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A STOCHASTIC MODEL OF INTERCONNECTED TWO QUEUES WITH A BALANCING POLICY

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Abstract

A Stochastic model of two queues (queue I and queue II) with an interconnection between them is considered. Customers arrive at a constant rate λ and join the shortest of the two queues if their sizes (including the customer undergoing service) are different and join the Queue I if otherwise. A waiting customer in a queue Q_i switches over to the other queue Q_j $i \neq j$ if the size of Q_j = size of $Q_i - 2$. The service times under the two servers are exponentially distributed with means $\frac{1}{\mu_1}$ and $\frac{1}{\mu_2}$ respectively. The steady-state behavior of the model is analyzed.

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1 Introduction

Several researchers have studied the Problem of Parallel Queues with customers jockeying. Koenigsburg(1996) [4] studied a queuing model with two servers where each server has its own queue, an incoming customer joins the shortest queue if the queues are of unequal length otherwise joins one specified queues exceeds 1, the last customer in the longer queue jockeys (switch over) to the shorter queue. Zhao and Grassmann (1990) [8] also studied the shortest queue model with n servers ($n \geq 2$) with jockeying and obtained explicit solutions of the equilibrium probabilities, the expected numbers of customers and the expected waiting time of a customer recently, Wang et al (1995) [7] analyzed a modification of the two server shortest queue model with jockeying where one server is primary and the other is secondary in the sense that the primary server is always available for service and if the length of the primary queue exceeds a certain number N the secondary server is activated for service at the epoch of which $\frac{(N+1)}{2}$ in case $N+1$ is even ($\frac{N}{2}$ in case $N+1$ is odd) customer of the primary queue go to service under the secondary server and the secondary server is deactivated at the epoch at which the system becomes empty However in several practical situation (such as banking and communication lines) it is observed that the secondary server is deactivated at the epoch at which the total number of customers in system becomes N. Accordingly, in this paper, we extend the results of Wang et al (1995) by analyzing the steady state behavior of a two-server parallel queue with customers jockeying to the shortest queue and also to the primary queue when the secondary server leaves the service at the epoch at which the system level becomes a threshold level N. The organization of the paper is as follows. In section 2, we describe the queuing model and notation used in this paper. We derive the steady-state equation and the equilibrium queue length distribution in section 4.

2 The shortest queue model

There are two servers in the system, one server called principal server and another server called secondary server. The primary server is always available whenever customer demand occurs. The secondary server opens his service when the total number of customers in the system exceeds a threshold value N and $N+1$ customer at that epoch are shared among the two servers. If $N+1$ is even, then $\frac{(N+1)}{2}$ customers (including the customer who is getting service under the principal server at that instant) are allotted to the principal queue and the remaining $\frac{(N+1)}{2}$ customers are allotted to the secondary server. If $N+1$ is odd, then $\left(\frac{N}{2}\right)+1$ customers (including the customer who is getting service under the principal server at that instant) area allotted to the principal queue and the remaining $\frac{N}{2}$ customers are allotted to the secondary server. Without loss of generality, we assume that N is odd. The secondary server is deactivated when the total number in the system comes down to N and again activated only when the number of customers in the system exceeds N. Customers arrive according to a Poisson process with rate λ and queue up before the principal server if the system size is not greater than N. when the system size is greater than n, an arriving customer joins the shortest queue if the two queues are of unequal length, if the queues are of equal length, the customer joins the primary queue. If, at the epoch of the departure of a customer, one queue length exceeds the other by more than 1. A customer from the longest queue jockeys to the shortest queue. The service times of the two servers are independent and exponentially distributed with rate μ_1 for primary server and μ_2 for secondary server.

3 Notation

In this section, we provide the notations and symbols used in this paper.
$\mu = \mu_1 + \mu_2$, $\rho_1 = \frac{\lambda}{\mu_1} = \frac{\lambda}{\mu}$, $\rho_2 = \frac{\lambda}{\mu_2}$, $\rho = \frac{\lambda}{\mu}$
EL_1 = mean queue length of the principal server
EL_2 = mean queue length of the secondary server and $EL = EL_1 + EL_2$
Q_n =the steadystate probability that there are n customers in the system and only the principal server is active .
(n,m) = the state of the system when both servers are busy. Here n denotes the number of customers in the principal queue and m denotes the number of customers in the secondary queue.
P_{nm} : the steady-state probability of the state (n.m)
Here $|n - m| \leq 1$, $n, m \geq K$, $n + m > N$, $K = \frac{(N+1)}{2}$.

4 The Steady-State Queue Length Distribution

Using the principle of flow balance, we have the following difference equations:
State 0:

$$\lambda q_0 = \mu_1 q_1; \tag{1}$$

State n:

$$(\lambda + \mu_1) q_n = \lambda q_{n-1} + \mu_1 q_{n+1}, \quad 1 \leq n \leq N - 1; \tag{2}$$

State N:

$$(\lambda + \mu_1) qN = \lambda_{qN-1} + \mu P_{k,k}; \tag{3}$$

State (K,K):

$$(\lambda + \mu) P_{k,k} = \lambda_{qN} + \mu \left\{ P_{k+1,k} + P_{k,k+1} \right\}; \tag{4}$$

State $(K + r, K + r + 1)$:

$$(\lambda + \mu) P_{K+r,K+r+1} = \mu_1 P_{K+r+1,K+r+1}, \quad r = 0, 1, 2, \cdots; \tag{5}$$

State $(K + r + 1, K + r)$;

$$(\lambda + \mu) P_{K+r+1,K+r} = \lambda P_{K+r,K+r} + \mu_2 P_{K+r+1,K+r+1}, r = 0, 1, 2, \cdots; \tag{6}$$

State $(K + r + 1, K + r + 1)$;

$$(\lambda+\mu) P_{K+r+1,K+r+1} = \lambda \left\{ P_{K+r,K+r+1} + P_{K+r+1,K+r} \right\} + \mu \left\{ P_{k+r+2,K+r+1} + P_{K+r+1,K+r+2} \right\},$$

$$r = 0, 1, 2, \cdots; \tag{7}$$

Defining the partial generating functions

$$F_1(z) = \sum_{n=0}^{N} q_n z^n$$

$$F_2(z) = \sum_{r=0}^{\infty} P_{K+r,K+r} Z^{K+r},$$

$$F_3(z) = \sum_{r=0}^{\infty} P_{K+r,K+r+1} Z^{K+r},$$

$$F_4(Z) = \sum_{r=0}^{\infty} P_{K+r+1,K+r} Z^{K+r},$$

and using the equation (1) to (7), we get

$$\left\{\lambda z^2 - (\lambda + \mu_1)z + \mu_1\right\} F_1(z) = \mu_1 q_0(1 - z) + \lambda_{qN} z^{N+2} - \mu P_{K,K} z^{N+1}. \quad (8)$$

$$(\lambda + \mu)F_2(z) = (\mu + \lambda z)\left\{F_3(z) + F_4(z)\right\} + \lambda_{qN} z^k, \quad (9)$$

$$(\lambda + \mu)F_3(z) = \frac{\mu_1}{z}\left\{F_2(z) - P_{K,K} z^K\right\}, \quad (10)$$

$$(\lambda + \mu)F_4(z) = \left(\lambda + \frac{\mu_2}{z}\right) F_2(z) - \frac{\mu_2}{z} P_{K,K} z^K, \quad (11)$$

From the equation (8), we get

$$F_1(z) = \frac{\mu_1 q_0(1 - z) + \lambda_{qN} z^{N+2} - \mu P_{K,K} z^{N+1}}{\lambda z^2 - (\lambda + \mu_1)z + \mu_1} \quad (12)$$

Since $F_1(z)$ is an analytic function and the denominator of (12) vanishes when $z = 1$, the numerator of (12) should vanish at $z = 1$ and so we have

$$P_{K,K} = \frac{\lambda}{\mu} qN. \quad (13)$$

Substituting (13) in (8), we get after simplification

$$F_1(z) = \frac{-\mu_1 q_0 + \lambda_{qN} z^{N+1}}{\lambda z - \mu_1} \quad (14)$$

Since the denominator of (14) vanishes when $z = \frac{\mu_1}{\lambda} = \frac{1}{\rho_1}$, the numerator of (14) should vanish at $z = \frac{1}{\rho_1}$ and so we have

$$qN = q_0 \rho_1^N \quad (15)$$

In the same way, we can obtain expressions for $F_2(z)$, $F_3(z)$ and $F_4(z)$.

5 Conclusion

The steady -state behavior of this model with $F_1(z)$ as an analytic function is obtained. The expression of the analytic function and the denominator vanishes when the system size is unity. Further the higher order expression can be obtained.

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