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Some Inequalities for the (p, q)-Digamma Function

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Abstract

Some inequalities involving the (p, q)-digamma function are presented. These results are the (p, q)-analogues of some recent results.

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1 Introduction and Preliminaries

The classical Euler's Gamma function $\Gamma(t)$ and the digamma function $\psi(t)$ are commonly defined as

$$\Gamma(t) = \int_0^\infty e^{-x} x^{t-1}\, dx, \quad \text{and} \quad \psi(t) = \frac{d}{dt}\ln(\Gamma(t)) = \frac{\Gamma'(t)}{\Gamma(t)}, \quad t > 0.$$

Also, the (p, q)-Gamma and (p, q)-digamma functions are defined as (see [1])

$$\Gamma_{p,q}(t) = \frac{[p]_q^t [p]_q!}{[t]_q [t+1]_q \dots [t+p]_q}, \quad t > 0, \quad p \in N, \quad q \in (0, 1).$$

and $\quad \psi_{p,q}(t) = \frac{d}{dt}\ln(\Gamma_{p,q}(t)) = \frac{\Gamma'_{p,q}(t)}{\Gamma_{p,q}(t)} \quad$ where $\quad [p]_q = \frac{1-q^p}{1-q}$.

The functions $\psi(t)$ and $\psi_{p,q}(t)$ as defined above exhibit the following series representations.

$$\psi(t) = -\gamma + (t-1)\sum_{n=0}^{\infty}\frac{1}{(1+n)(n+t)}, \quad t>0$$
$$\psi_{p,q}(t) = \ln[p]_q + (\ln q)\sum_{n=1}^{p}\frac{q^{nt}}{1-q^n}, \qquad t>0.$$

where γ is the Euler-Mascheroni's constant.

By taking the m-th derivative of the above functions, we arrive at the following statements for $m \in N$.

$$\psi^{(m)}(t) = (-1)^{m+1}m!\sum_{n=0}^{\infty}\frac{1}{(n+t)^{m+1}}, \quad t>0$$
$$\psi_{p,q}^{(m)}(t) = (\ln q)^{m+1}\sum_{n=1}^{p}\frac{n^m q^{nt}}{1-q^n}, \quad t>0.$$

In 2011, Sulaiman [3] presented the following results.

$$\psi(t+s) \geq \psi(t) + \psi(s) \tag{1}$$

where $t>0$ and $0<s<1$.

$$\psi^{(m)}(t+s) \leq \psi^{(m)}(t) + \psi^{(m)}(s) \tag{2}$$

where m is a positive odd integer and $t,s>0$.

$$\psi^{(m)}(t+s) \geq \psi^{(m)}(t) + \psi^{(m)}(s) \tag{3}$$

where m is a positive even integer and $t,s>0$.

In a recent paper, Sroysang [2] established the following geralizations of the above inequalities.

$$\psi(t+\sum_{i=1}^{\alpha}\beta_i s_i) \geq \psi(t) + \sum_{i=1}^{\alpha}\beta_i\psi(s_i) \tag{4}$$

where $t>0$, $\beta_i>0$ and $0<s_i<1$ for all $i \in N_\alpha$.

$$\psi^{(m)}(t+\sum_{i=1}^{\alpha}\beta_i s_i) \leq \psi^{(m)}(t) + \sum_{i=1}^{\alpha}\beta_i\psi^{(m)}(s_i) \tag{5}$$

where m is a positive odd integer, $t > 0$, $\beta_i > 0$ and $s_i > 0$ for all $i \in N_\alpha$.

$$\psi^{(m)}(t + \sum_{i=1}^{\alpha} \beta_i s_i) \geq \psi^{(m)}(t) + \sum_{i=1}^{\alpha} \beta_i \psi^{(m)}(s_i) \tag{6}$$

where m is a positive even integer, $t > 0$, $\beta_i > 0$ and $s_i > 0$ for all $i \in N_\alpha$.

The objective of this paper is to establish that the inequalities (4), (5) and (6) still hold true for the (p,q)-digamma function.

2 Main Results

We now present our results.

Theorem 2.1. *Let $p \in N$, $q \in (0,1)$, $t > 0$, $\beta_i > 0$ and $0 < s_i < 1$ for all $i \in N_\alpha$. Then the following inequality is valid.*

$$\psi_{p,q}(t + \sum_{i=1}^{\alpha} \beta_i s_i) \geq \psi_{p,q}(t) + \sum_{i=1}^{\alpha} \beta_i \psi_{p,q}(s_i) \tag{7}$$

Proof. Let $U(t) = \psi_{p,q}(t + \sum_{i=1}^{\alpha} \beta_i s_i) - \psi_{p,q}(t) - \sum_{i=1}^{\alpha} \beta_i \psi_{p,q}(s_i)$. Then fixing s_i for each i we have,

$$\begin{aligned} U'(t) &= \psi'_{p,q}(t + \sum_{i=1}^{\alpha} \beta_i s_i) - \psi'_{p,q}(t) \\ &= (\ln q)^2 \sum_{n=1}^{p} \left[\frac{nq^{n(t+\sum_{i=1}^{\alpha} \beta_i s_i)}}{1-q^n} - \frac{nq^{nt}}{1-q^n} \right] \\ &= (\ln q)^2 \sum_{n=1}^{p} \frac{nq^{nt}(q^{n\sum_{i=1}^{\alpha} \beta_i s_i} - 1)}{1-q^n} \leq 0. \end{aligned}$$

That implies U is non-increasing. Besides,

$$\begin{aligned} \lim_{t\to\infty} U(t) &= \lim_{t\to\infty} \left[\psi_{p,q}(t + \sum_{i=1}^{\alpha} \beta_i s_i) - \psi_{p,q}(t) - \sum_{i=1}^{\alpha} \beta_i \psi_{p,q}(s_i) \right] \\ &= -\ln[p]_q \sum_{i=1}^{\alpha} \beta_i \\ &\quad + (\ln q) \lim_{t\to\infty} \sum_{n=1}^{p} \left[\frac{q^{n(t+\sum_{i=1}^{\alpha} \beta_i s_i)}}{1-q^n} - \frac{q^{nt}}{1-q^n} - \sum_{i=1}^{\alpha} \frac{\beta_i q^{ns_i}}{1-q^n} \right] \\ &= -\ln[p]_q \sum_{i=1}^{\alpha} \beta_i - (\ln q) \sum_{n=1}^{p} \sum_{i=1}^{\alpha} \frac{\beta_i q^{ns_i}}{1-q^n} \geq 0. \end{aligned}$$

Therefore $U(t) \geq 0$ yielding the result.

Theorem 2.2. *Let $p \in N$, $q \in (0,1)$, $t > 0$, $\beta_i > 0$ and $s_i > 0$ for all $i \in N_\alpha$. Suppose that m is a positive odd integer, then the following inequality is valid.*

$$\psi_{p,q}^{(m)}(t + \sum_{i=1}^{\alpha} \beta_i s_i) \leq \psi_{p,q}^{(m)}(t) + \sum_{i=1}^{\alpha} \beta_i \psi_{p,q}^{(m)}(s_i) \tag{8}$$

Proof. Let $V(t) = \psi_{p,q}^{(m)}(t + \sum_{i=1}^{\alpha} \beta_i s_i) - \psi_{p,q}^{(m)}(t) - \sum_{i=1}^{\alpha} \beta_i \psi_{p,q}^{(m)}(s_i)$. Then fixing s_i for each i we have,

$$\begin{aligned}
V'(t) &= \psi_{p,q}^{(m+1)}(t + \sum_{i=1}^{\alpha} \beta_i s_i) - \psi_{p,q}^{(m+1)}(t) \\
&= (\ln q)^{m+2} \sum_{n=1}^{p} \left[\frac{n^{m+1} q^{n(t + \sum_{i=1}^{\alpha} \beta_i s_i)}}{1 - q^n} - \frac{n^{m+1} q^{nt}}{1 - q^n} \right] \\
&= (\ln q)^{m+2} \sum_{n=1}^{p} \left[\frac{n^{m+1} q^{nt} (q^{n \sum_{i=1}^{\alpha} \beta_i s_i} - 1)}{1 - q^n} \right] \geq 0. \text{ (since } m \text{ is odd)}
\end{aligned}$$

That implies V is non-decreasing. Besides,

$$\begin{aligned}
\lim_{t \to \infty} V(t) &= \lim_{t \to \infty} \left[\psi_{p,q}^{(m)}(t + \sum_{i=1}^{\alpha} \beta_i s_i) - \psi_{p,q}^{(m)}(t) - \sum_{i=1}^{\alpha} \beta_i \psi_{p,q}^{(m)}(s_i) \right] \\
&= (\ln q)^{m+1} \lim_{t \to \infty} \sum_{n=1}^{p} \left[\frac{n^m q^{n(t + \sum_{i=1}^{\alpha} \beta_i s_i)}}{1 - q^n} - \frac{n^m q^{nt}}{1 - q^n} - \sum_{i=1}^{\alpha} \beta_i \frac{n^m q^{n s_i}}{1 - q^n} \right] \\
&= -(\ln q)^{m+1} \sum_{n=1}^{\infty} \sum_{i=1}^{\alpha} \beta_i \frac{n^m q^{n s_i}}{1 - q^n} \leq 0. \text{ (since } m \text{ is odd)}
\end{aligned}$$

Therefore $V(t) \leq 0$ yielding the result.

Theorem 2.3. *Let $p \in N$, $q \in (0,1)$, $t > 0$, $\beta_i > 0$ and $s_i > 0$ for all $i \in N_\alpha$. Suppose that m is a positive even integer, then the following inequality is valid.*

$$\psi_{p,q}^{(m)}(t + \sum_{i=1}^{\alpha} \beta_i s_i) \geq \psi_{p,q}^{(m)}(t) + \sum_{i=1}^{\alpha} \beta_i \psi_{p,q}^{(m)}(s_i) \tag{9}$$

Proof. Let $W(t) = \psi_{p,q}^{(m)}(t + \sum_{i=1}^{\alpha} \beta_i s_i) - \psi_{p,q}^{(m)}(t) - \sum_{i=1}^{\alpha} \beta_i \psi_{p,q}^{(m)}(s_i)$. Then fixing

s_i for each i we have,

$$\begin{aligned} W'(t) &= \psi_{p,q}^{(m+1)}(t+\sum_{i=1}^{\alpha}\beta_i s_i) - \psi_{p,q}^{(m+1)}(t) \\ &= (\ln q)^{m+2}\sum_{n=1}^{p}\left[\frac{n^{m+1}q^{n(t+\sum_{i=1}^{\alpha}\beta_i s_i)}}{1-q^n} - \frac{n^{m+1}q^{nt}}{1-q^n}\right] \\ &= (\ln q)^{m+2}\sum_{n=1}^{p}\left[\frac{n^{m+1}q^{nt}(q^{n\sum_{i=1}^{\alpha}\beta_i s_i}-1)}{1-q^n}\right] \leq 0. \text{ (since } m \text{ is even)} \end{aligned}$$

That implies W is non-increasing. Besides,

$$\begin{aligned} \lim_{t\to\infty} W(t) &= \lim_{t\to\infty}\left[\psi_{p,q}^{(m)}(t+\sum_{i=1}^{\alpha}\beta_i s_i) - \psi_{p,q}^{(m)}(t) - \sum_{i=1}^{\alpha}\beta_i\psi_{p,q}^{(m)}(s_i)\right] \\ &= (\ln q)^{m+1}\lim_{t\to\infty}\sum_{n=1}^{p}\left[\frac{n^m q^{n(t+\sum_{i=1}^{\alpha}\beta_i s_i)}}{1-q^n} - \frac{n^m q^{nt}}{1-q^n} - \sum_{i=1}^{\alpha}\beta_i\frac{n^m q^{ns_i}}{1-q^n}\right] \\ &= -(\ln q)^{m+1}\sum_{n=1}^{p}\sum_{i=1}^{\alpha}\beta_i\frac{n^m q^{ns_i}}{1-q^n} \geq 0. \text{ (since } m \text{ is even)} \end{aligned}$$

Therefore $W(t) \geq 0$ yielding the result.

Remark 2.4. If we let $p \to \infty$ as $q \to 1^-$ in inequalities (7), (8) and (9) then we repectively recover the inequalities (4), (5) and (6).

References

[1] V. Krasniqi and F. Merovci, *Some Completely Monotonic Properties for the (p,q)-Gamma Function*, Mathematica Balkanica, New Series **26**(2012), 1-2.

[2] B. Sroysang, *More on some inequalities for the digamma function*, Math. Aeterna, **4**(2)(2014), 123-126.

[3] W. T. Sulaiman, *Turan inequalites for the digamma and polygamma functions*, South Asian J. Math. **1**(2)(2011), 49-55.

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