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The Killing Forms of Symplectic Ternary Algebras

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Abstract

Symplectic ternary algebras are introduced in connection with Lie triple systems. The authors obtain by means of the Killing forms two criterions for semisimplicity and for solvability respectively, and investigate the relationship among the Killing forms of a real Symplectic Ternary Algebras U_0, the complexification U, and the realification U_R of U.

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1 Introduction

The algebras studied here are a generalization of the class of ternary algebras given in [1], in turn, a variation of Freudenthal triple systems[1]. The advantage of the latest algebras,which we call Symplectic ternary algebras, is that they are defined by identities and hence admit direct sums. JOHN.R.FAULKENER and JOSEPH.C.FERRAR study the structure of this algebra, discussed the semisimple Symplectic ternary algebras and give a classification of the simple algebras over algebraically closed fields of characteristic 0. They construct a good connection between Lie triple system and Symplectic ternary algebras, so it is naturally that we can generalize the conclusion from Lie triple system to Symplectic ternary algebras.

The theory of symplectic killing forms are well proved very useful in the study of Lie algebra and so on. The purpose of this paper is to give some properties for symplectic ternary algebras. The fundamental facts about symplectic ternary algebras are collected in section 2. Then we give the definition of killing forms and establish some properties in section 3. Section 4 is devoted to investigation of Killing forms of a real Symplectic ternary algebras U_0, the complexification U, and the realification U_R of U.

In this article, we will be concerned with symplectic ternary algebras and Lie triple systems with finite dimension over a field F of characteristic 0.

A symplectic ternary algebra U is a vector space with a trilinear product $< x, y, z >$ and satisfies the following identities:

$$S(x, y) = L(x, y) - L(y, x) = R(x, y) - R(y, x), \tag{1}$$

$$S(x, y)R(z, w) = R(z, w)S(x, y) = R(zS(x, y), w) = R(z, wS(x, y)), \tag{2}$$

$$[R(x, y), R(z, w)] = R(xR(z, w), y) = R(x, yR(w, z)), \tag{3}$$

where $x, y, z, u, v \in U$, and define $L(x, y), U(x, y) \in EndU$ by $< x, y, z >= L(x, y)z = R(y, z)x = U(x, z)y$.

Example Let U be a vector space with non-degenerate skew form $<, >$ with product defined by

$$< x, y, z >= \frac{1}{2}(< x, y > z+ < y, z > x+ < x, z > y), x, y, z \in U.$$

We can verify U is a symplectic ternary algebra by direct calculation.

An ideal of a symplectic ternary algebra U is a subspace I if $< U, I, U >\subseteq I$ and $< I, U, U >\subseteq I$ or $< U, U, I >\subseteq I$.An ideal $I$of U is called solvable if there is a positive integer k for which $I^{(k)} \neq 0, I^{(k+1)} = 0$,where $I^{(0)} = I, \cdots, I^{(s+1)} = \Sigma_{\pi\in S_3} < I^{(s)}_{\pi_1}, U_{\pi_2}, I^{(s)}_{\pi_3} >$. U is called semisimple if the radical $R(U)$ of U is zero.

A derivation of U is a linear transformation D of U into U such that

$$D < x, y, z >=< d(x), y, z > + < x, D(y), z > + < x, y, D(z) > .$$

A isomorphism σ of $U \to U$ is called automorphism, if it satisfied

$$\sigma < x, y, z >=< \sigma(x), \sigma(y), \sigma(z) > .$$

A Lie triple system over a field F is a vector space T, with a trilinear product [, ,], satisfying the following conditions:

$$[x, y, y] = 0, \tag{4}$$

$$[x, y, z] + [y, z, x] + [z, x, y] = 0, \tag{5}$$

$$[[x, y, z], u, v] = [[x, u, v], y, z] + [x, [y, u, v], z] + [x, y, [z, u, v]], \tag{6}$$

where $x, y, z, u, v \in T$. Define $L(x, y), R(x, y)$by $[x, y, z] = L(x, y)z = R(y, z)x$.

We consider the vector space direct sum $T(U) = U \oplus U$, define for elements $X = (a, b), a, b \in U$, a bracket

$$\left[\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix} \right] = \begin{pmatrix} R(c, f) - R(e, d) & S(e, c) \\ S(d, f) & R(f, c) - R(d, e) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$
$$= \begin{pmatrix} < a, c, f > - < a, e, d > + < e, c, b > - < c, e, b > \\ < d, f, a > - < f, d, a > + < b, f, c > - < b, d, e > \end{pmatrix} \tag{7}$$

It is easy to prove that $T(U)$ with the above bracket is a Lie triple system which is called the Lie triple system associated with U .

Now we record an important result which will be needed later on.

Theorem 1.1 U is solvable if and only if the Lie triple system $T(U)$ associated with U is solvable.

2 The Killing forms of Symplectic ternary algebra

Following the Lie triple system theories, we define, in this section, Killing forms $(,)$ of a Symplectic ternary algebra and enumerate several elementary results concerning $(,)$.

Definition 2.1 The killing form of a Symplectic ternary algebra is the bilinear form

$$(a, b) = tr(\frac{1}{2}S(a, b) + U(a, b) - U(b, a).$$

Clearly, the killing form (a, b) is anti-symmetric.

Definition 2.2 The killing form of a Lie triple system T is the bilinear form

$$\rho(x, y) := \frac{1}{2}tr(L(x, y) + L(y, x), \forall x, y \in T.$$

It is easy to see that the killing form $\rho(x, y)$ is symmetric.

Let $(,), \rho(,)$ be the killing form of a Symplectic ternary algebra, of the Lie triple system $T(U)$ associated with U respectively, we have get the following identity which describes the relationship between them(refee to [3]):

$$\rho((a_1, b_1), (a_2, b_2)) = (a_1, b_2) + (a_2, b_1).$$

For the killing form $\rho(x,y)$ of Lie triple system T , some properties have been obtained. we enumerate them as following(refee to [4]):

(1)$\rho(Ax,Ay)=\rho(x,y), for A\in AutT;$

(2)$\rho(Dx,y)+\rho(x,Dy)=0, for D\in DerT;$

(3)$\rho(x,y)$ is right invariant and left invariant, i.e.

$$\rho(R(a,b)x,y)=\rho(x,R(b,a)y),\rho(L(a,b)x,y)=\rho(x,L(b,a)y);$$

(4)If $\rho(x,y)$ is non-degenerate, then $\rho(x,y)=trL(x,y)$.

There is an analogous properties in the case of Symplectic ternary algebras:

Theorem 2.1 suppose $(,)$is the killing form of a Symplectic ternary algebra U. Then

(1)$(A(x),A(y))=(x,y), for A\in AutU;$

(2)$(D(x),y)+(x,D(y))=0, for D\in DerU;$

(3)(x,y) is right invariant and left invariant, i.e.,

$$(R(a,b)x,y)=-(x,R(b,a)y),(L(a,b)x,y)=-(x,L(b,a)y),(U(a,b)x,y)=-(x,U(b,a)y).$$

(4)If $(,)$ is non-degenerate, then $(a,b)=trL(a,b)+2U(b,a)$.

proof (1)$\forall A\in AutU$, it is easy to see that $\begin{pmatrix} A & 0\\ 0 & A\end{pmatrix}\in AutT(U)$ then we get

$$\begin{aligned}(Ax,Ay)=\rho(\begin{pmatrix} Ax\\ 0\end{pmatrix},\begin{pmatrix} 0\\ Ay\end{pmatrix}) \qquad (8)\\ = \rho(\begin{pmatrix} A & 0\\ 0 & A\end{pmatrix}\begin{pmatrix} x\\ 0\end{pmatrix},\begin{pmatrix} A & 0\\ 0 & A\end{pmatrix}\begin{pmatrix} 0\\ y\end{pmatrix})\\ = \rho(\begin{pmatrix} x\\ 0\end{pmatrix},\begin{pmatrix} 0\\ y\end{pmatrix})=(x,y).\end{aligned}$$

(2) $\forall D\in DerU$, then $\begin{pmatrix} D & 0\\ 0 & D\end{pmatrix}\in DerT(U)$. Based on the properties of derivations for Lie triple system, similarly to the proof of (1), we can easily conclude (2).

(3)Notice that the killing form $\rho(,)$ of Lie triple system $T(U)$ is invariant and $\rho((a_1,b_1),(a_2,b_2))=(a_1,b_2)+(a_2,b_1)$. For $a,b,x,y\in U$,we have

$$\rho([\begin{pmatrix} x\\ 0\end{pmatrix},\begin{pmatrix} y\\ 0\end{pmatrix},\begin{pmatrix} 0\\ z\end{pmatrix}],\begin{pmatrix} 0\\ w\end{pmatrix}) = \rho(\begin{pmatrix} 0\\ z\end{pmatrix},\begin{pmatrix} <y,x,w>\\ 0\end{pmatrix}),$$

$$right = \rho(\begin{pmatrix} 0\\ z\end{pmatrix},\begin{pmatrix} <y,x,w>\\ 0\end{pmatrix}=(<y,x,w>,z)$$

$$left = \rho(\begin{pmatrix} < x,y,z > \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ w \end{pmatrix}) = (< x,y,z >, w),$$

so we have $(L(x,y)z, w) = -(z, L(y,x)w)$, the others can be got by similar way.

(4) suppose (x,y) is nondegenerate. Let $L(x,y)^*, U(x,y)^*$ denote the adjoint endomorphism of $L(x,y), U(x,y)$ in U with respect to $(,)$, by the invariancy of $(,)$, then we get $L(x,y)^* = -L(y,x), U(x,y)^* = -U(y,x)$, which implies $(x,y) = tr(L(x,y)+2U(y,x))$, because $trL(x,y)^* = L(x,y), U(x,y)^* = U(x,y)$.

A Lie triple system T is solvable if and only if $\rho(x,y) = 0$, for $x \in T, y \in T^{(1)} = [T,T,T]$, where $\rho(,)$ is the killing form of T. Now we will obtain an analogous theorem for Symplectic ternary algebra U. For this purpose, we first prove the following conclusion as base:

Theorem 2.2 Let $(,)$ and $\rho(,)$ be as in theorem. The following are equivalent:

$(1)\rho(x,y) = 0$, for all $x \in T(U), y \in T(U)^{(1)}$;

$(2)(x,y) = 0$, for all $x \in U, y \in U^{(1)}$;

proof If we assume (1), then we obtain (2) by using

$$\begin{aligned} (x, < a,c,b >) &= \rho(\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ < a,c,b > \end{pmatrix}) \\ &= \rho(\begin{pmatrix} x \\ 0 \end{pmatrix}, [\begin{pmatrix} a \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ b \end{pmatrix}, \begin{pmatrix} 0 \\ c \end{pmatrix}]) = 0 \end{aligned}$$

for all $x \in U$, and $< a,c,b > \in U^{(1)}$.

conversely, suppose (2) holds, then for $x \in T(U), y \in T(U)^{(1)}$, by (7), we get $T(U)^{(1)} \subseteq T(U^{(1)})$, so $\rho(T(U), T(U)^{(1)}) = (U, U^{(1)}) = 0$, it finished the proof of (1).

Corollary2.1 A Symplectic ternary algebra U is solvable if and only if it's killing form $(x,y) = 0, \forall x \in U, y \in U^{(1)}$.

proof This follows from theorem 1.1, 2.2 and the Cartan's Criterion for Lie triple systems.

It is easy to prove the following two lemmas .

Lemma2.1 Let I be an ideal of U, then $I^\perp$ is also an ideal of U, where

$$I^\perp = \{x \in U \mid (x,y) = 0, \forall y \in U\}.$$

Lemma2.2 Let I be an ideal of U, and $(x,y)_I, (x,y)$ the killing forms of I and U respectively. Then $(x,y)_I = (x,y), \forall x,y \in U$.

Theorem2.3 If U is a Symplectic ternary algebra with finite dimension over a field of characteristic 0, then U is semisimple if and only if (x,y) is non-degenerate.

proof Suppose $R(U)=0$. It is easy to see that $(U,U^{\perp})=0$, particularly $(U^{\perp},U^{\perp(1)})=0$. By lemma2.1 , we conclude that $U^{\perp}$ is solvable, then $U^{\perp}\subseteq R(U)=0$, which means $(,)$ is non-degenerate.

On the other hand, if the killing form $(,)$ is non-degenerate,i.e., $U^{\perp}=\{0\}$. In order to prove that U is semi-simple it suffices to prove that every solvable ideal I is included in $U^{\perp}$.

Because I is a solvable ideal of U, i.e.$I^{(m+1)}=0, I^{(m)}\neq 0$, where $I^{(m+1)}=(I^{(m)})^{(1)}$. If we denote $I^m=J$, we get an ideal J such that $J^{(1)}=0$. Chose the basis $\{e_1,e_2,\cdots,e_r\}$ of J, and extend it to be the basis, $\{e_1,e_2,\cdots,e_r,e_{r+1},\cdots,e_n\}$ for U. Notice that

$$<J,U,J>\subseteq<J,U,J>+<J,J,U>+<U,J,J>=J^{(1)}=0,$$

so as $<J,J,U>=<U,J,J>=0$. Since J is an ideal, then $<U,U,J>,<U,J,U>,<J,U,U>\subseteq J$.

Let $e_{i_1}\in U, e_{i_2}\in J$, we examine the matrix of linear transformation $L(e_{i_1},e_{i_2})$ and $U(e_{i_1},e_{i_2})$. It is easy to get $trL(e_{i_1},e_{i_2})=0, trU(e_{i_1},e_{i_2})=0$.

In other words, $trL(J,U)=trU(J,U)=0$, so $(J,U)=0$, which means $J\subseteq U^{\perp}=0$.

3 The complexification of real Symplectic ternary algebra

In this section we assume all Symplectic ternary algebras real or complex. First we give a definition of a Symplectic ternary algebra which is similar to the case of Lie triple systems[4].

Definition3.1 Let U be a Symplectic ternary algebra over R. An R-linear endomorphism J of U is called a compatible complex structure, if J satisfies :

(1)$J^2=-id$;

(2)$J<x,y,z>=<Jx,y,z>=<x,Jy,z>=<x,y,Jz>,\forall x,y,z\in U$.

Let U^0 be a real Symplectic ternary algebra with a compatible complex structure J. Define:

$$(a+bi)x=ax+bJx,\forall x\in U_0,a,b\in R,$$

it is easy to verify that U^0 is a complex Symplectic ternary algebra denoted by $\overline{U_0}$ with the ternary product inherited form U_0.

We form the tensor product $U_0^C:=C\otimes_R U_0$. It is easy to verify U_0^C is a vector space over C with the product $\alpha(\beta\otimes x)=\alpha\beta\otimes x$, for $\alpha,\beta\in C, x\in U_0$, and it is a Symplectic ternary algebra with $<\alpha\otimes x,\beta\otimes y,\gamma\otimes z>=\alpha\beta\gamma\otimes<x,y,z>$. We call it the complexification of U_0. We can think of U_0^C as

$$U_0^C=\{x+yi|x,y\in U_0,i=\sqrt{-1}\}.$$

A real Symplectic ternary algebra U_0 is called a real form of a complex Symplectic ternary algebra U if its comlexification is isomorphic to U. On the contrary, we call a real Symplectic ternary algebra U^R the realification of U, where U^R is obtained from a complex Symplectic ternary algebra U by restricting the ground field to the real field.

Let U_0 be a real Symplectic ternary algebra, U the complxification of U_0, U^R the realification of U. It is easy to see that $U = U_0 + iU_0$ and $U^R = U_0 + JU_0$, where J is a compatible complex structure of U_0.

Now, we discuss the relationship among the killing forms of them.

Theorem3.1 Let $(,),(,)_0$ and $(,)^R$ be the killing forms of Symplectic ternary algebras U, U_0, U^R respectively. Then :

(1)$(x, y) = (x, y)_0$,for $x, y \in U_0$.

(2)$(x, y)^R = 2Re(x, y), \forall x, y \in U^R$(Re=real part).

Proof Suppose $\{x_1, x_2, \cdots, x_n\}$ be a basis of a real Symplectic ternary algebra U_0. From the knowledge of linear algebra, it is also a basis of U, and $\{x_1, x_2, \cdots, x_n, J(x_1), J(x_2), \cdots, J(x_n)\}$ is a basis of U^R.

Let $x, y \in U_0$, the endomorphism $L(x, y)$ has the same matrix expression either viewed as acting on U or as acting on $U_0$with respect to the basis $\{x_1, x_2, \cdots, x_n\}$, so as $U(x, y)$. Therefore the first formular is proved.

For the second, for $x, y \in U^R$, let $A + iB$ denote the matrix of $L(x, y)$ with respect to the basis $\{x_1, x_2, \cdots, x_n\}$, where A, B is real matrix. Now we consider the matrix of $L(x, y)$ with respect to $Jx_1, Jx_2, \cdots, Jx_n$.

Because

$$L(x, y)(Jx_i) =< x, y, Jx_i >= J < x, y, x_i >= JL(x, y)(x_i),$$

so we have that the matrix of $L(x, y)$ with respect to $\{Jx_1, Jx_2, \cdots, Jx_n\}$ is $-B + iA$. This shows the matrix of $L(x, y)$ with respect to the basis of $\{x_1, x_2, \cdots, x_n, J(x_1), J(x_2), \cdots, J(x_n)\}$ is

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix}.$$

Similar conclusion for the endomorphism $U(x, y)$ is also obtained. Then $(x, y)^R = 2Re(x, y)$.

Theorem3.2 Let $(,),(,)_0$ and $(,)_R$ be as in theorem 3.1, then they are all non-degenerate if and only if one of them is.

Proof First, if $(,)_0$ is non-degenerate. Suppose $x+iy \in U$, $x, y \in U_0$, such that $(x + iy, U) = 0$. As $U_0 \subseteq U$, then $(x, U_0) + i(y, U_0) = 0$. According to the theorem 3.1, we have $(x, U_0)_0 = 0, (y, U_0)_0 = 0$. Since $(,)_0$ is non-degenerate, so $x = y = 0$, in other words $x + iy = 0$. So $(,)$ is non-degenerate.

Now we assume $(,)$ is non-degenerate, by the proof of theorem 3.1(2), it is easy to find the following :

$$(x, y)^R = 2Re(x, y), (Jx, y)^R = 2Re(ix, y) = -2Im(x, y), \forall x, y \in U^R.$$

Let $y \in U^R$ such that $(x, y)^R = 0, \forall x \in U^R$, then we have $Re(x, y) = 0$, and $Im(x, y) = -\frac{1}{2}(Jx, y)^R = \frac{1}{2}(y, Jx)^R = 0$, which means that $(x, y) = 0, \forall x \in U^R$. Then $(y, U^R) = 0$, so $(y, U) = 0$, and $y = 0$, which means that $(,)^R$ is non-degenerate.

Finally, suppose$(,)^R$ is non-degenerate, and $(x, y)_0 = 0, \forall y \in U_0$. By theorem 3.1(1),$(x, U) = (x, U_0) + i(x, U_0) = 0$, i.e., $(x, U) = 0$, so we have $Re(x, U) = 0$, i.e.$(x, U^R)^R = 0$. Since $(,)^R$ is non-degenerate,so we have $x = 0$, which finish the proof.

Corollary 3.1 U_0, U, U^R are semi-simple if and only if one of them is.

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