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Weak Monotonicity for Weak Solutions to Elliptic Equations with Degenerate Coercivity

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Abstract

Weak monotonicity result is obtained for weak solutions to elliptic equations of the type

$$-\mathrm{div}(a(x,u)|Du|^{p-2}Du) = 0, \quad x \in \Omega, \quad p > 1,$$

where $a(x,s) : \Omega \times \mathrm{R} \to \mathrm{R}$ satisfies

$$\frac{\alpha}{(1+|s|)^\theta} \le a(x,s) \le \beta, \quad \alpha \le \beta < \infty, 0 < \theta < 1.$$

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1 Introduction and Statement of Result.

Let Ω be a bounded open subset of R^N, $N \ge 2$. Consider the elliptic equation

$$-\mathrm{div}(a(x,u)|Du|^{p-2}Du) = 0, \quad x \in \Omega, \ p > 1, \tag{1.1}$$

where $a(x,s) : \Omega \times \mathrm{R} \to \mathrm{R}$ is a measurable Carathéodory function satisfying

$$\frac{\alpha}{(1+|s|)^\theta} \le a(x,s) \le \beta, \quad 0 < \theta < 1, \tag{1.2}$$

where $0 < \alpha \le \beta < \infty$.

Definition 1.1 *A function $u \in W^{1,p}_{loc}(\Omega)$ is called a weak solution to (1.1) if*

$$\int_\Omega a(x,u)|Du|^{p-2}DuD\psi dx = 0 \tag{1.3}$$

holds true for any $\psi \in W^{1,p}(\Omega)$ with compact support.

Some regularity properties for weak solutions to equation (1.1) with condition (1.2) have been obtained in [1].

Definition 1.2 *A real valued function $u \in W^{1,1}_{loc}(\Omega)$ is said to be weakly monotone if for every ball $B \subset \Omega$ and all constants $m \le M$ such that*

$$\varphi = (u-M)^+ - (m-u)^+ \in W^{1,1}_0(B), \tag{1.4}$$

we have

$$m \le u(x) \le M \tag{1.5}$$

for almost every $x \in B$.

The concept of weakly monotone function was introduced by Manfredi in 1994, see [2]. For continuous functions (1.4) holds if and only if $m \le u(x) \le M$ on ∂B. Then (1.5) says we want the same condition in B, that is the maximum and minimum principles. For some results related to weakly monotone functions, see [3-4].

This paper deals with weak monotonicity property for weak solutions to (1.1) with $a(x,s)$ satisfying the degenerate coercivity condition (1.2). The main result of this paper is the following theorem.

Theorem 1.3 *Under the condition (1.2), any weak solution $u \in W^{1,p}_{loc}(\Omega)$ to (1.1) is weakly monotone.*

2 Proof of Theorem 1.3.

Let $u \in W^{1,p}_{loc}(\Omega)$ be a weak solution to (1.1), $B \subset \Omega$ and (1.4) holds true. Take $\psi = (u-M)^+ \in W^{1,p}_0(B)$ as a test function in (1.3), and notice that

$$D\psi = \begin{cases} Du, & u < M, \\ 0, & u \ge M \end{cases} \tag{2.1}$$

we arrive at

$$\int_{B\cap\{u<M\}} a(x,u)|Du|^{p-2}DuDudx = 0. \tag{2.2}$$

Note that condition (1.2) ensures the integrability of the integrand of (2.2). Condition (1.2) implies

$$\int_{B\cap\{u<M\}} \frac{\alpha}{(1+|u|)^{\theta}}|Du|^{p}dx = 0,$$

from which we derive

$$|Du| = 0, \text{ a.e. } B\cap\{u<M\}.$$

This result together with (2.1) implies $D\psi = 0$ a.e. B. Since $\psi \in W_0^{1,p}(B)$, then $\psi = 0$ a.e. B. That is, $u(x) \le M$ a.e. B.

The same reasoning applies to $\psi = (m-u)^{+}$ implies $m \le u(x)$ a.e. B, completing the proof of Theorem 1.3.

References

[1] D.Giachetti, M.M.Porzio, Elliptic equations with degenerate coercivity: gradient regularity, Acta Math. Sin. (Engl. Ser.), 2003, 19(2), 349-370.

[2] J.Manfredi, Weakly monotone functions, J. Geom. Anal., 1994, 3, 393-402.

[3] T.Iwaniec, P.Koskela, J.Onninen, Mappings of finite distortion: monotonicity and continuity, Invent Math., 2001, 144, 507-531.

[4] P.Koskela, J.Manfredi, E.Villamor, Regularity theory and traces of $\mathcal{A}$-harmonic functions, Trans. Amer. Math. Soc., 1996, 348(2), 755-766.

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