Mathematica Aeterna, Vol. 4, 2014, no. 5, 521 - 525

Some Inequalities for the k-Digamma Function

Kwara Nantomah

Department of Mathematics, University for Development Studies,
Navrongo Campus, P. O. Box 24, Navrongo, UE/R, Ghana.
mykwarasoft@yahoo.com, knantomah@uds.edu.gh

Edward Prempeh

Department of Mathematics, Kwame Nkrumah University of
Science and Technology, Kumasi, Ghana.
eprempeh.cos@knust.edu.gh

Abstract

Some inequalities involving the k-digamma function are presented. These results are the k-analogues of some recent results.

Mathematics Subject Classification: 33B15, 26A48.

Keywords: digamma function, k-digamma function, Inequality.

1 Introduction and Preliminaries

The classical Euler's Gamma function $\Gamma(t)$ and the digamma function $\psi(t)$ are commonly defined as

$$\Gamma(t) = \int_0^\infty e^{-x} x^{t-1}\, dx, \quad \text{and} \quad \psi(t) = \frac{d}{dt}\ln(\Gamma(t)) = \frac{\Gamma'(t)}{\Gamma(t)}, \quad t > 0.$$

Similarly the k-Gamma and k-digamma functions are defined as (see [1])

$$\Gamma_k(t) = \lim_{k\to\infty} \frac{n! k^n (nk)^{\frac{t}{k}-1}}{(t)_{n,k}} = \int_0^\infty e^{-\frac{x^k}{k}} x^{t-1}\, dx, \quad k > 0, \quad t > 0.$$

where $(t)_{n,k} = t(t+k)(t+2k)\dots(t+(n-1)k)$ and

$$\psi_k(t) = \frac{d}{dt}\ln(\Gamma_k(t)) = \frac{\Gamma_k'(t)}{\Gamma_k(t)}, \quad k > 0, \quad t > 0.$$

The functions $\psi(t)$ and $\psi_k(t)$ as defined above exhibit the following series representations.

$$\psi(t) = -\gamma + (t-1)\sum_{n=0}^{\infty} \frac{1}{(1+n)(n+t)}, \quad t > 0$$
$$\psi_k(t) = \frac{\ln k - \gamma}{k} - \frac{1}{t} + \sum_{n=1}^{\infty} \frac{t}{nk(nk+t)}, \quad k > 0, \quad t > 0$$

where γ is the Euler-Mascheroni's constant.

By taking the m-th derivative of the above functions, we arrive at the following statements for $m \in N$.

$$\psi^{(m)}(t) = (-1)^{m+1} m! \sum_{n=0}^{\infty} \frac{1}{(n+t)^{m+1}}, \quad t > 0$$
$$\psi_k^{(m)}(t) = (-1)^{m+1} m! \sum_{n=0}^{\infty} \frac{1}{(nk+t)^{m+1}}, \quad k > 0, \quad t > 0.$$

In 2011, Sulaiman [3] presented the following results.

$$\psi(t+s) \geq \psi(t) + \psi(s) \tag{1}$$

where $t > 0$ and $0 < s < 1$.

$$\psi^{(m)}(t+s) \leq \psi^{(m)}(t) + \psi^{(m)}(s) \tag{2}$$

where m is a positive odd integer and $t, s > 0$.

$$\psi^{(m)}(t+s) \geq \psi^{(m)}(t) + \psi^{(m)}(s) \tag{3}$$

where m is a positive even integer and $t, s > 0$.

In a recent paper, Sroysang [2] presented the following geralizations of the above inequalities.

$$\psi(t + \sum_{i=1}^{\alpha} \beta_i s_i) \geq \psi(t) + \sum_{i=1}^{\alpha} \beta_i \psi(s_i) \tag{4}$$

where $t > 0$, $\beta_i > 0$ and $0 < s_i < 1$ for all $i \in N_\alpha$.

$$\psi^{(m)}(t + \sum_{i=1}^{\alpha} \beta_i s_i) \leq \psi^{(m)}(t) + \sum_{i=1}^{\alpha} \beta_i \psi^{(m)}(s_i) \tag{5}$$

where m is a positive odd integer, $t > 0$, $\beta_i > 0$ and $s_i > 0$ for all $i \in N_\alpha$.

$$\psi^{(m)}(t + \sum_{i=1}^{\alpha} \beta_i s_i) \geq \psi^{(m)}(t) + \sum_{i=1}^{\alpha} \beta_i \psi^{(m)}(s_i) \tag{6}$$

where m is a positive even integer, $t > 0$, $\beta_i > 0$ and $s_i > 0$ for all $i \in N_\alpha$.

The objective of this paper is to establish that the inequalities (4), (5) and (6) still hold true for the k-digamma function.

2 Main Results

We now present our results.

Theorem 2.1. *Let $k > 0$, $t > 0$, $\beta_i > 0$ and $0 < s_i < 1$ for all $i \in N_\alpha$. Then the following inequality is valid.*

$$\psi_k(t + \sum_{i=1}^{\alpha} \beta_i s_i) \geq \psi_k(t) + \sum_{i=1}^{\alpha} \beta_i \psi_k(s_i) \tag{7}$$

Proof. Let $\mu(t) = \psi_k(t + \sum_{i=1}^{\alpha} \beta_i s_i) - \psi_k(t) - \sum_{i=1}^{\alpha} \beta_i \psi_k(s_i)$. Then fixing s_i for each i we have,

$$\begin{aligned} \mu'(t) &= \psi_k'(t + \sum_{i=1}^{\alpha} \beta_i s_i) - \psi_k'(t) \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{(nk + t + \sum_{i=1}^{\alpha} \beta_i s_i)^2} - \frac{1}{(nk+t)^2} \right] \leq 0 \end{aligned}$$

That implies μ is non-increasing. Furthermore,

$$\begin{aligned} \lim_{t\to\infty} \mu(t) &= \lim_{t\to\infty} \left[\psi_k(t + \sum_{i=1}^{\alpha} \beta_i s_i) - \psi_k(t) - \sum_{i=1}^{\alpha} \beta_i \psi_k(s_i) \right] \\ &= \lim_{t\to\infty} \Bigg[\sum_{i=1}^{\alpha} \beta_i \frac{1}{s_i} + \frac{1}{t} - \frac{1}{t + \sum_{i=1}^{\alpha} \beta_i s_i} - (\frac{\ln k - \gamma}{k}) \sum_{i=1}^{\alpha} \beta_i + \\ &\sum_{n=1}^{\infty} \left(\frac{t + \sum_{i=1}^{\alpha} \beta_i s_i}{nk(nk + t + \sum_{i=1}^{\alpha} \beta_i s_i)} - \frac{t}{nk(nk+t)} - \sum_{i=1}^{\alpha} \beta_i \frac{s_i}{nk(nk+s_i)} \right) \Bigg] \\ &= \sum_{i=1}^{\alpha} \beta_i \left[\frac{1}{s_i} - \frac{\ln k - \gamma}{k} - \sum_{n=1}^{\infty} \frac{s_i}{nk(nk+s_i)} \right] \\ &= -\sum_{i=1}^{\alpha} \beta_i \psi_k(s_i) \geq 0. \quad \text{(Note that } \psi_k(t) < 0 \text{ for } 0 < t \leq 1) \end{aligned}$$

Therefore $\mu(t) \geq 0$ yielding the result.

Theorem 2.2. *Let $k > 0$, $t > 0$, $\beta_i > 0$ and $s_i > 0$ for all $i \in N_\alpha$. Suppose that m is a positive odd integer, then the following inequality is valid.*

$$\psi_k^{(m)}(t+\sum_{i=1}^{\alpha}\beta_i s_i) \leq \psi_k^{(m)}(t) + \sum_{i=1}^{\alpha}\beta_i \psi_k^{(m)}(s_i) \tag{8}$$

Proof. Let $\eta(t) = \psi_k^{(m)}(t+\sum_{i=1}^{\alpha}\beta_i s_i) - \psi_k^{(m)}(t) - \sum_{i=1}^{\alpha}\beta_i\psi_k^{(m)}(s_i)$. Then fixing s_i for each i we have,

$$\begin{aligned}
\eta'(t) &= \psi_k^{(m+1)}(t+\sum_{i=1}^{\alpha}\beta_i s_i) - \psi_k^{(m+1)}(t) \\
&= (-1)^{m+2}(m+1)!\sum_{n=0}^{\infty}\left[\frac{1}{(nk+t+\sum_{i=1}^{\alpha}\beta_i s_i)^{m+2}} - \frac{1}{(nk+t)^{m+2}}\right] \\
&= -(m+1)!\sum_{n=0}^{\infty}\left[\frac{1}{(nk+t+\sum_{i=1}^{\alpha}\beta_i s_i)^{m+2}} - \frac{1}{(nk+t)^{m+2}}\right] \text{ (for odd } m) \\
&\geq 0.
\end{aligned}$$

That implies η is non-decreasing. Furthermore,

$$\begin{aligned}
\lim_{t\to\infty}\eta(t) &= \lim_{t\to\infty}\left[\psi_k^{(m)}(t+\sum_{i=1}^{\alpha}\beta_i s_i) - \psi_k^{(m)}(t) - \sum_{i=1}^{\alpha}\beta_i\psi_k^{(m)}(s_i)\right] \\
&= (-1)^{m+1}m! \quad \times \\
&\lim_{t\to\infty}\sum_{n=0}^{\infty}\left[\frac{1}{(nk+t+\sum_{i=1}^{\alpha}\beta_i s_i)^{m+1}} - \frac{1}{(nk+t)^{m+1}} - \sum_{i=1}^{\alpha}\frac{\beta_i}{(nk+s_i)^{m+1}}\right] \\
&= (-1)^{m+1}m!\sum_{n=0}^{\infty}\sum_{i=1}^{\alpha}\left[-\frac{\beta_i}{(nk+s_i)^{m+1}}\right] \\
&= -m!\sum_{n=0}^{\infty}\sum_{i=1}^{\alpha}\left[\frac{\beta_i}{(nk+s_i)^{m+1}}\right] \leq 0. \quad \text{(since } m \text{ is odd)}
\end{aligned}$$

Therefore $\eta(t) \leq 0$ yielding the result.

Theorem 2.3. *Let $k > 0$, $t > 0$, $\beta_i > 0$ and $s_i > 0$ for all $i \in N_\alpha$. Suppose that m is a positive even integer, then the following inequality is valid.*

$$\psi_k^{(m)}(t+\sum_{i=1}^{\alpha}\beta_i s_i) \geq \psi_k^{(m)}(t) + \sum_{i=1}^{\alpha}\beta_i \psi_k^{(m)}(s_i) \tag{9}$$

Proof. Let $\lambda(t) = \psi_k^{(m)}(t + \sum_{i=1}^{\alpha} \beta_i s_i) - \psi_k^{(m)}(t) - \sum_{i=1}^{\alpha} \beta_i \psi_k^{(m)}(s_i)$. Then fixing s_i for each i we have,

$$\begin{aligned}
\lambda'(t) &= \psi_k^{(m+1)}(t + \sum_{i=1}^{\alpha} \beta_i s_i) - \psi_k^{(m+1)}(t) \\
&= (-1)^{m+2}(m+1)! \sum_{n=0}^{\infty} \left[\frac{1}{(nk + t + \sum_{i=1}^{\alpha} \beta_i s_i)^{m+2}} - \frac{1}{(nk+t)^{m+2}} \right] \\
&= (m+1)! \sum_{n=0}^{\infty} \left[\frac{1}{(nk + t + \sum_{i=1}^{\alpha} \beta_i s_i)^{m+2}} - \frac{1}{(nk+t)^{m+2}} \right] \text{ (for even } m) \\
&\leq 0.
\end{aligned}$$

That implies λ is non-increasing. Furthermore,

$$\begin{aligned}
\lim_{t\to\infty} \lambda(t) &= \lim_{t\to\infty} \left[\psi_k^{(m)}(t + \sum_{i=1}^{\alpha} \beta_i s_i) - \psi_k^{(m)}(t) - \sum_{i=1}^{\alpha} \beta_i \psi_k^{(m)}(s_i) \right] \\
&= (-1)^{m+1} m! \quad \times \\
&\lim_{t\to\infty} \sum_{n=0}^{\infty} \left[\frac{1}{(nk + t + \sum_{i=1}^{\alpha} \beta_i s_i)^{m+1}} - \frac{1}{(nk+t)^{m+1}} - \sum_{i=1}^{\alpha} \frac{\beta_i}{(nk+s_i)^{m+1}} \right] \\
&= (-1)^{m+1} m! \sum_{n=0}^{\infty} \sum_{i=1}^{\alpha} \left[-\frac{\beta_i}{(nk+s_i)^{m+1}} \right] \\
&= m! \sum_{n=0}^{\infty} \sum_{i=1}^{\alpha} \left[\frac{\beta_i}{(nk+s_i)^{m+1}} \right] \geq 0. \quad \text{(since } m \text{ is even)}
\end{aligned}$$

Therefore $\lambda(t) \geq 0$ yielding the result.

Remark 2.4. If we let $k \to 1$ in inequalities (7), (8) and (9) then we repectively recover the inequalities (4), (5) and (6).

References

[1] R. Díaz and E. Pariguan, *On hypergeometric functions and Pachhammer k-symbol*, Divulgaciones Matemtícas **15**(2)(2007), 179-192.

[2] B. Sroysang, *More on some inequalities for the digamma function*, Math. Aeterna, **4**(2)(2014), 123-126.

[3] W. T. Sulaiman, *Turan inequalites for the digamma and polygamma functions*, South Asian J. Math. **1**(2)(2011), 49-55.

Received: May, 2014