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Energy decay of a thermoelastic system with nonlinear feedback

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Abstract

Using the multiplier method and the abstract setting from [7], we derive different stability results for an isotropic thermoelastic system with combined nonlinear internal and boundary feedbacks.

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1 Introduction

Let Ω be a non empty bounded open subset of $\mathbb{R}^n, n \geq 1$, with a boundary Γ of class C^2. We denote by $\nu = (\nu_1, \cdots, \nu_n)$ the unit outward normal vector along Γ. For a fixed $x_0 \in \mathbb{R}^n$ we define the function $m(x) = x - x_0$; $x \in \mathbb{R}^n$ and the following partition of the boundary Γ:

$$\Gamma_1 = \{x \in \Gamma : m(x) \cdot \nu(x) \leq 0\}, \tag{1}$$

$$\Gamma_2 = \{x \in \Gamma : m(x) \cdot \nu(x) > 0\}. \tag{2}$$

In this paper we consider the system of isotropic thermoelasticity:

$$\begin{cases} u'' - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div}\, u + \alpha\nabla\theta + f(u') &= 0 \text{ in } Q := \Omega\times\mathbb{R}^+, \\ \theta' - \Delta\theta + \beta \mathrm{div}\, u' &= 0 \text{ in } Q, \\ \theta &= 0 \text{ on } \Gamma\times\mathbb{R}^+, \\ u &= 0 \text{ on } \Gamma_1\times\mathbb{R}^+, \\ \mu\partial_\nu u + (\lambda+\mu)\mathrm{div}\, u\nu + am\cdot\nu u + m\cdot\nu g(u') &= 0 \text{ on } \Gamma_2\times\mathbb{R}^+, \\ u(\cdot,0) = u_0,\ u'(\cdot,0) = u_1\ \theta(\cdot,0) = \theta_0 \quad \text{in } \Omega, \end{cases} \tag{3}$$

where $u = u(x,t) = (u_1(x,t),\cdots,u_n(x,t))$ denotes the displacement vector field,
$\theta = \theta(x,t)$ the temperature.

The function a is non negative and belongs to $C^1(\Gamma_2)$; the functions $f(u) = (f_1(u),...,f_n(u))$ and $g(u) = (g_1(u),...,g_n(u))$ are continuous and satisfy

$$f(0) = g(0) = 0 \tag{4}$$
$$(f(x) - f(y))\cdot(x-y) \ge 0,\ \forall x,y\in\mathbb{R}^n, \tag{5}$$
$$(g(x) - g(y))\cdot(x-y) \ge 0,\ \forall x,y\in\mathbb{R}^n. \tag{6}$$

The coupling parameters α and β are supposed to be positive.
Theses assumptions guarantee that the system (3) is dissipative since its energy defined by

$$E(t) = \frac{1}{2}\int_\Omega \left\{|u'|^2 + \mu|\nabla u|^2 + (\lambda+\mu)|\mathrm{div}\, u|^2 + \frac{\alpha}{\beta}|\theta|^2\right\}dx + \frac{1}{2}\int_{\Gamma_2} am\cdot\nu|u|^2 d\Gamma \tag{7}$$

is nonincreasing

The stabilization of different variant of the system (3) has been studied in the literature, notably in [2, 4, 5, 6, 8, 10, 11] (see also [3] in the anisotropic case). In [4], Liu considered the case $f = 0$ in the linear feedback, i.e., $g(x) = x$ on $\Gamma_2 \neq \emptyset$ and give exponential decay of energy. Still in the case $f = 0$ Liu and Zuazua [5] have established exponential, polynomial and logarithmic decay for some nonlinearities g.
The aims of this work is to generalize these results to the case $f \neq 0$. For this purpose, in the linear case we establish integral inequalities as in [4] leading to the exponential decay and in the nonlinear case, we use the theorical results established in [7].

2 Main Results

In the remainder of our paper we suppose that

$$\Gamma_1 \neq \emptyset \text{ or } a(x) > 0, \forall x\in\Gamma_2. \tag{8}$$

Furthermore, in order to avoid regularity problems related to the change of boundary conditions we assume that

$$\overline{\Gamma_1} \cap \overline{\Gamma_2} = \emptyset. \tag{9}$$

We finally suppose that there exist positive constants $C > 0$ and $\sigma \geq 0, \sigma' \geq 0$ such that

$$|f(x)| \leq \begin{cases} C[1+|x|^{\frac{n+2}{n-2}}] & \text{if } n \geq 3, \\ C[1+|x|^{\sigma'}] & \text{if } n \leq 2, \end{cases} \tag{10}$$

$$|g(x)| \leq \begin{cases} C[1+|x|^{\frac{n}{n-2}}] & \text{if } n \geq 3, \\ C[1+|x|^{\sigma}] & \text{if } n \leq 2. \end{cases} \tag{11}$$

We define the following Hilbert spaces:

$$\begin{aligned} H^1_{\Gamma_1}(\Omega) &= \{u \in H^1(\Omega);\ u = 0 \quad \text{on } \Gamma_1\}, \\ D_{\Gamma_1} &= \{(u, v, \theta) \in (H^2(\Omega) \cap H^1_{\Gamma_1}(\Omega))^n \times (H^1_{\Gamma_1}(\Omega))^n \times (H^2(\Omega) \cap H^1_0(\Omega)) : \\ &\quad \mu\partial_\nu u + (\lambda+\mu)\mathrm{div}\, u\nu + am \cdot \nu u + m \cdot \nu v = 0 \text{ on } \Gamma_2\}, \\ W &= (H^1_{\Gamma_1}(\Omega))^n \times (L^2(\Omega))^n, \\ \mathcal{H} &= W \times L^2(\Omega). \end{aligned}$$

The space W is equipped with the natural norm:

$$\|(u,v)\|^2_W = \int_\Omega [|v|^2 + \mu|\nabla u|^2 + (\lambda+\mu)|\mathrm{div}\, u|^2]dx + \int_{\Gamma_2} am \cdot \nu |u|^2 d\Gamma.$$

In the sequel, we denote by $< \cdot, \cdot >$ the duality pairing between $(H^1_{\Gamma_1}(\Omega))^n$ and $[(H^1_{\Gamma_1}(\Omega))^n]'$ or between $H^1_0(\Omega)$ and $H^{-1}(\Omega)$, and by $(\cdot, \cdot)$ the inner product in $(H^1_{\Gamma_1}(\Omega))^n$.

Theorem 2.1 *Let Γ_1 and Γ_2 be given by (1)-(2) and satisfying (8) and (9). Assume that the functions f and g satisfy (4), (5), (6), (10) and (11). Then for initial data $(u_0, u_1, \theta_0) \in \mathcal{H}$, the system (3) has a unique (weak) solution (u, θ) satisfying*

$$(u, u', \theta) \in C([0, \infty); \mathcal{H}). \tag{12}$$

The main result of our paper is the next theorem

Theorem 2.2 *Let Γ_1 and Γ_2 given by (1) , (2) and satisfying (8) and (9). Assume that the functions f and g satisfy (10), (11) and the inequalities*

$$g(x) \cdot x \geq m_g|x|^2 \ \forall x \in \mathbb{R}^n \ |x| \geq 1, \tag{13}$$

$$|x|^2 + |g(x)|^2 \leq G(g(x) \cdot x) \ \forall x \in \mathbb{R}^n \ |x| \leq 1, \tag{14}$$

$$|x|^2 + |f(x)|^2 \leq G(f(x) \cdot x) \ \forall x \in \mathbb{R}^n \ |x| \leq 1, \tag{15}$$

where m_g is a positive constant and G a concave function defined on $\mathbb{R}_+$ such that G(0)=0. Then there exist positive constants τ, r_1, r_2 and a time $T_1 > 0$ (depending on $\tau, E(0), |\Gamma_2|, |\Omega|$) such that the energy of any solution of (3) satisfies

$$E(t) \leq r_2 G(\frac{\Psi^{-1}(r_1 t)}{r_1 \tau t}), \ \forall t \geq T_1, \tag{16}$$

where Ψ is given by

$$\Psi(t) = \int_t^1 \frac{1}{\Phi(s)} ds, \quad \textit{with } \Phi(s) = \tau R_1 G^{-1}(\frac{s}{r_2}) \textit{ and } R_1 = \min(|\Gamma_2|, |\Omega||). \tag{17}$$

Explicit decays are presented in Section 4.

Remark 2.1 The previous theorem still hold if $f = 0$ and g satisfies the previous hypotheses (case of boundary feedback only) or conversely if $\Gamma_2 = \emptyset$ and f satisfies the previous hypotheses (case of internal feedback).

3 Well-posedness of the problem

In this Section we prove Theorem 2.1 by reducing system (3) to a first order evolution equation. Let us define the operators
$A : (H^1_{\Gamma_1}(\Omega))^n \longmapsto [(H^1_{\Gamma_1}(\Omega))^n]'$ and $A_0 : H^1_0(\Omega) \longmapsto H^{-1}(\Omega)$ by

$$\begin{aligned} < Au, v > &= \int_\Omega [\mu \nabla u \cdot \nabla v + (\lambda + \mu) \mathrm{div}\, u \mathrm{div}\, v] dx, \forall u, v \in (H^1_{\Gamma_1}(\Omega))^n, \\ < A_0 u, v > &= \int_\Omega \nabla u \cdot \nabla v \, dx, \forall u, v \in H^1_0(\Omega). \end{aligned}$$

We further introduce the nonlinear operator B_0 from $(H^1_{\Gamma_1}(\Omega))^n$ to $[(H^1_{\Gamma_1}(\Omega))^n]'$ by

$$< B_0 u, v >= \int_{\Gamma_2} m \cdot \nu g(u) \cdot v d\Gamma + \int_\Omega f(u) \cdot v dx, \forall u, v \in (H^1_{\Gamma_1}(\Omega))^n.$$

Lemma 3.1 *If the functions fonctions f et g satisfy (10) and (11), then the operator B_0 is well defined.*

The proof of this lemma is similar to the one of lemma 3.1 of [5] (see also section 6 of [7]).

To obtain the abstract formulation of (3), we multiply the first identity of the system (3) by $v \in (H^1_{\Gamma_1}(\Omega))^n$ and we integrate by parts on Ω, this yields

$$\begin{aligned}
0 &= \int_\Omega [u'' - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div}\, u + \alpha\nabla\theta + f(u')]\cdot v\, dx \\
&= \int_\Omega u''\cdot v dx - \mu\int_\Gamma \frac{\partial u}{\partial\nu}\cdot v d\Gamma - (\lambda+\mu)\int_\Gamma v\cdot\nu \mathrm{div}\, u d\Gamma \\
&+ \int_\Omega [(\mu\nabla u\nabla v + (\lambda+\mu)\mathrm{div}\, u \mathrm{div}\, v] dx + \int_\Omega (\alpha\nabla\theta\cdot v) dx + \int_\Omega f(u').v dx \\
&= \int_\Omega u''\cdot v dx + \int_\Gamma a.m\cdot\nu u.v d\Gamma + \int_\Gamma m\cdot\nu g(u').v d\Gamma \\
&+ \int_\Omega [(\mu\nabla u\nabla v + (\lambda+\mu)\mathrm{div}\, u \mathrm{div}\, v] dx + \int_\Omega \alpha\nabla\theta\cdot v dx + \int_\Omega f(u')\cdot v dx \\
&= < u'', v > + < Au, v > + < B_0 u', v > + < \alpha\nabla\theta, v > .
\end{aligned}$$

This leads to the identity

$$u'' + Au + B_0 u' + \alpha\nabla\theta = 0.$$

In a similar manner, if we multiply the second identity of system (3) by $v \in (H^1_{\Gamma_1}(\Omega))^n$ and if we integrate by parts on Ω, we obtain

$$\theta' + A_0\theta + \beta \mathrm{div}\,(u') = 0.$$

Setting

$$\Phi = (u, u', \theta)$$

and

$$\mathcal{A}\Phi = (-u', Au + B_0 u' + \alpha\nabla\theta, A_0\theta + \beta\mathrm{div}\,(u')), \tag{18}$$

the system (3) reduce to

$$\left\{ \begin{array}{l} \Phi' + \mathcal{A}\Phi = 0, \\ \Phi(0) = (u_0, u_1, \theta_0). \end{array} \right. \tag{19}$$

Lemma 3.2 *Under the hypothese (4), (5), (6), (8), (10) and (11), the operator $\mathcal{A}$ defined on $\mathcal{H}$ by (18) with domain*

$$D(\mathcal{A}) = \{(u, v, \theta) \in \mathcal{H} : v \in (H^1_{\Gamma_1})^n, Au + B_0 v \in (L^2(\Omega))^n, \theta \in H^2(\Omega)\cap H^1_0(\Omega)\}$$

is maximal monotone. Morever, $D(\mathcal{A})$ is dense in $\mathcal{H}$.

The proof of this lemma is similar to the one of lemma 3.2 of [5]. The theory of nonlinear semi-groups (see [12] for example) leads to Theorem 2.1. Thus the energy of the solution of (3) is given by

$$E(t) = E(u, \theta, t) = \frac{1}{2}\|(u(t), u'(t), \theta(t))\|^2_{\mathcal{H}}.$$

4 Proof of Theorem 2.2

Deriving (7) in time and integrating by parts in space, we readily see that

$$E'(t) = -\frac{\alpha}{\beta}\int_\Omega |\nabla\theta(x,t)|^2\,dx - \int_{\Gamma_2} m\cdot\nu g(u'(t))\cdot u'(t) - \int_\Omega f(u'(t))\cdot u'(t)dx,$$

and consequently

$$\begin{aligned} E(T)-E(S) &= -\frac{\alpha}{\beta}\int_S^T\int_\Omega |\nabla\theta|^2 dxdt \qquad (20)\\ &- \int_S^T\int_{\Gamma_2} m\cdot\nu g(u'(t))\cdot u'(t)dxdt \\ &- \int_S^T\int_\Omega f(u'(t))\cdot u'(t)d\Gamma dt, \forall 0\le S\le T<\infty. \end{aligned}$$

The hypotheses (4), (5) and (6) lead to the decay of the energy.

Under additional hypotheses on f and g, we will now obtain different types of decay. For that purpose introduce the constant

$$\begin{aligned} R_0 &= \max_{x\in\overline{\Omega}}(\sum_{k=1}^n (x_k-x_{0k})^2)^{1/2},\\ R_1 &= \min(|\Gamma_2|;|\Omega|),\\ K(a) &= \max_{x\in\Gamma_2}|\frac{2R_0^2 a(x)}{\mu}+(2-n)|. \end{aligned}$$

Further let γ and λ_0 be the smallest positive constantes such that for all $u\in(H^1_{\Gamma_1}(\Omega))^n$

$$\int_{\Gamma_2}|u|^2 d\Gamma \le \gamma^2\left(\int_\Omega\{\mu|\nabla u|^2+(\lambda+\mu)|\operatorname{div} u|^2\}dx + \int_{\Gamma_2} am\cdot\nu|u|^2 d\Gamma\right), \quad (21)$$

and

$$\|u\|^2_{(L^2(\Omega))^n} \le \lambda_0^2\left(\int_\Omega\{\mu|\nabla u|^2+(\lambda+\mu)|\operatorname{div} u|^2\}dx + \int_{\Gamma_2} am\cdot\nu|u|^2 d\Gamma\right) \quad (22)$$

respectively.

To prove Theorem 2.2, we are reduced to check the sufficient conditions of Theorem 5.3 of [7]. In our case it remains to show that the linear system associated with (3) is exponentially stable. This system takes the form

$$\begin{cases} u'' - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div}\, u + \alpha\nabla\theta + u' & = \ 0 \ \text{in} \ Q, \\ \theta' - \Delta\theta + \beta \mathrm{div}\, u' & = \ 0 \ \text{in} \ Q, \\ \theta & = \ 0 \ \text{on} \ \Gamma, \\ u & = \ 0 \ \text{on} \ \Gamma_1, \\ \mu\partial_\nu u + (\lambda+\mu)\mathrm{div}\, u\nu + am\cdot\nu u + m\cdot\nu u' & = \ 0 \ \text{on} \ \Gamma_2, \\ u(.,0) = u_0, \ u'(.,0) = u_1 \ \theta(.,0) = \theta_0 \quad \text{in } \Omega. \end{cases} \tag{23}$$

We start with technical lemma

Lemma 4.1 *For all $\varepsilon_0 > 0$ and $T > 0$, there exists a positive constant $C(\varepsilon_0)$ such that for all (u, u', θ) solution of (23)*

$$\int_{\Sigma_{2T}} am\cdot\nu |u|^2 d\Sigma \le C(\varepsilon_0)E(0) + \varepsilon_0 \int_0^T E(t)dt.$$

Proof: We proceed as in [1]. For $t \ge 0$, consider the solution $z = z(t)$ of

$$\begin{cases} -\mu\Delta z - (\lambda+\mu)\nabla\mathrm{div}\, z = 0 \ \text{ in } \ \Omega, \\ z = u \ \text{ on } \Gamma. \end{cases} \tag{24}$$

this solution is characterized by $z = \omega + u$ where $\omega \in (H_0^1(\Omega))^n$ is the unique solution of

$$\int_\Omega (\mu\nabla\omega\nabla v + (\lambda+\mu)\mathrm{div}\,\omega \mathrm{div}\, v)dxdt = -\int_\Omega (\mu\nabla u\nabla v + (\lambda+\mu)\mathrm{div}\, u \mathrm{div}\, v)dxdt,$$

$$\forall v \in (H_0^1(\Omega))^n.$$

this identity means that

$$\int_\Omega (\mu\nabla u\nabla z dxdt + (\lambda+\mu)\mathrm{div}\, u\mathrm{div}\, z)dxdt = \int_\Omega (\mu|\nabla z|^2 dxdt + (\lambda+\mu)|\mathrm{div}\, z|^2)dxdt \ge 0. \tag{25}$$

Morever by Korn's inequality we have

$$\int_\Omega |z|^2 dx \le C_0 \int_\Gamma |u|^2 d\Sigma \tag{26}$$

and

$$\int_\Omega |z'|^2 dx \le C_0 \int_\Gamma |u'|^2 d\Sigma \le C_0' \int_\Gamma m\cdot\nu |u'|^2 d\Sigma \tag{27}$$

where C_0, C_0' are positive constants.
For $0 < T < \infty$, we set

$$Q_T = \Omega \times [0,T],$$
$$\Sigma_T = \Gamma \times [0,T]; \ \Sigma_{1T} = \Gamma_1 \times [0,T]; \quad \Sigma_{2T} = \Sigma_T \setminus \Sigma_{1T}.$$

Multiplying the first identity of (23) by z and integrating on Q_T we obtain

$$\int_{Q_T} z(u'' - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div}\, u + \alpha\nabla\theta + u')dxdt = 0.$$

Applying Green's formula and taking into account the boundary conditions in (23) and (24), we get

$$\int_{Q_T} (zu'' + \mu\nabla u\nabla z + (\lambda+\mu)\mathrm{div}\, u \mathrm{div}\, z + \alpha z\nabla\theta + u'z)dxdt +$$

$$+ \int_{\Sigma_{2T}} am\cdot\nu|u|^2 d\Sigma + \int_{\Sigma_{2T}} m\cdot\nu uu' d\Sigma = 0.$$

Integrating by parts in t and using (25), we obtain

$$\begin{aligned} \int_{\Sigma_{2T}} am\cdot\nu|u|^2 d\Sigma &\leq -\int_{\Sigma_{2T}} m\cdot\nu uu' d\Sigma + \int_{Q_T} z'u'dxdt \\ &- \alpha\int_{Q_T} z\nabla\theta dxdt - \int_{Q_T} u'zdxdt - \int_{\Omega} zu'|_0^T. \end{aligned}$$

Fix an arbitrary $\varepsilon_0 > 0$. Using several times (20)(with $f(x) = g(x) = x$), (26), (27) and Young's inequality, we can estimate the different integrals of the right-hand side of the above inequality as follows:

$$\begin{aligned} -\int_{\Sigma_{2T}} m\cdot\nu uu' d\Sigma &\leq \varepsilon_0\int_{\Sigma_{2T}} m\cdot\nu|u|^2 d\Sigma + \frac{1}{4\varepsilon_0}\int_{\Sigma_{2T}} m\cdot\nu|u'|^2 d\Sigma \\ &\leq 2\varepsilon_0 R_0\gamma^2\int_0^T E(t)dt + \frac{1}{4\varepsilon_0}E(0), \end{aligned}$$

$$\begin{aligned} -\int_{Q_T} z'u'dxdt &\leq \varepsilon_0\int_{Q_T} |u'|^2 dxdt + \frac{1}{4\varepsilon_0}\int_{Q_T} |z'|^2 dxdt \\ &\leq \varepsilon_0\int_0^T E(t)dt + \frac{C_0'}{4\varepsilon_0}E(0), \end{aligned}$$

$$\begin{aligned} -\int_{Q_T} \alpha\nabla\theta.zdxdt &\leq \frac{\alpha^2}{4\varepsilon_0}\int_{Q_T} |\nabla\theta|^2 dxdt + \varepsilon_0\int_{Q_T} |z|^2 d\Sigma \\ &\leq \frac{\alpha\beta}{4\varepsilon_0}E(0) + 2\varepsilon_0 C_0\gamma^2\int_0^T E(t)dt, \end{aligned}$$

$$\begin{aligned}\int_{Q_T} u' z dx dt &\leq \varepsilon_0 \int_{Q_T} |z|^2 dx dt + \frac{1}{4\varepsilon_0} \int_{Q_T} |u'|^2 d\Sigma \\ &\leq \varepsilon_0 C_0 \gamma^2 \int_0^T E(t) dt + \frac{1}{4\varepsilon_0} E(0),\end{aligned}$$

$$\int_\Omega z u'|_0^T \leq 4(1 + C_0\gamma^2) E(0).$$

Using these different estimates, we arrive at the requested estimate ■

Proof of Theorem 2.2: Let us introduce the following constant

$$\begin{aligned}c_1 &= \alpha\beta(n-1)^2 + 4\alpha\beta R_0^2, \\ N &= \lambda_0^2 + \frac{2}{\mu} + \frac{\gamma^2 \lambda_0^2 R_0^2 (n-1)^2}{c_1} + 4\frac{R_0^2}{\mu c_1}.\end{aligned}$$

Fix $\varepsilon > 0$ such that

$$0 < \varepsilon < \frac{2}{1+N}$$

and define the constant

$$\begin{aligned}k_1 &= 1 + \frac{2R_0^2}{\mu} + \frac{c_1}{4\varepsilon}, \\ k_2 &= \frac{\alpha\beta(n-1)^2}{4\varepsilon} + \frac{\alpha\beta R_0^2}{\varepsilon} + \lambda_0^2.\end{aligned}$$

Multiplying the first identity of (23) by $M_i = 2m_k \frac{\partial u_i}{\partial x_k} + (n-1)u_i$ and integrating by parts on Q_T (the convention of repeated indices is adopted), we obtain

$$\begin{aligned}&\int_{Q_T} u_i'' M_i dx dt \\ =\ & (u_i'(t), 2m_k \frac{\partial u_i}{\partial x_k})|_0^T - \int_{\Sigma_T} m_k \nu_k |u_i'|^2 d\Sigma + n \int_{Q_T} |u_i'|^2 dx dt \\ +\ & (n-1)(u_i', u_i)|_0^T - (n-1) \int_{Q_T} |u_i'|^2 dx dt \\ =\ & 2(u_i'(t), m_k \frac{\partial u_i}{\partial x_k} + \frac{n-1}{2} u_i)|_0^T - \int_{\Sigma_{2T}} m_k \nu_k |u_i'|^2 d\Sigma + \int_{Q_T} |u_i'|^2 dx dt.\end{aligned}$$

$$\begin{aligned}
&\int_{Q_T} \Delta u_i M_i dxdt \\
= &\ 2\int_{\Sigma_{2T}} \frac{\partial u_i}{\partial \nu} m_k \frac{\partial u_i}{\partial x_k} d\Sigma - \int_{\Sigma_T} m_k \nu_k |\nabla u_i|^2 d\Sigma + (n-2)\int_{Q_T} |\nabla u_i|^2 dxdt \\
+ &\ (n-1)\int_{\Sigma_T} \frac{\partial u_i}{\partial \nu} u_i - (n-1)\int_{Q_T} |\nabla u|^2 dxdt \\
= &\ 2\int_{\Sigma_{2T}} \frac{\partial u_i}{\partial \nu} m_k \frac{\partial u_i}{\partial x_k} d\Sigma - \int_{\Sigma_T} m_k \nu_k |\nabla u_i|^2 d\Sigma + (n-1)\int_{\Sigma_T} \frac{\partial u_i}{\partial \nu} u_i \\
- &\ \int_{Q_T} |\nabla u|^2 dxdt.
\end{aligned}$$

$$\begin{aligned}
\int_{Q_T} \frac{\partial}{\partial x_i}(\operatorname{div} u) M_i dxdt &= 2\int_{\Sigma_T} \operatorname{div} u m_k \frac{\partial u_i}{\partial x_k} \nu_i d\Sigma \\
&- \int_{\Sigma_T} m_k \nu_k |\operatorname{div} u|^2 d\Sigma + (n-2)\int_{Q_T} |\operatorname{div} u|^2 dxdt \\
&+ (n-1)\int_{\Sigma_T} \operatorname{div} u u_i \nu_i - (n-1)\int_{Q_T} |\operatorname{div} u|^2 \\
&= 2\int_{\Sigma_T} \operatorname{div} u m_k \frac{\partial u_i}{\partial x_k} \nu_i d\Sigma - \int_{\Sigma_T} m_k \nu_k |\operatorname{div} u|^2 d\Sigma. \\
&+ (n-1)\int_{\Sigma_T} \operatorname{div} u u_i \nu_i - \int_{Q_T} |\operatorname{div} u|^2 dxdt.
\end{aligned}$$

Using these different identities, we obtain

$$\begin{aligned}
&2\int_0^T E(t)dt = \int_{\Sigma_T} [|u_i'|^2 - \mu|\nabla u_i|^2 - (\lambda+\mu)|\operatorname{div} u|^2] m\cdot\nu d\Sigma \\
+ &\ 2\int_{\Sigma_T} [\mu \frac{\partial u_i}{\partial \nu} + (\lambda+\mu)\operatorname{div} u \nu_i] m_k \frac{\partial u_i}{\partial x_k} \\
+ &\ (n-1)\int_{\Sigma_T} [\mu \frac{\partial u_i}{\partial \nu} + (\lambda+\mu)\operatorname{div} u \nu_i] u_i d\Sigma \\
- &\ 2(u'_i, m_k \frac{\partial u_i}{\partial x_k} + \frac{n-1}{2} u_i)_0^T - 2\alpha \int_{Q_T} \frac{\partial \theta}{\partial x_i}(m_k \frac{\partial u_i}{\partial x_k} + \frac{n-1}{2} u_i) dxdt \\
+ &\ \frac{\alpha}{\beta}\int_{Q_T} \theta^2 dxdt + \int_{\Sigma_T} a m\cdot\nu |u|^2 - 2\int_{Q_T} u_i' m_k \frac{\partial u_i}{\partial x_k} - (n-1)\int_{Q_T} u_i u_i' dxdt
\end{aligned}$$

Taking into account the boundary conditions (23)(implying in particular

$\dfrac{\partial u_i}{\partial x_k} = \dfrac{\partial u_i}{\partial \nu}\nu_k$ sur Σ_{1T}), we arrive at

$$2\int_0^T E(t)dt = \sum_{i=1}^{7} I_i,$$

where we have set

$$\begin{aligned}
I_1 &:= \int_{\Sigma_{1T}} m\cdot\nu[\mu|\frac{\partial u_i}{\partial \nu}|^2 + (\lambda+\mu)|\operatorname{div} u|^2]d\Sigma,\\
I_2 &:= \int_{\Sigma_{2T}} m\cdot\nu[|u_i'|^2 - \mu|\nabla u_i|^2 - (\lambda+\mu)|\operatorname{div} u|^2]d\Sigma,\\
I_3 &:= -2\int_{\Sigma_{2T}} m\cdot\nu[au_i + u_i']m_k\frac{\partial u_i}{\partial x_k}d\Sigma,\\
I_4 &:= -(n-1)\int_{\Sigma_{2T}} m\cdot\nu[au_i+u_i']u_i d\Sigma + \int_{\Sigma_{2T}} am\cdot\nu|u_i|^2 d\Sigma,\\
I_5 &:= -2({u'}_i, m_k\frac{\partial u_i}{\partial x_k} + \frac{n-1}{2}u_i)_0^T - (n-1)\int_{Q_T} u_i u_i' dxdt,\\
I_6 &:= -2\alpha\int_{Q_T}\nabla\theta(m_k\frac{\partial u_i}{\partial x_k} + \frac{n-1}{2}u_i)dxdt + \frac{\alpha}{\beta}\int_{Q_T}|\theta|^2 dxdt,\\
I_7 &:= -2\int_{Q_T} u_i' m_k\frac{\partial u_i}{\partial x_k}.
\end{aligned}$$

It the remains to estimate each term I_i:
$I_1 \le 0$ since $m\cdot\nu \le 0$ on Σ_1 and also

$$I_2 \le \int_{\Sigma_{2T}} m\cdot\nu(|u'|^2 - \mu|\nabla u|^2)d\Sigma.$$

Young's inequality and definition of R_0 imply

$$\begin{aligned}
I_3 &\le 2\frac{R_0^2}{\mu}\int_{\Sigma_{2T}} m\cdot\nu a^2 u_i^2 + \frac{\mu}{2}\int_{\Sigma_{2T}} m\cdot\nu|\nabla u_i|^2\\
&+ \frac{R_0^2}{\mu}\int_{\Sigma_{2T}} m\cdot\nu|u_i'|^2 + \frac{\mu}{2}\int_{\Sigma_{2T}} m\cdot\nu|\nabla u_i|^2.
\end{aligned}$$

Thus we have

$$I_3 \le 2\frac{R_0^2}{\mu}\int_{\Sigma_{2T}} m\cdot\nu a^2 u_i^2 d\Sigma + \frac{2R_0^2}{\mu}\int_{\Sigma_{2T}} m\cdot\nu|u_i'|^2 d\Sigma + \mu\int_{\Sigma_{2T}} m\cdot\nu|\nabla u_i|^2 d\Sigma.$$

Similarly

$$\begin{aligned} I_4 &\leq \frac{c_1}{4\varepsilon}\int_{\Sigma_{2T}} m\cdot\nu|u_i'|^2 d\Sigma + \frac{(n-1)^2}{c_1}\varepsilon\int_{\Sigma_{2T}} m\cdot\nu|u_i|^2 d\Sigma \\ &+ (2-n)\int_{\Sigma_{2T}} am\cdot\nu|u_i|^2 d\Sigma \\ &\leq \frac{c_1}{4\varepsilon}\int_{\Sigma_{2T}} m\cdot\nu|u_i'|^2 d\Sigma + (2-n)\int_{\Sigma_{2T}} am\cdot\nu|u_i|^2 d\Sigma \\ &+ \frac{(n-1)^2\varepsilon\gamma^2\lambda_0^2R_0^2}{c_1}\int_0^T E(t)dt. \end{aligned}$$

The inequalities

$$|2\int_\Omega u_i' m\cdot\nabla u_i dx| \leq \frac{R_0^2}{\mu^{\frac{1}{2}}}[\|u_i'(t)\|^2 + \mu\|\nabla u_i(t)\|^2] \leq \frac{2R_0^2}{\mu^{\frac{1}{2}}}E(t),$$

$$|(n-1)\int_\Omega u_i' u_i dx| \leq \frac{n-1}{2}\lambda_0[\|u_i'(t)\|^2 + \mu\| \ u(t)\|^2_{(H^1_{\Gamma_1})^n}] \leq (n-1)\lambda_0 E(t),$$

$$|\frac{(n-1)}{2}\int_\Omega u_i^2| \leq (n-1)\lambda_0^2 E(t),$$

and the definition of k_1 lead to

$$I_5 \leq k_1 E(0).$$

By Young's inequality, the definition of R_0 and of λ_0, and taking into account (20) (with $f(x) = g(x) = x$), we have successively

$$\begin{aligned} I_6 &\leq \frac{\alpha^2(n-1)^2}{4\varepsilon}\int_{Q_T}|\nabla\theta|^2 + \varepsilon\int_{Q_T}|u|^2 + \frac{\alpha^2R_0^2}{\varepsilon}\int_{Q_T}|\nabla\theta|^2 \\ &+ \frac{\alpha}{\beta}\int_{Q_T}|\theta|^2 dxdt + \varepsilon\int_{Q_T}|\nabla u|^2, \\ &\leq [\frac{\alpha\beta(n-1)^2}{4\varepsilon} + \frac{\alpha\beta R_0^2}{\varepsilon} + \lambda_0^2]\int_{Q_T}\frac{\alpha}{\beta}|\nabla\theta|^2 + [\varepsilon\lambda_0^2 + \frac{2\varepsilon}{\mu}]\int_0^T E(t)dt \\ &\leq k_2E(0) + [\varepsilon\lambda_0^2 + \frac{2\varepsilon}{\mu}]\int_0^T E(t)dt. \\ I_7 &= -2\int_{Q_T} u_i' m_k\frac{\partial u_i}{\partial x_k} \leq \frac{c_1}{4\varepsilon}\int_{Q_T}|u'|^2 + \frac{4R_0^2\varepsilon}{\mu c_1}\int_{Q_T}\mu|\nabla u|^2. \end{aligned}$$

All together we have

$$\begin{aligned} 2\int_0^T E(t)dt &\leq I_9 + k_1(\int_{\Sigma_{2T}} m\cdot\nu|u'|^2 d\Sigma + \int_{Q_T}|u'|^2 dxdt) \\ &+ k_2E(0) + \varepsilon N\int_0^T E(t)dt, \end{aligned}$$

where we have set

$$I_9 = \int_{\Sigma_{2T}} [\frac{2R_0^2 a^2}{\mu} + (2-n)a] m \cdot \nu |u|^2 d\Sigma.$$

The definition of $K(a)$ leads to

$$I_9 \leq K(a) \int_{\Sigma_{2T}} am \cdot \nu |u|^2 d\Sigma.$$

Applying Lemma 4.1 with $\varepsilon_0 = \frac{\varepsilon}{K(a)}$, there exist a positive constant $C(\varepsilon)$ such that

$$I_9 \leq C(\varepsilon)E(0) + \varepsilon \int_0^T E(t)dt.$$

Finally, setting

$$C_1 = \frac{k_2 + C(\varepsilon)}{2 - \varepsilon(1+N)},\ C_2 = \frac{k_1}{2 - \varepsilon(1+N)},$$

we conclude that

$$\int_0^T E(t) \leq C_1 E(0) + C_2 (\int_{\Sigma_{2T}} m \cdot \nu |u'|^2 d\Sigma dt + \int_{Q_T} |u'|^2 dx dt). \qquad (28)$$

This estimate remains valid for weak solutions by a density argument.
We then conclude by Theorem 5.3 of [7].
To complete the proof, we now define as in formalism of [7] the operators A_1 and $\mathcal{I}_U$ associed to (23) as follows: A_1 is defined on

$$\mathcal{V} := (H^1_{\Gamma_1})^n \times (H^1_{\Gamma_1})^n \times H^1_0(\Omega)$$

by

$$A_1 \Phi = (-v, Au + \alpha \nabla \theta, \beta \mathrm{div}\, v).$$

Taking into account the feedbacks in (23) and identity (11) of [7], we set

$$U = (L^2(\Gamma_2))^n \times (L^2(\Omega))^n \times L^2(\Omega)$$

and we define the application

$$I_U : \mathcal{V} \longmapsto U$$

$$(u, v, \theta) \longmapsto (v_{|\Gamma_2}, v, \theta)$$

and $\mathcal{I}_U$ from $\mathcal{V}$ to$\mathcal{V}'$ by

$$< \mathcal{I}_U(u,v,\theta), (u^*, v^*, \theta^*) >= (I_U(u,v,\theta), I_U(u^*,v^*,\theta^*))$$
$$= \int_{\Gamma_2} m \cdot \nu v.v^* d\Gamma + \int_\Omega v.v^* dx + \frac{\alpha}{\beta} \int_\Omega \nabla\theta \cdot \nabla\theta^* dx.$$

5 examples

1. If we assume that $f = 0$ and g satisfy (5), (11), (13), (14) as well as

$$x \cdot g(x) \geq c_0|x|^{p+1},\ \forall |x| \leq 1, \tag{29}$$

$$|g(x)| \leq C_0|x|^{\alpha},\ \forall |x| \leq 1, \tag{30}$$

where c_0, C_0 are positive constants, $\alpha \in (0,1]$ and $p \geq \alpha$ then making the choice

$$G(x) = x^{\frac{2}{q+1}} \text{ and } q = \frac{p+1}{\alpha} - 1$$

we obtain decays similar to the ones from Theorem 2.3 to [5]. Indeed, if $p = \alpha = 1$, then $\Psi^{-1}(t) = e^{-t}$ and we conclude and exponential decay. Conservely, if $p + 1 \geq 2\alpha$, then $\Psi^{-1}(t) = t^{\frac{2}{1-q}}$ and we obtain a decay of order $t^{-\frac{2\alpha}{p+1-2\alpha}}$.
2. In a similar manner as in examples 5.6 et 5.8 of [7] good choices of f and g allow to obtain logarithmic, double logarithmic decay etc..

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