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Some Inequalities for the (q, k)-Digamma Function

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Abstract

Some inequalities involving the (q, k)-digamma function are presented. These results are the (q, k)-analogues of some recent results.

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1 Introduction and Preliminaries

The classical Euler's Gamma function $\Gamma(t)$ and the digamma function $\psi(t)$ are commonly defined as

$$\Gamma(t) = \int_0^\infty e^{-x} x^{t-1}\, dx, \quad \text{and} \quad \psi(t) = \frac{d}{dt} \ln(\Gamma(t)) = \frac{\Gamma'(t)}{\Gamma(t)}, \quad t > 0.$$

The (q, k)-Gamma and (q, k)-digamma functions are defined as [1],

$$\Gamma_{q,k}(t) = \frac{(1-q^k)_{q,k}^{\frac{t}{k}-1}}{(1-q)^{\frac{t}{k}-1}} = \frac{(1-q^k)_{q,k}^{\infty}}{(1-q^t)_{q,k}^{\infty}(1-q)^{\frac{t}{k}-1}}, \quad t > 0,\, q \in (0,1),\, k > 0.$$

and

$$\psi_{q,k}(t) = \frac{d}{dt} \ln \Gamma_{q,k}(t) = \frac{\Gamma'_{q,k}(t)}{\Gamma_{q,k}(t)}.$$

The functions $\psi(t)$ and $\psi_{q,k}(t)$ as defined above exhibit the following series representations.

$$\psi(t) = -\gamma + (t-1)\sum_{n=0}^{\infty}\frac{1}{(1+n)(n+t)}, \quad t>0$$

$$\psi_{q,k}(t) = \frac{-\ln(1-q)}{k} + (\ln q)\sum_{n=1}^{\infty}\frac{q^{nkt}}{1-q^{nk}}, \qquad t>0.$$

where γ is the Euler-Mascheroni's constant.

By taking the m-th derivative of the above functions, we arrive at the following statements for $m \in N$.

$$\psi^{(m)}(t) = (-1)^{m+1}m!\sum_{n=0}^{\infty}\frac{1}{(n+t)^{m+1}}, \quad t>0$$

$$\psi_{q,k}^{(m)}(t) = (\ln q)^{m+1}\sum_{n=1}^{\infty}\frac{n^m k^m q^{nkt}}{1-q^{nk}}, \quad t>0$$

In 2011, Sulaiman [3] presented the following results.

$$\psi(t+s) \geq \psi(t) + \psi(s) \tag{1}$$

where $t>0$ and $0<s<1$.

$$\psi^{(m)}(t+s) \leq \psi^{(m)}(t) + \psi^{(m)}(s) \tag{2}$$

where m is a positive odd integer and $t,s>0$.

$$\psi^{(m)}(t+s) \geq \psi^{(m)}(t) + \psi^{(m)}(s) \tag{3}$$

where m is a positive even integer and $t,s>0$.

In a recent paper, Sroysang [2] presented the following geralizations of the above inequalities.

$$\psi(t+\sum_{i=1}^{\alpha}\beta_i s_i) \geq \psi(t) + \sum_{i=1}^{\alpha}\beta_i\psi(s_i) \tag{4}$$

where $t>0$, $\beta_i>0$ and $0<s_i<1$ for all $i \in N_\alpha$.

$$\psi^{(m)}(t+\sum_{i=1}^{\alpha}\beta_i s_i) \leq \psi^{(m)}(t) + \sum_{i=1}^{\alpha}\beta_i\psi^{(m)}(s_i) \tag{5}$$

where m is a positive odd integer, $t > 0$, $\beta_i > 0$ and $s_i > 0$ for all $i \in N_\alpha$.

$$\psi^{(m)}(t + \sum_{i=1}^{\alpha} \beta_i s_i) \geq \psi^{(m)}(t) + \sum_{i=1}^{\alpha} \beta_i \psi^{(m)}(s_i) \tag{6}$$

where m is a positive even integer, $t > 0$, $\beta_i > 0$ and $s_i > 0$ for all $i \in N_\alpha$.

The objective of this paper is to establish that the inequalities (4), (5) and (6) still hold true for the (q,k)-digamma function.

2 Main Results

We now present our results.

Theorem 2.1. *Let $q \in (0,1)$, $k > 0$, $t > 0$, $\beta_i > 0$ and $0 < s_i < 1$ for all $i \in N_\alpha$. Then the following inequality is valid.*

$$\psi_{q,k}(t + \sum_{i=1}^{\alpha} \beta_i s_i) \geq \psi_{q,k}(t) + \sum_{i=1}^{\alpha} \beta_i \psi_{q,k}(s_i) \tag{7}$$

Proof. Let $F(t) = \psi_{q,k}(t + \sum_{i=1}^{\alpha} \beta_i s_i) - \psi_{q,k}(t) - \sum_{i=1}^{\alpha} \beta_i \psi_{q,k}(s_i)$. Then fixing s_i for each i we have,

$$\begin{aligned} F'(t) &= \psi'_{q,k}(t + \sum_{i=1}^{\alpha} \beta_i s_i) - \psi'_{q,k}(t) = (\ln q)^2 \sum_{n=1}^{\infty} \left[\frac{nkq^{nk(t+\sum_{i=1}^{\alpha} \beta_i s_i)}}{1-q^{nk}} - \frac{nkq^{nkt}}{1-q^{nk}} \right] \\ &= (\ln q)^2 \sum_{n=1}^{\infty} \frac{nkq^{nkt}(q^{nk\sum_{i=1}^{\alpha} \beta_i s_i} - 1)}{1-q^{nk}} \leq 0. \end{aligned}$$

That implies F is non-increasing. Also,

$$\begin{aligned} \lim_{t\to\infty} F(t) &= \lim_{t\to\infty} \left[\psi_{q,k}(t + \sum_{i=1}^{\alpha} \beta_i s_i) - \psi_{q,k}(t) - \sum_{i=1}^{\alpha} \beta_i \psi_{q,k}(s_i) \right] \\ &= \sum_{i=1}^{\alpha} \beta_i \frac{\ln(1-q)}{k} \\ &\quad + (\ln q) \lim_{t\to\infty} \sum_{n=1}^{\infty} \left[\frac{q^{nk(t+\sum_{i=1}^{\alpha} \beta_i s_i)}}{1-q^{nk}} - \frac{q^{nkt}}{1-q^{nk}} - \sum_{i=1}^{\alpha} \beta_i \frac{q^{nks_i}}{1-q^{nk}} \right] \\ &= -\sum_{i=1}^{\alpha} \beta_i \left[\frac{-\ln(1-q)}{k} + (\ln q) \sum_{n=1}^{\infty} \frac{q^{nks_i}}{1-q^{nk}} \right] \\ &= -\sum_{i=1}^{\alpha} \beta_i \psi_{q,k}(s_i) \geq 0. \quad \text{(Note: } \psi_{q,k}(t) < 0 \text{ for } 0 < t \leq 1) \end{aligned}$$

Therefore $F(t) \geq 0$ concluding the proof.

Theorem 2.2. *Let $q \in (0,1)$, $k > 0$, $t > 0$, $\beta_i > 0$ and $s_i > 0$ for all $i \in N_\alpha$. Suppose that m is a positive odd integer, then the following inequality is valid.*

$$\psi_{q,k}^{(m)}(t + \sum_{i=1}^{\alpha} \beta_i s_i) \leq \psi_{q,k}^{(m)}(t) + \sum_{i=1}^{\alpha} \beta_i \psi_{q,k}^{(m)}(s_i) \tag{8}$$

Proof. Let $G(t) = \psi_{q,k}^{(m)}(t + \sum_{i=1}^{\alpha} \beta_i s_i) - \psi_{q,k}^{(m)}(t) - \sum_{i=1}^{\alpha} \beta_i \psi_{q,k}^{(m)}(s_i)$. Then fixing s_i for each i we have,

$$\begin{aligned} G'(t) &= \psi_{q,k}^{(m+1)}(t + \sum_{i=1}^{\alpha} \beta_i s_i) - \psi_{q,k}^{(m+1)}(t) \\ &= (\ln q)^{m+2} \sum_{n=1}^{\infty} \left[\frac{n^{m+1} k^{m+1} q^{nk(t+\sum_{i=1}^{\alpha} \beta_i s_i)}}{1 - q^{nk}} - \frac{n^{m+1} k^{m+1} q^{nkt}}{1 - q^{nk}} \right] \\ &= (\ln q)^{m+2} \sum_{n=1}^{\infty} \left[\frac{n^{m+1} k^{m+1} q^{nkt} (q^{nk \sum_{i=1}^{\alpha} \beta_i s_i} - 1)}{1 - q^{nk}} \right] \geq 0. \text{ (since } m \text{ is odd)} \end{aligned}$$

That implies G is non-decreasing. Also,

$$\begin{aligned} \lim_{t\to\infty} G(t) &= \lim_{t\to\infty} \left[\psi_{q,k}^{(m)}(t + \sum_{i=1}^{\alpha} \beta_i s_i) - \psi_{q,k}^{(m)}(t) - \sum_{i=1}^{\alpha} \beta_i \psi_{q,k}^{(m)}(s_i) \right] \\ &= (\ln q)^{m+1} \quad \times \\ &\lim_{t\to\infty} \sum_{n=1}^{\infty} \left[\frac{n^m k^m q^{nk(t+\sum_{i=1}^{\alpha} \beta_i s_i)}}{1 - q^{nk}} - \frac{n^m k^m q^{nkt}}{1 - q^{nk}} - \sum_{i=1}^{\alpha} \beta_i \frac{n^m k^m q^{nks_i}}{1 - q^{nk}} \right] \\ &= -(\ln q)^{m+1} \sum_{n=1}^{\infty} \sum_{i=1}^{\alpha} \beta_i \frac{n^m k^m q^{nks_i}}{1 - q^{nk}} \leq 0. \text{ (since } m \text{ is odd)} \end{aligned}$$

Therefore $G(t) \leq 0$ concluding the proof.

Theorem 2.3. *Let $q \in (0,1)$, $k > 0$, $t > 0$, $\beta_i > 0$ and $s_i > 0$ for all $i \in N_\alpha$. Suppose that m is a positive even integer, then the following inequality is valid.*

$$\psi_{q,k}^{(m)}(t + \sum_{i=1}^{\alpha} \beta_i s_i) \geq \psi_{q,k}^{(m)}(t) + \sum_{i=1}^{\alpha} \beta_i \psi_{q,k}^{(m)}(s_i) \tag{9}$$

Proof. Let $H(t) = \psi_{q,k}^{(m)}(t + \sum_{i=1}^{\alpha} \beta_i s_i) - \psi_{q,k}^{(m)}(t) - \sum_{i=1}^{\alpha} \beta_i \psi_{q,k}^{(m)}(s_i)$. Then fixing

s_i for each i we have,

$$\begin{aligned} H'(t) &= \psi_{q,k}^{(m+1)}(t+\sum_{i=1}^{\alpha}\beta_i s_i) - \psi_{q,k}^{(m+1)}(t) \\ &= (\ln q)^{m+2}\sum_{n=1}^{\infty}\left[\frac{n^{m+1}k^{m+1}q^{nk(t+\sum_{i=1}^{\alpha}\beta_i s_i)}}{1-q^{nk}} - \frac{n^{m+1}k^{m+1}q^{nkt}}{1-q^{nk}}\right] \\ &= (\ln q)^{m+2}\sum_{n=1}^{\infty}\left[\frac{n^{m+1}k^{m+1}q^{nkt}(q^{nk\sum_{i=1}^{\alpha}\beta_i s_i}-1)}{1-q^{nk}}\right] \leq 0. \text{ (since } m \text{ is even)} \end{aligned}$$

That implies H is non-increasing. Also,

$$\begin{aligned} \lim_{t\to\infty} H(t) &= \lim_{t\to\infty}\left[\psi_{q,k}^{(m)}(t+\sum_{i=1}^{\alpha}\beta_i s_i) - \psi_{q,k}^{(m)}(t) - \sum_{i=1}^{\alpha}\beta_i\psi_{q,k}^{(m)}(s_i)\right] \\ &= (\ln q)^{m+1} \quad \times \\ &\lim_{t\to\infty}\sum_{n=1}^{\infty}\left[\frac{n^m k^m q^{nk(t+\sum_{i=1}^{\alpha}\beta_i s_i)}}{1-q^{nk}} - \frac{n^m k^m q^{nkt}}{1-q^{nk}} - \sum_{i=1}^{\alpha}\beta_i\frac{n^m k^m q^{nks_i}}{1-q^{nk}}\right] \\ &= -(\ln q)^{m+1}\sum_{n=1}^{\infty}\sum_{i=1}^{\alpha}\beta_i\frac{n^m k^m q^{nks_i}}{1-q^{nk}} \geq 0. \text{ (since } m \text{ is even)} \end{aligned}$$

Therefore $H(t) \geq 0$ concluding the proof.

Remark 2.4. If we let $q \to 1^-$ as $k \to 1$ in inequalities (7), (8) and (9) then we repectively recover the inequalities (4), (5) and (6).

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