, t_2]$ and strictly increasing in $[t_2, +\infty)$.

From (2.13) and (2.14) together with the monotonicity of $g_4(t)$ in $[1, t_2]$ and in $[t_2, +\infty)$, we clearly see that there exists $t_3 > t_2$ such that $g_4(t) < 0$ for $t \in [1, t_3)$ and $g_4(t) > 0$ for $t \in (t_3, +\infty)$, hence we know that $g_3(t)$ is strictly decreasing in $[1, t_3]$ and strictly increasing in $[t_3, +\infty)$.

It follows from (2.10) and (2.11) together with the monotonicity of $g_3(t)$ in $[1, t_3]$ and in $[t_3, +\infty)$ that there exists $t_4 > t_3$ such that $g_3(t) < 0$ for $t \in [1, t_4)$ and $g_3(t) > 0$ for $t \in (t_4, +\infty)$, hence we know that $g_2(t)$ is strictly decreasing in $[1, t_4]$ and strictly increasing in $[t_4, +\infty)$.

From (2.7) and (2.8) together with the monotonicity of $g_2(t)$ in $[1, t_4]$ and in $[t_4, +\infty)$, we clearly see that there exists $t_5 > t_4$ such that $g_2(t) < 0$ for $t \in [1, t_5)$ and $g_2(t) > 0$ for $t \in (t_5, +\infty)$, hence we know that $g_1(t)$ is strictly decreasing in $[1, t_5]$ and strictly increasing in $[t_5, +\infty)$.

It follows from (2.4) and (2.5) together with the monotonicity of $g_1(t)$ in $[1, t_5]$ and in $[t_5, +\infty)$ that there exists $t_6 > t_5$ such that $g_1(t) < 0$ for $t \in [1, t_6)$

and $g_1(t) > 0$ for $t \in (t_6, +\infty)$, hence we know that $g(t)$ is strictly decreasing in $[1, t_6]$ and strictly increasing in $[t_6, +\infty)$.

Now from (2.1), (2.2)and the the monotonicity of $g(t)$ in $[1, t_6]$ and in $[t_6, +\infty)$ imply that there exist $\lambda \in (1, +\infty)$, such that $g(t) < 0$ for $t \in [1, \lambda)$ and $g(t) > 0$ for $t \in (\lambda, +\infty)$.

Theorem 2.2 *If r_1 is the solution of equation $\frac{1}{r-1}\ln r = \ln\pi$, then the double inequality*

$$L_{r_1}(a, b) < P(a, b) < L_1(a, b) = I(a, b)$$

holds for all $a, b > 0$, and the given parameter r_1 and 1 in each inequality are best possible.

Proof. Firstly, we prove

$$L_{r_1}(a, b) < P(a, b), \tag{2.27}$$

for all $a, b > 0$ with $a \neq b$.

Without loss of generality, we assume $a > b$. Let $t = \sqrt{a/b} > 1$ and $r = r_1$. Then

$$P(a, b)/L_r(a, b) = \frac{t^2 - 1}{4\arctan t - \pi}\left(\frac{r(1 - t^2)}{1 - t^{2r}}\right)^{\frac{1}{r-1}}. \tag{2.28}$$

Let

$$f(t) = \ln\left\{\frac{t^2 - 1}{4\arctan t - \pi}\left(\frac{r(1 - t^2)}{1 - t^{2r}}\right)^{\frac{1}{r-1}}\right\} = \ln\frac{t^2 - 1}{4\arctan t - \pi} + \frac{1}{r-1}\ln\frac{r(1 - t^2)}{1 - t^{2r}}. \tag{2.29}$$

Simple computations lead to

$$\lim_{t\to 1^+} f(t) = 0, \tag{2.30}$$

$$\lim_{t\to +\infty} f(t) = \frac{1}{r-1}\ln r - \ln\pi, \tag{2.31}$$

$$f'(t) = \frac{2(t + t^3)}{(t^4 - 1)(4\arctan t - \pi)(1 - t^{2r})}g(t) \tag{2.32}$$

where

$$g(t) = 4\arctan t - \pi + \frac{1}{1-r}(rt^{2r-2} - 1)(4\arctan t - \pi) - \frac{2(t^2 - 1)(1 - t^{2r})}{t + t^3}. \tag{2.33}$$

If $r = r_1$, from (2.32) and Lemma we know that there exists $\lambda \in (1, +\infty)$ such that $f'(t) > 0$ for $t \in [1, \lambda)$ and $f'(t) < 0$ for $t \in (\lambda, +\infty)$, hence $f(t)$ is strictly increasing in $[1, \lambda]$ and strictly decreasing in $[\lambda, +\infty)$. From (2.30) and (2.31) together with the monotonicity of $f(t)$ in $[1, \lambda]$ and in $[\lambda, +\infty)$, we clearly see that $f(t) > 0$ for $t \in (1, +\infty)$, and from (2.28) and (2.29) we know that $L_{r_1}(a, b) < p(a, b)$ holds for all $a, b > 0$ with $a \neq b$.

The other inequality of the theorem $P(a, b) < L_1(a, b) = I(a, b)$ has been proved in [9].

Secondly, we prove that the parameters r_1 and 1 cannot be improved in each inequality.

For any $\varepsilon > 0$ and $x > 1$, we have

$$\lim_{x\to+\infty} \frac{P(1,x)}{L_{r_1+\varepsilon}(1,x)} = \frac{1}{\pi}(r_1+\varepsilon)^{\frac{1}{r_1+\varepsilon-1}}. \tag{2.34}$$

But

$$\ln[\frac{1}{\pi}(r_1+\varepsilon)^{\frac{1}{r_1+\varepsilon-1}}] = -\ln\pi + \frac{1}{r_1+\varepsilon-1}\ln(r_1+\varepsilon), \tag{2.35}$$

by simple computations we can get

$$\frac{1}{r_1+\varepsilon-1}\ln(r_1+\varepsilon) < \frac{1}{r_1-1}\ln r_1, \tag{2.36}$$

where r_1 is the solution of equation $\frac{1}{r-1}\ln r = \ln\pi$, together with (2.34), (2.35) and (2.36) we have

$$\lim_{x\to+\infty} \frac{P(1,x)}{L_{r_1+\varepsilon}(1,x)} < 1. \tag{2.37}$$

Inequality (2.37) implies that for any $\varepsilon > 0$ there exists $X = X(\varepsilon) > 1$ such that $P(1, x) < L_{r_1}(1, x)$ for $x \in (X, +\infty)$. Hence the parameter r_1 cannot be improved, it is best possible.

Next we prove the parameter 1 in right-side inequality (the result in [9]) cannot be improved.

For any $0 < \varepsilon < 1$, let $0 < x < 1$, then we have

$$\begin{aligned} P(1+x,1) - L_{1-\varepsilon}(1+x,1) &= \frac{x}{4\arctan\sqrt{1+x}-\pi} - [\frac{(1+x)^{1-\varepsilon}-1}{(1-\varepsilon)x}]^{-\frac{1}{\varepsilon}} \\ &= \frac{h(x)}{4\arctan\sqrt{1+x}-\pi}, \end{aligned} \tag{2.38}$$

where $h(x) = x - [\frac{(1+x)^{1-\varepsilon}-1}{(1-\varepsilon)x}]^{-\frac{1}{\varepsilon}}(4\arctan\sqrt{1+x} - \pi)$.

Let $x \to 0$, making use of the Taylor expansion we get

$$h(x) = \frac{\varepsilon}{24}(x^3 + o(x^3)), \tag{2.39}$$

(2.38) and (2.39) imply that for any $0 < \varepsilon < 1$ there exists $0 < \delta = \delta(\varepsilon) < 1$ such that $P(1 + x, 1) > L_{1-\varepsilon}(1 + x, 1)$. Hence the parameter 1 cannot be improved in the right-side inequality.

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