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Local Regularity for Very Weak Solutions to Elliptic Equations with Degenerate Coercivity

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Abstract

This paper deals with very weak solutions to elliptic equations

$$-\mathrm{div}(a(x,u)Du) = -\mathrm{div}F, \quad x \in \Omega,$$

with degenerate coercivity. A local regularity result is obtained under appropriate assumptions.

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1 Introduction and Statement of Result.

Let Ω be a bounded open subset of R^N, $N \geq 2$. Consider the elliptic equation

$$-\mathrm{div}(a(x,u)Du) = -\mathrm{div}F, \quad x \in \Omega, \tag{1.1}$$

where $a(x,s) : \Omega \times \mathrm{R} \to \mathrm{R}$ is a measurable Carathéodory function satisfying

$$\frac{\alpha}{(1+|s|)^\theta} \leq a(x,s) \leq \frac{\beta}{(1+|s|)^\theta}, \quad 0 < \theta < 1, \tag{1.2}$$

where $0 < \alpha \leq \beta < \infty$.

Definition 1.1 *A function $u \in W^{1,r}_{loc}(\Omega)$, $1 < r < 2$, is called a very weak solution to (1.1) if*

$$\int_\Omega a(x,u)DuD\varphi dx = \int_\Omega FD\varphi dx \tag{1.3}$$

holds true for any $\varphi \in W^{1,\frac{r}{r-1}}(\Omega)$ with compact support.

This paper considers local regularity property for very weak solutions to (1.1) with $a(x,s)$ satisfying (1.2). Local regularity theory is important among the regularity theories of elliptic PDEs. Some local regularity results can be found in the literature [1-6].

The main result of this paper is the following theorem.

Theorem 1.2 *Let $F \in L^m_{loc}(\Omega)$, $r < m < N$. There exists a constant $\varepsilon_0 = \varepsilon_0(\alpha,\beta,N)$ such that every very weak solution $u \in W^{1,r}_{loc}(\Omega)$ with $r \geq 2-\varepsilon_0$ is actually in $L^{m^*(1-\theta)}_{loc}(\Omega)$, where $m^* = \frac{Nm}{N-m}$.*

In order to prove the above theorem, we need two preliminary lemmas. The first lemma can be found in [5].

Lemma 1.3 *Let $f(\tau)$ be a non-negative bounded function defined for $0 \leq R_0 \leq t \leq R_1$. Suppose that for $R_0 \leq \tau < t \leq R_1$ we have*

$$f(\tau) \leq A(t-\tau)^{-\alpha} + B + \gamma f(t),$$

where A, B, α, γ are non-negative constants, and $\gamma < 1$. Then there exists a constant c, depending only on α and γ such that for every ρ, R, $R_0 \leq \rho < R \leq R_1$ we have

$$f(\rho) \leq c[A(R-\rho)^{-\alpha} + B].$$

The following lemma comes from [1].

Lemma 1.4 *Let $v \in W^{1,r}_{loc}(\Omega)$, $\phi_0 \in L^m_{loc}(\Omega)$, where $1 < r < N$ and m satisfies*

$$r < m < N.$$

Assume that for all $B_{R_1} \subset\subset \Omega$ the following integral estimate holds

$$\int_{A_{k,\rho}} |Dv|^r dx \leq c_1 \left[\int_{A_{k,R}} \phi_0 dx + (R-\rho)^{-\lambda} \int_{A_{k,R}} |v|^r dx \right],$$

for every $k \in N$ and $R_0 \leq \rho < R \leq R_1$, where $A_{k,\rho} = B_\rho \cap \{|v| > k\}$. Here $c_1 = c_1(N,r,m,R_0,R_1,|\Omega|)$, and λ is a positive constant. Then we have $v \in L^{m^*}_{loc}(\Omega)$.

2 Proof of Theorem 1.2.

In the following, the symbol $C(*,\cdots,*)$ will denote a constant depends only on the quantities $*,\cdots,*$, it value may vary from line to line. Define

$$v(x)=\frac{1}{1-\theta}\mathrm{sign}(u)((1+|u|)^{1-\theta}-1).$$

It is easy to see that

$$|Dv|=\frac{|Du|}{(1+|u|)^{\theta}}. \tag{2.1}$$

Let $0<R_0<R_1$ be such that $B_{R_1}\subset\subset\Omega$, and $R_0<\tau<t<R_1$ be arbitrarily fixed. Let $\eta\in C_0^\infty(B_t)$ be a cut-off function such that $0\le\eta\le1$, $\eta\equiv1$ on B_τ and $|D\eta|\le C(t-\tau)^{-1}$. For any $k\in\mathrm{N}$, we let $T_k(v)$ be the usual truncation of v at level k, that is,

$$T_k(v)=\min\{k,\max\{v,-k\}\}.$$

We introduce the Hodge decomposition of the vector field $|D(\eta(v-T_k(v)))|^{r-2}$ $D(\eta(v-T_k(v)))\in L^{r/(r-1)}(\Omega)$. Accordingly,

$$|D(\eta(v-T_k(v)))|^{r-2}D(\eta(v-T_k(v)))=D\varphi+h, \tag{2.2}$$

with $\varphi\in W_0^{1,r/(r-1)}(B_t)$ and h a divergence free vector field of class $L^{r/(r-1)}(\Omega,\mathrm{R}^n)$. The reader is referred to [7,8] for estimates concerning such decomposition. We have

$$\|D\varphi\|_{\frac{r}{r-1}}\le C(n)\|D(\eta(v-T_k(v)))\|_r^{r-1} \tag{2.3}$$

and

$$\|h\|_{\frac{r}{r-1}}\le C(n)|r-2|\|D(\eta(v-T_k(v)))\|_r^{r-1}. \tag{2.4}$$

Let

$$E=|D(\eta(v-T_k(v)))|^{r-2}D(\eta(v-T_k(v)))-|\eta D(v-T_k(v))|^{r-2}\eta D(v-T_k(v)).$$

By an elementary inequality (see [9])

$$||X|^{-\varepsilon}X-|Y|^{-\varepsilon}Y|\le\frac{2^{\varepsilon}(1+\varepsilon)}{1-\varepsilon}|X-Y|^{1-\varepsilon},\quad X,Y\in\mathrm{R},\quad 0<\varepsilon<1,$$

one has

$$|E|\le\frac{2^{2-r}(3-r)}{r-1}|(v-T_k(v))D\eta|^{r-1}. \tag{2.5}$$

Take φ in the Hodge decomposition (2.2) as a test function in (1.3) we arrive at

$$\begin{aligned}&\int_{A_{k,t}}a(x,u)Du|\eta D(v-T_k(v))|^{r-2}\eta D(v-T_k(v))dx\\ =\;&-\int_{A_{k,t}}a(x,u)DuEdx+\int_{A_{k,t}}a(x,u)Duhdx+\int_{A_{k,t}}FD\varphi dx\\ =\;&I_1+I_2+I_3.\end{aligned} \tag{2.6}$$

Using (1.2), and noticing (2.1), the left-hand side of the above inequality can be estimated as

$$\begin{aligned}&\int_{A_{k,t}} a(x,u)Du|\eta D(v-T_k(v))|^{r-2}\eta D(v-T_k(v))dx\\ \geq\ & \alpha\int_{A_{k,t}}\frac{Du}{(1+|u|)^{\theta}}|\eta Dv|^{r-2}\eta Dv dx\geq\alpha\int_{A_{k,\tau}}|Dv|^r dx.\end{aligned}\tag{2.7}$$

Using (1.2), (2.1), (2.5) and Young inequality, $|I_1|$ can be estimated as

$$\begin{aligned}|I_1| &= \left|-\int_{A_{k,t}} a(x,u)DuEdx\right|\leq\beta\int_{A_{k,t}}\frac{|Du|}{(1+|u|)^{\theta}}|E|dx\\ &= \beta\int_{A_{k,t}}|Dv||E|dx\leq\beta\|Dv\|_r\|E\|_{\frac{r}{r-1}}\\ &\leq \beta\frac{2^{2-r}(3-r)}{r-1}\|Dv\|_r\|(v-T_k(v))D\eta\|_r^{r-1}\\ &\leq \beta\frac{2^{2-r}(3-r)}{r-1}\left[\varepsilon\|Dv\|_r^r+C(\varepsilon)\|(v-T_k(v))D\eta\|_r^r\right]\\ &\leq \beta\frac{2^{2-r}(3-r)}{r-1}\left[\varepsilon\|Dv\|_r^r+\frac{C(\varepsilon)}{(t-\tau)^r}\|v\|_r^r\right],\end{aligned}\tag{2.8}$$

where $\|\cdot\|_r=\|\cdot\|_{r,A_{k,t}}$, and have used the fact $|v-T_k(v)|\leq|v|$.

Using (1.2) and (2.1) again and the estimate (2.4), $|I_2|$ can be estimated as

$$\begin{aligned}|I_2| &= \left|\int_{A_{k,t}} a(x,u)Duhdx\right|\leq\beta\int_{A_{k,t}}\frac{|Du|}{(1+|u|)^{\theta}}|h|dx\\ &= \beta\int_{A_{k,t}}|Dv||h|dx\leq\beta\|Dv\|_r\|h\|_{\frac{r}{r-1}}\\ &\leq C\beta|r-2|\|Dv\|_r\|D(\eta(v-T_k(v)))\|_r^{r-1}\end{aligned}$$

Since

$$\|D(\eta(v-T_k(v)))\|_r=\|\eta Dv+(v-T_k(v))D\eta\|_r\leq\|Dv\|_r+\frac{C}{t-\tau}\|v\|_r,\tag{2.9}$$

then Young inequality yields

$$|I_2|\leq C(\varepsilon)\beta|r-2|\|Dv\|_r^r+\varepsilon\|Dv\|_r^r+\frac{C}{(t-\tau)^r}\|v\|_r^r.\tag{2.10}$$

(2.3), (2.9) together with Young inequality yield

$$\begin{aligned}|I_3| &= \left|\int_{A_{k,t}} FD\varphi dx\right|\leq\|F\|_r\|D\varphi\|_r^{r-1}\\ &\leq C\|F\|_r\|D(\eta(v-T_k(v)))\|_r^{r-1}\\ &\leq C(\varepsilon)\|F\|_r^r+\varepsilon\|Dv\|_r^r+\frac{C}{(t-\tau)^r}\|v\|_r^r.\end{aligned}\tag{2.11}$$

Combining (2.6)-(2.8), (2.10)-(2.11) we arrive at

$$\alpha \int_{A_{k,\tau}} |Dv|^r dx \leq \left[\beta \frac{2^{2-r}(3-r)}{r-1}\varepsilon + C(\varepsilon)\beta|r-2| + \varepsilon\right] \int_{A_{k,t}} |Dv|^r dx$$
$$+C(\varepsilon) \int_{A_{k,t}} |F|^r dx + \frac{C(\varepsilon)}{(t-\tau)^r} \int_{A_{k,t}} |v|^r dx.$$

Take ε sufficiently small, and then r sufficiently close to 2 such that $\beta \frac{2^{2-r}(3-r)}{r-1}\varepsilon + C(\varepsilon)\beta|r-2| + \varepsilon < \alpha$, then Lemma 1.3 yields that for any $R_0 \leq \rho < R \leq R_1$,

$$\int_{A_{k,\rho}} |Dv|^r dx \leq C \int_{A_{k,R}} |F|^r dx + \frac{C}{(R-\rho)^r} \int_{A_{k,R}} |v|^r dx.$$

By Lemma 1.4 we have $v \in L^{m^*}_{loc}(\Omega)$. This is equivalent to $u \in L^{m^*(1-\theta)}_{loc}(\Omega)$. This ends the proof of Theorem 1.2.

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