Various operators and fuzzy relation in complete residuated lattices

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Abstract

In this paper, we investigate the properties of modal, necessity, sufficiency and co-sufficiency operators in a complete residuated lattice. In particular, we study the relationships between fuzzy relations and various operators.

Mathematics Subject Classification: 03E72, 06A15, 06B30

Keywords: Complete residuated lattices, modal, necessity, sufficiency and co-sufficiency operators

1 Introduction

Wille [9] introduced the structures on lattices by allowing some uncertainty in data. The structures on lattices are important mathematical tools for data analysis and knowledge processing [1,2,5-7,9]. Kim [5,6] investigated the properties of modal, necessity, sufficiency and co-sufficiency operators on sets. Hájek [3] introduced a complete residuated lattice which is an algebraic structure for many valued logic. It is an important mathematical tool for algebraic structure of fuzzy contexts [1,4,8].

In this paper, we investigate the properties of modal, necessity, sufficiency and co-sufficiency operators in a complete residuated lattice. In particular, we study the relationships between fuzzy relations and various operators.

2 Preliminaries

Definition 2.1 [3] A triple \((L, \lor, \land, \odot, \rightarrow, 0, 1)\) is called a complete residuated lattice iff it satisfies the following properties:
(L1) \( (L, \lor, \land, 1, 0) \) is a complete lattice where 1 is the universal upper bound and 0 denotes the universal lower bound;

(L2) \( (L, \circ, 1) \) is a commutative monoid;

(L3) It has an adjointness, i.e.
\[
x \leq y \rightarrow z \text{ iff } x \circ y \leq z.
\]

A map \( * : L \rightarrow L \) defined by \( a^* = a \rightarrow 0 \) is called a strong negation if \( b^* \leq a^* \) for \( a \leq b \) and \( a^{**} = a \).

Example 2.2 \([1,3,4,8]\) (1) Each frame \((L, \lor, \land, \circ) = (\land, \rightarrow, 0, 1)\) is a complete residuated lattice.

(2) The unit interval with a left-continuous t-norm \( \circ \), \([0,1], \lor, \land, \circ \rightarrow , 0, 1\), is a complete residuated lattice.

In this paper, we assume that \((L, \lor, \land, \circ, \rightarrow, ^*, 0, 1)\) be a complete residuated lattice with strong negation \( ^* \).

Lemma 2.3 \([1,3,4,8]\) For each \( x, y, z, x_i, y_i \in L \), we define \( x \rightarrow y = \lor \{ z \in L \mid x \circ z \leq y \} \). Then the following properties hold.

1. If \( y \leq z \), \((x \circ y) \leq (x \circ z) \) and \( x \rightarrow y \leq x \rightarrow z \) and \( z \rightarrow x \leq y \rightarrow x \).
2. \( x \circ y \leq x \land y \) and \( x \rightarrow (x \rightarrow y) \leq y \).
3. \( x \circ (\lor_{i \in F} y_i) = \lor_{i \in F} (x \circ y_i) \).
4. \( (\land_{i \in G} y_i) = \land_{i \in G} (x \rightarrow y_i) \).
5. \( (\lor_{i \in F} x_i) \rightarrow y = \land_{i \in G} (x_i \rightarrow y) \).
6. \( x \rightarrow (\lor_{i \in F} y_i) \geq \lor_{i \in F} (x \rightarrow y_i) \).
7. \( (\land_{i \in G} x_i) \rightarrow y \geq \lor_{i \in F} (x_i \rightarrow y) \).
8. \( \land_{i \in G} y_i^* = (\lor_{i \in G} y_i)^* \) and \( \lor_{i \in F} y_i^* = (\land_{i \in G} y_i)^* \).
9. \( (x \circ y) \rightarrow z = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z) \).
10. \( 1 \rightarrow x = x \) and \( x \rightarrow y = y^* \rightarrow x^* \).
11. \( x \leq y \text{ iff } x \rightarrow y = 1 \).
12. \( (x \rightarrow y) \circ (y \rightarrow z) \leq x \rightarrow z \).
13. \( (x \rightarrow y)^* = x \circ y^* \).

3 Various operators and fuzzy relation in complete residuated lattices

Definition 3.1 A map \( F : L^X \rightarrow L^Y \) is called:

1. modal operator if \( F(\lor_{i \in F} A_i) = \lor_{i \in F} F(A_i) \), \( F(\alpha \circ A) = \alpha \circ F(A) \),
2. necessity operator if \( F(\land_{i \in G} A_i) = \land_{i \in G} F(A_i) \), \( F(\alpha \rightarrow A) = \alpha \rightarrow F(A) \),

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(3) sufficiency operator if \( F(\bigvee_{i \in I} A_i) = \bigwedge_{i \in I} F(A_i), \) \( F(\alpha \odot A) = \alpha \rightarrow F(A), \)
(4) co-sufficiency operator if \( F(\bigwedge_{i \in I} A_i) = \bigvee_{i \in I} F(A_i), \) \( F(\alpha \rightarrow A) = \alpha \odot F(A). \)
(5) If \( F : L^X \rightarrow L^X \) is a map, then its dual operator \( F^\partial \) is defined by \( F^\partial(A) = F(A^\ast)^c \) where \( A^\ast(x) = A(x) \rightarrow 0. \) Moreover, its complementary counterpart \( F^\ast(A) = (F(A))^c \) and \( F^c(A) = F(A^\ast). \)

Remark 3.2 \([2,5,6,7]\) Let \( L = \{0, 1\} \) be given. We regard \( L^X, L^Y \) as \( P(X), P(Y), \) respectively. Then a map \( F : P(X) \rightarrow P(Y) \) is called
(1) a modal operator if \( F(\bigcup_{i \in I} A_i) = \bigvee_{i \in I} F(A_i), \) \( F(\emptyset) = \emptyset. \)
(2) a necessity operator if \( F(\bigcap_{i \in I} A_i) = \bigwedge_{i \in I} F(A_i), \) \( F(X) = Y. \)
(3) a sufficiency operator if \( F(\bigcup_{i \in I} A_i) = \bigvee_{i \in I} F(A_i), \) \( F(\emptyset) = Y. \)
(4) a co-sufficiency operator if \( F(\bigcap_{i \in I} A_i) = \bigwedge_{i \in I} F(A_i), \) \( F(X) = \emptyset. \)
(5) a dual operator \( F^\partial \) is defined by \( F^\partial(A) = F(A^\ast)^c. \) Moreover, its complementary counterpart \( F^\ast(A) = (F(A))^c \) and \( F^c(A) = F(A^\ast). \)

Definition 3.3 Let \( R \in L^{X \times Y} \) be a fuzzy relation. For each \( A \in L^X, \) we define operations \( R^{-1}(y, x) = R(x, y) \) and \([R], [[R)], \langle R \rangle, \langle\langle R \rangle\rangle, [R]^c, (R)^c : L^X \rightarrow L^Y \) as follows:
\[
[R](A)(y) = \bigwedge_{x \in X} (R(x, y) \rightarrow A(x)), \quad [[R]](A)(y) = \bigwedge_{x \in X} (A(x) \rightarrow R(x, y)), \\
\langle R \rangle(A)(y) = \bigvee_{x \in X} (R(x, y) \odot A(x)), \quad \langle\langle R \rangle\rangle(A)(y) = \bigvee_{x \in X} (R^\ast(x, y) \odot A^\ast(x)), \\
[R]^c(A)(y) = \bigwedge_{x \in X} (R(x, y) \rightarrow A^\ast(x)), \quad (R)^c(A)(y) = \bigvee_{x \in X} (R(x, y) \odot A^\ast(x)).
\]

Theorem 3.4 (1) A map \( F : L^X \rightarrow L^X \) is a modal operator iff \( F^\partial : L^X \rightarrow L^X \) is a necessity operator.
(2) A map \( F : L^X \rightarrow L^X \) is a sufficiency operator iff \( F^\partial : L^X \rightarrow L^X \) is a co-sufficiency operator.
(3) A map \( F : L^X \rightarrow L^X \) is a modal operator iff \( F^c : L^X \rightarrow L^X \) is a sufficient operator.
(4) A map \( F : L^X \rightarrow L^X \) is a sufficiency operator iff \( F^c : L^X \rightarrow L^X \) is a necessity operator.
(5) A map \( F : L^X \rightarrow L^X \) is a modal operator iff \( F^* : L^X \rightarrow L^X \) is a cosufficient operator.
(6) A map \( F : L^X \rightarrow L^X \) is a sufficiency operator iff \( F^* : L^X \rightarrow L^X \) is a necessity operator.
Proof. (1) Let \( F : L^X \to L^X \) be a modal operator.

\[
F^\delta(\bigwedge_{i \in \Gamma} A_i) = (F(\bigvee_{i \in \Gamma} A_i^*)^*) = (\bigvee_{i \in \Gamma} F(A_i^*))^* = \bigwedge_{i \in \Gamma}(F(A_i^*))^* = \bigwedge_{i \in \Gamma} F^\delta(A_i)
\]

\[
F^\delta(\alpha \to A) = (F((\alpha \to A)^*))^* = (F(\alpha \circ A^*))^* \quad (by \text{Lemma 2.3 (13)})
\]

\[
= (\alpha \circ F(A^*))^* = \alpha \to F(A)^* = \alpha \to F^\delta(A).
\]

Conversely, \((F^\delta)^\delta(A) = (F^\delta(A^*))^* = F(A)\).

\[
F(\bigvee_{i \in \Gamma} A_i) = (F(\bigwedge_{i \in \Gamma} A_i^*))^* = (\bigwedge_{i \in \Gamma} F^\delta(A_i^*))^* = \bigvee_{i \in \Gamma}(F^\delta(A_i^*))^* = \bigvee_{i \in \Gamma} F(A_i)
\]

\[
F(\alpha \circ A) = (F((\alpha \circ A)^*))^* = (F^\delta(\alpha \to A^*))^* = (\alpha \to F^\delta(A^*))^* = \alpha \circ F^\delta(A) = \alpha \circ F(A).
\]

(4) Let \( F : L^X \to L^X \) be a sufficiency operator.

\[
F^c(\bigwedge_{i \in \Gamma} A_i) = F((\bigwedge_{i \in \Gamma} A_i^*)^*) = F(\bigvee_{i \in \Gamma} A_i^*) = \bigwedge_{i \in \Gamma} F^c(A_i)
\]

\[
F^c(\alpha \to A) = F((\alpha \to A)^*) = F(\alpha \circ A^*) = (\alpha \to F(A)^*) = \alpha \to F^c(A).
\]

Conversely,

\[
F(\bigvee_{i \in \Gamma} A_i) = F^c(\bigwedge_{i \in \Gamma} A_i^*) = \bigwedge_{i \in \Gamma} F^c(A_i^*)
\]

\[
F(\alpha \circ A) = F^c((\alpha \circ A)^*) = F^c(\alpha \to A^*) = (\alpha \to F^c(A^*)) = \alpha \to F(A).
\]

Other cases are similarly proved.

Theorem 3.5 Let \( F, G : L^X \to L^Y \) be operators. Then the following properties hold:

1. \((F^\delta)^\delta = F, (F^*)^* = F \) and \((F^c)^c = F\).
2. \((F^\delta)^* = (F^*)^\delta, (F^\delta)^c = (F^c)^\delta \) and \((F^*)^c = (F^c)^* = F^\delta\).
3. \((F \lor G)^\delta = F^\delta \land G^\delta, (F \lor G)^* = F^* \lor G^* \) and \((F \lor G)^c = F^c \lor G^c\).
4. \( F, G : L^X \to L^Y \) are modal operators, then \( F \lor G \) is a model operator and its dual operator \( F^\delta \land G^\delta \) is a necessity operator.
5. \( F, G : L^X \to L^Y \) are necessity operators, then \( F \land G \) is a necessity operator and its dual operator \( F^\delta \lor G^\delta \) is a model operator.
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(1) \((F^\circ)^\circ(A) = (F^\circ(A^*))^* = F(A)\).

(2) \((F^\circ)^* = (F^\circ(A))^* = (F^\circ(A^*))^* = (F^\circ)^\circ(A)\).

(3) \((F \lor G)^\circ = (F \lor G)(A^*)^* = (F^\circ(A^*))^* = F^\circ(A) \land G^\circ(A)\).

(4) Since \(F, G\) are modal operators, then \((F \lor G)(\bigvee_{i \in \Gamma} A_i) = \bigvee_{i \in \Gamma} G(A_i) \lor F(A_i)\).

(5) Since \(F, G\) are necessity operators, then \((F \land G)(\bigwedge_{i \in \Gamma} A_i) = \bigwedge_{i \in \Gamma} F(A_i) \land G(A_i)\).

Theorem 3.6 Let \(R \in L^X \times Y\) be a fuzzy relation.

1. \(\langle R \rangle\) is a modal operator and \([R]\) is a necessity operator with \(\langle R \rangle(A) = ([R](A^*))^* = [R]^\circ(A)\), for each \(A \in L^X\).

2. If \(F : L^X \to L^Y\) is a modal operator on \(L^X\), there exists a unique fuzzy relation \(R_F \in L^X \times Y\) such that \(\langle R_F \rangle = F\) and \([R_F] = F^\circ\) where \(R_F(x, y) = F(1_x)\).

3. \(R(A) = \bigwedge_{i \in \Gamma} (R^\circ)(A_i)\).

Proof. (1) By Lemma 2.3, since

\[
[R] \bigwedge_{i \in \Gamma} A_i = \bigwedge_{j \in X} (R(x, y) \to A_i)(x) = \bigwedge_{i \in \Gamma} \bigwedge_{j \in X} (R(x, y) \to A_i(x))
\]

\[
[R] \bigwedge_{i \in \Gamma} A_i = \bigwedge_{i \in \Gamma} \bigwedge_{j \in X} (R(x, y) \to \bigwedge_{i \in \Gamma} A_i(y)) = \bigwedge_{i \in \Gamma} \bigwedge_{j \in X} (R(x, y) \to A_i(y))
\]

then \([R]\) is a necessity operator. We easily show \(\langle R \rangle\) is a modal operator. We have \(\langle R \rangle = [R]^\circ\) from:

\[
([R](A^*))^* = \bigwedge_{i \in \Gamma} (R(x, y) \to A_i(x))
\]

\[
([R](A^*))^* = \bigwedge_{i \in \Gamma} (R(x, y) \to A^i(x))
\]

\[
([R](A^*))^* = \bigwedge_{i \in \Gamma} (R(x, y) \to A^i(x))
\]

\[
([R](A^*))^* = \bigwedge_{i \in \Gamma} (R(x, y) \to A^i(x))
\]

(2) Since \(A = \bigvee_{x \in X} A(x) \odot 1_x\) and \(F\) is a modal operator, then \(F(A) = F(\bigvee_{x \in X} A(x) \odot 1_x) = \bigvee_{x \in X} F(1_x) \odot A(x)\).

Thus

\[
\langle R_F \rangle(A)(y) = \bigvee_{x \in X} (R_F(x, y) \to A(x)) = \bigvee_{x \in X} (F(1_x)(y) \to A(x))
\]

\[
\langle R_F \rangle(A)(y) = \bigvee_{x \in X} (R_F(x, y) \to A(x)) = \bigvee_{x \in X} (F(1_x)(y) \to A(x))
\]

\[
\langle R_F \rangle(A)(y) = \bigvee_{x \in X} (R_F(x, y) \to A(x)) = \bigvee_{x \in X} (F(1_x)(y) \to A(x))
\]

\[
(R_F(A))^* = F(\bigvee_{x \in X} 1_x \odot A(x) = F(A).
\]

\[
(F^\circ)^\circ = (F^\circ)^* = F(A).
\]

\[
(F^\circ)^* = (F^\circ)^* = F(A).
\]

\[
(F^\circ)^ = (F^\circ)^* = F(A).
\]

\[
(F^\circ)^* = (F^\circ)^* = F(A).
\]
(3) \( R_i(x, y) = (R)\(1_x\) (y) = V_{z \in X} (R(z, y) \cap 1_x(z)) = R(x, y). \)

**Theorem 3.7** Let \( R \in L^{X \times Y} \) be a fuzzy relation.

1. \( (\langle R \rangle)^c \) is a modal operator and \( [[R]]^c \) is a necessity operator with \( (\langle R \rangle)^c(A) = \left(\left[[R]\left(A^*\right)\right]\right)^c = [[R]]^c(A) \) for each \( A \in L^X \).
2. If \( F : L^X \to L^Y \) is a modal operator on \( L^X \), there exists a unique fuzzy relation \( R_F \in L^{X \times Y} \) such that \( (\langle R_F \rangle)^c = F \) and \( [[R_F]]^c = F^0 \) where \( R_F(x, y) = F(1_x)^c(y) \).
3. \( R_{\langle(R)\rangle^c} = R. \)

**Proof.**

1. Since \( [[R]]^c(A_i \Gamma) = \Lambda_{z \in X}(A_i \Gamma(A_i(x))^\ast \to R(x, y)) = \Lambda_{i \in \Gamma} \Lambda_{z \in X}(A_i \Gamma(A_i(x))^\ast \to R(x, y)) = \Lambda_{i \in \Gamma} \Lambda_{z \in X}(\left[[R]\left(A_i\right)\right] \to (\alpha \to A)(y) = \Lambda_{z \in X}(\left[[R]\left(A_i\right)\right](\alpha \to A)_\ast \to (\alpha \to A)(x)) = \alpha \to \left[[R]\left(A_i\right)\right]^c(y), \)
   \( [[R]]^c \) is a necessity operator. We easily show \( (\langle R \rangle)^c \) is a modal operator. We have \( (\langle R \rangle)^c(A) = (\left[[R]\left(A^*\right)\right]^c = [[R]]^c(A) \) from:
   \[
   (\left[[R]\left(A^*\right)\right]^c(y) = \left(\Lambda_{z \in X}(A(x) \to R(x, y))\right)^c = \left(\Lambda_{z \in X}(R^\ast(x, y) \circ A(x))^\ast \right)^c = \left(\Lambda_{z \in X}(R^\ast(x, y) \circ A(x))^\ast \right) = \left(\Lambda_{z \in X}(1_x \circ A(x))\right)(y) = R(x, y).
   \]

2. Since \( A = V_{z \in X} A(x) \circ 1_x \) and \( F(A) = V_{z \in X} A(x) \circ F(1_x) \), we have
   \[
   (\langle R_F \rangle)^c(A)(y) = \Lambda_{z \in X}(R^\ast_F(x, y) \circ A(x)) = \left(\Lambda_{z \in X}(F(1_x)(y) \circ A(x))\right)(y) = F(A)(y).
   \]

3. \( R_{\langle(R)\rangle^c} = R = F(1_x)^c(y). \)

**Theorem 3.8** Let \( R \in L^{X \times Y} \) be a fuzzy relation.

1. \( [[R]]^c \) is a sufficiency operator and \( \langle R \rangle \) is a co-sufficiency operator with \( \langle R \rangle(A) = (\left[[R]\left(A^*\right)\right]^c = [[R]]^c(A) \) for each \( A \in L^X \).
2. If \( F : L^X \to L^Y \) is a sufficiency operator on \( L^X \), there exists a unique fuzzy relation \( R_F \in L^{X \times Y} \) such that \( [[R_F]] = F \) and \( \langle R_F \rangle = F^0 \) where \( R_F(x, y) = F(1_x)(y) \).
3. \( R_{[[R]]^c} = R. \)
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**Proof.** (1) Since $\langle\langle R \rangle\rangle^c$ is a modal operator and $[[R]]^c$ is a necessity operator, by Theorem, $[[R]]^c$ is a sufficiency operator and $\langle\langle R \rangle\rangle$ is a co-sufficiency operator. We have $\langle\langle R \rangle\rangle(A) = (\langle\langle R \rangle\rangle(A^*))^* = [[R]]^0(A)$ from:

\[
\begin{align*}
\langle\langle R \rangle\rangle(A^*)(x) &= \left(\bigwedge_{y \in X}(A^*(y) \rightarrow R(x, y))\right)^* \\
&= \left(\bigwedge_{y \in X}(R^*(x, y) \circ A^*(y))^*\right)^* \\
&= \bigvee_{y \in X}(R^*(x, y) \circ A^*(y)) = \langle\langle R \rangle\rangle(A)(x)
\end{align*}
\]

(2) Since $F(\forall x \in X(A(x) \circ 1_x)) = \bigwedge_{x \in X}(A(x) \rightarrow F(1_x))$, we have

\[
[[R_F]](A)(y) = \bigwedge_{x \in X}(A(x) \rightarrow R_F(x, y)) = \bigwedge_{x \in X}(A(x) \rightarrow F(1_x)(y)) \\
= \bigwedge_{x \in X}(F(A(x) \circ 1_x)(y)) = F(\forall x \in X(A(x) \circ 1_x))(y) = F(A)(y).
\]

\[
\langle\langle R_F \rangle\rangle(A)(y) = \forall x \in X(R_F^*(x, y) \circ A^*(x)) = \forall x \in X(F(1_x)^*(y) \circ A^*(x)) \\
= \left(\bigwedge_{x \in X}(A^*(x) \rightarrow F(1_x)(y))^*\right)^* \\
= \left(F(\forall x \in X 1_x \circ A^*(x))(y))^* = F^0(A)(y).
\]

(3) $R_{[[R]]}(x, y) = [[R]](1_x)(y) = \bigwedge_{x \in X}(1_x(z) \rightarrow R(z, y)) = R(x, y).

**Theorem 3.9** Let $R \in L^X \times Y$ be a fuzzy relation.

(1) $[R]^c$ is a sufficiency operator and $\langle R \rangle^c$ is a co-sufficiency operator with $[R]^c(A) = (\langle R \rangle^c(A^*))^*$.

(2) If $F : L^X \rightarrow L^X$ is a sufficiency operator on $L^X$, there exists a unique fuzzy relation $R_F \in L^X \times Y$ such that $[R_F]^c = F$ and $\langle R_F \rangle^c = F^0$ where $R_F(x, y) = F(1_x)^*(y)$.

(3) $[R]^c = R$.

**Proof.** (1) Since $[R]$ is a necessity operator and $\langle R \rangle$ is a modal operator, by Theorem, $[R]^c$ is a sufficiency operator and $\langle R \rangle^c$ is a co-sufficiency operator. We have $[R]^c(A) = (\langle R \rangle^c(A^*))^*$ from

\[
\begin{align*}
\langle\langle R \rangle\rangle(A^*)(x) &= \left(\bigvee_{x \in X}(A(x) \circ R(x, y))\right)^* \\
&= \left(\bigwedge_{y \in X}(R(x, y) \rightarrow A^*(x))\right) \\
&= [R]^c(A)(x)
\end{align*}
\]

(2)

\[
\begin{align*}
[R_F]^c(A)(y) &= \bigwedge_{x \in X}(R_F(x, y) \rightarrow A^*(x)) \\
&= \bigwedge_{x \in X}(A(x) \rightarrow F(1_x)(y)) \\
&= \bigwedge_{x \in X}(F(A(x) \circ 1_x)(y)) \\
&= F(\forall x \in X(A(x) \circ 1_x))(y) = F(A)(y)
\end{align*}
\]
\[ (R_F)c(A)(y) = \bigvee_{x \in X}(R_F(x, y) \odot A^*(x)) \]
\[ = \bigvee_{x \in X}(F(1_x)^*(y) \odot A^*(x)) \]
\[ = \left( \bigwedge_{x \in X}(A^*(x) \rightarrow F(1_x)(y)) \right)^* \]
\[ = \left( F(\bigvee_{x \in X} 1_x \odot A^*(x))(y) \right)^* \]
\[ = F^\partial(A)(y) \]

(3) \( R_{[R^c]}(x, y) = [R^c(1_x)]^*(y) = \left( \bigwedge_{x \in X}(R(z, y) \rightarrow 1_x^*(z)) \right)^* = R(x, y). \)

**Theorem 3.10** Let \( R \in L^X \times Y \) be a fuzzy relation.

(1) If \( F : L^X \rightarrow L^Y \) is a necessity operator on \( L^X \), there exists a unique fuzzy relation \( R_F \in L^X \times Y \) such that [\( R_F \) = \( F \)] and \( (R_F) = F^\partial \) where \( R_F(x, y) = F(1_x^*)^*(y) \).

(2) \( R_{[R]} = R \).

**Proof.** (1)

\[ [R_F](A)(y) = \bigwedge_{x \in X}(R_F(x, y) \rightarrow A(x)) \]
\[ = \bigwedge_{x \in X}(F(1_x)^*(y) \rightarrow A(x)) \]
\[ = \bigwedge_{x \in X}(A^*(x) \rightarrow F(1_x^*)(y)) \]
\[ = \bigwedge_{x \in X} F(\bigvee_{x \in X}(A^*(x) \rightarrow 1_x^*(y))) \]
\[ = F((\bigwedge_{x \in X} A^*(x) \odot 1_x^*)^*)(y) \]
\[ = F((A^*)^*)(y) = F(A)(y) \]

(3) \( R_{[R]}(x, y) = [R](1_x^*)^*(y) = \left( \bigwedge_{x \in X}(R(z, y) \rightarrow 1_x^*(z)) \right)^* = R(x, y). \)

**Theorem 3.11** Let \( R \in L^X \times Y \) be a fuzzy relation.

(1) If \( F : L^X \rightarrow L^Y \) is a co-sufficiency operator on \( L^X \), there exists a unique fuzzy relation \( R_F \in L^X \times Y \) such that \( \langle R_F \rangle = F \) and \( [[R_F]] = F^\partial \) where \( R_F(x, y) = F(1_x^*)^*(y) \).

(2) \( R_{\langle R_F \rangle} = R \).
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Proof. (1)

\[ \langle (R_F) \rangle(A)(x) = \bigvee_{x \in X} (R_F^*(x, y) \odot A^*(x)) \]
\[ = \bigvee_{x \in X} (F(1^*_x)(y) \odot A^*(x)) \]
\[ = F(\bigwedge_{x \in X} (A^*(x) \rightarrow 1^*_x(y))) \]
\[ = F(\bigwedge_{x \in X} (A^*(x) \rightarrow 1^*_x))(y) \]
\[ = F((A^*)^*(y) = F(A)(y) \]

\[ [[R_F]](A)(y) = \bigwedge_{x \in X} (A(x) \rightarrow R_F(x, y)) \]
\[ = \bigwedge_{x \in X} (A(x) \rightarrow F(1^*_x)(y)) \]
\[ = \bigwedge_{x \in X} (A^*(x) \rightarrow F(1^*_x))(y) \]
\[ = \bigwedge_{x \in X} (A^*(x) \rightarrow 1^*_x))(y) \]
\[ = F((A^*)^*(y) = F(\partial A)(y) \]

(2) \( R_{\langle (R) \rangle}(x, y) = \langle (R_F) \rangle(1^*_x)(y) = (\bigvee_{y \in X} (R^*(z, y) \odot 1^*_x(z)))^* = R(x, y) \).

Theorem 3.12 Let \( R \in L^{X \times Y} \) be a fuzzy relation.

1. If \( F : L^X \rightarrow L^Y \) is a necessity operator on \( L^X \), there exists a unique fuzzy relation \( R_F \in L^{X \times Y} \) such that \( [[R_F]]^c = F \) and \( \langle (R_F) \rangle^c = F^\partial \) where \( R_F(x, y) = F(1^*_x)(y) \).

Proof. (1) Since \( A = \bigwedge_{x \in X} (A^*(x) \rightarrow 1^*_x) \), we have:

\[ [[R_F]]^c(A)(y) = \bigwedge_{x \in X} (A^*(x) \rightarrow R_F(x, y)) \]
\[ = \bigwedge_{x \in X} (A^*(x) \rightarrow F(1^*_x)(y)) \]
\[ = F(\bigwedge_{x \in X} (A^*(x) \rightarrow 1^*_x))(y) \]
\[ = F(A)(y). \]

Since \( A^* = \bigwedge_{x \in X} (A(x) \rightarrow 1^*_x) \), we have:

\[ \langle (R_F) \rangle^c(A)(y) = \bigvee_{x \in X} (R_F^*(x, y) \odot A(x)) \]
\[ = \bigvee_{x \in X} (F(1^*_x)(y) \odot A(x)) \]
\[ = \bigwedge_{x \in X} (A(x) \rightarrow F(1^*_x))(y) \]
\[ = \bigwedge_{x \in X} (A(x) \rightarrow 1^*_x))(y) \]
\[ = F((A^*)^*(y) = F^\partial(A)(y). \]

(3) \( R_{[[R_F]]^c}(x, y) = [[R_F]]^c(1^*_x)(y) = \bigwedge_{x \in X} (1^*_x(z) \rightarrow R(z, y)) = R(x, y). \)
Theorem 3.13 Let $R \in L^{X \times Y}$ be a fuzzy relation.

(1) If $F : L^X \to L^Y$ is a co-sufficiency operator on $L^X$, there exists a unique fuzzy relation $R_F \in L^{X \times Y}$ such that $\langle R_F \rangle^c = F$ and $[R_F]^c = F^\partial$ where $R_F(x, y) = F(1^*_x)(y)$.

(2) $R(\langle R_F \rangle^c) = R$.

Proof. (1)

$$\langle R_F \rangle^c(y) = \bigvee_{x \in X} (R_F(x, y) \odot A^*(x))$$

$$= \bigvee_{x \in X} (F(1^*_x)(y) \odot A^*(x))$$

$$= F(\bigwedge_{x \in X} (A^*(x) \rightarrow 1^*_y))(y)$$

$$= F(A)(y)$$

$$[R_F]^c(A)(y) = \bigwedge_{x \in X} (R(x, y) \rightarrow A^*(x))$$

$$= \left( \bigvee_{x \in X} (A(x) \odot F(1^*_x)(y)) \right)^*$$

$$= F(\bigwedge_{x \in X} (A(x) \rightarrow 1^*_y))(y)^*$$

$$= F(A^*)^c(y) = F^\partial(A)(y)$$

(2) $R(\langle R_F \rangle^c)(x, y) = \langle R_F \rangle^c(1^*_x)(y) = \bigvee_{z \in X} (R(z, y) \odot 1_x(z)) = R(x, y)$.

Example 3.14 Let $X = \{a, b, c\}$ and $Y = \{x, y\}$ be a set and $(L = [0, 1], \odot)$ with $a \odot b = \max\{0, a + b - 1\}$ and $a \rightarrow b = \min\{1, 1 - a + b\}$. Define $F, G : L^X \rightarrow L^Y$ as

$$F(1_a)(x) = 0.6, F(1_b)(x) = 0.2, F(1_c)(x) = 0.7$$

$$F(1_a)(y) = 0.5, F(1_b)(y) = 1.0, F(1_c)(y) = 0.9$$

$$G(1^*_a)(x) = 0.8, G(1^*_b)(x) = 0.6, G(1^*_c)(x) = 0.5$$

$$G(1^*_a)(y) = 0.7, G(1^*_b)(y) = 0.9, G(1^*_c)(y) = 0.4$$

(1) If $F$ is a modal operator, then, by Theorem 3.6,

$$R_F = \begin{pmatrix} 0.6 & 0.5 \\ 0.2 & 1.0 \\ 0.7 & 0.9 \end{pmatrix}$$

$$\langle R_F \rangle(A) = F(A) = \begin{pmatrix} 0.6 \odot A(a) \lor 0.2 \odot A(b) \lor 0.7 \odot A(c) \\ 0.5 \odot A(a) \lor A(b) \lor 0.9 \odot A(c) \end{pmatrix}$$

$$[R_F](A) = F^\partial(A) = \begin{pmatrix} (0.6 \rightarrow A(a)) \land (0.2 \rightarrow A(b)) \land (0.7 \rightarrow A(c)) \\ (0.5 \rightarrow A(a)) \land (1.0 \rightarrow A(b)) \land (0.9 \rightarrow A(c)) \end{pmatrix}$$
(2) If $F$ is a modal operator, then, by Theorem 3.7,

$$R_F = \begin{pmatrix} 0.4 & 0.5 \\ 0.8 & 0.0 \\ 0.3 & 0.1 \end{pmatrix}$$

$$\langle \langle R_F \rangle \rangle^c(A) = F(A) = \begin{pmatrix} 0.6 \odot A(a) \lor 0.2 \odot A(b) \lor 0.7 \odot A(c) \\ 0.5 \odot A(a) \lor A(b) \lor 0.9 \odot A(c) \end{pmatrix}$$

Since $[[R_F]]^c(A)(y) = \bigwedge_{x \in X}(R^*(x, y) \rightarrow A(x))$, we have

$$[[R_F]]^c(A) = F^{\partial}(A) = \begin{pmatrix} (0.6 \rightarrow A(a)) \land (0.2 \rightarrow A(b)) \land (0.7 \rightarrow A(c)) \\ (0.5 \rightarrow A(a)) \land (1.0 \rightarrow A(b)) \land (0.9 \rightarrow A(c)) \end{pmatrix}$$

(3) If $F$ is a sufficiency operator, then, by Theorem 3.8,

$$R_F = \begin{pmatrix} 0.6 & 0.5 \\ 0.2 & 1.0 \\ 0.7 & 0.9 \end{pmatrix}$$

$$[[R_F]](A) = F(A) = \begin{pmatrix} (A(a) \rightarrow 0.6) \land (A(b) \rightarrow 0.2) \land (A(c) \rightarrow 0.7) \\ (A(a) \rightarrow 0.5) \land (A(c) \rightarrow 0.9) \end{pmatrix}$$

$$\langle \langle R_F \rangle \rangle(A) = F^{\partial}(A) = \begin{pmatrix} 0.4 \odot A^*(a) \lor 0.8 \odot A^*(b) \lor 0.3 \odot A^*(c) \\ 0.5 \odot A^*(a) \lor 0.1 \odot A^*(c) \end{pmatrix}$$

(4) If $F$ is a sufficiency operator, then, by Theorem 3.9,

$$R_F = \begin{pmatrix} 0.4 & 0.5 \\ 0.8 & 0.0 \\ 0.3 & 0.1 \end{pmatrix}$$

$$[R_F]^c(A) = F(A) = \begin{pmatrix} (A(a) \rightarrow 0.6) \land (A(b) \rightarrow 0.2) \land (A(c) \rightarrow 0.7) \\ (A(a) \rightarrow 0.5) \land (A(c) \rightarrow 0.9) \end{pmatrix}$$

$$\langle \langle R_F \rangle \rangle^c(A) = F^{\partial}(A) = \begin{pmatrix} 0.4 \odot A^*(a) \lor 0.8 \odot A^*(b) \lor 0.3 \odot A^*(c) \\ 0.5 \odot A^*(a) \lor 0.1 \odot A^*(c) \end{pmatrix}$$

(5) If $G$ is a necessity operator, then, by Theorem 3.10,

$$R_G = \begin{pmatrix} 0.8 & 0.7 \\ 0.6 & 0.9 \\ 0.5 & 0.4 \end{pmatrix}$$

$$[R_G](A) = G(A) = \begin{pmatrix} (0.8 \rightarrow A(a)) \land (0.6 \rightarrow A(b)) \land (0.5 \rightarrow A(c)) \\ (0.7 \rightarrow A(a)) \land (0.9 \rightarrow A(b)) \land (0.4 \rightarrow A(c)) \end{pmatrix}$$
\[ \langle R_G \rangle^c(A) = G^\partial(A) = \left( \begin{array}{c} 0.8 \odot A(a) \lor 0.4 \odot A(b) \lor 0.5 \odot A(c) \\ 0.7 \odot A(a) \lor 0.9 \odot A(b) \lor 0.4 \odot A(c) \end{array} \right) \]

(6) If \( G \) is a necessity operator, then, by Theorem 3.12,

\[
R_G = \left( \begin{array}{cc} 0.2 & 0.3 \\ 0.4 & 0.1 \\ 0.5 & 0.6 \end{array} \right)
\]

\[ [[R_G]]^c(A) = G(A) = \left( \begin{array}{c} (0.8 \rightarrow A(a)) \land (0.6 \rightarrow A(b)) \land (0.5 \rightarrow A(c)) \\ (0.7 \rightarrow A(a)) \land (0.9 \rightarrow A(b)) \land (0.4 \rightarrow A(c)) \end{array} \right) \]

\[ \langle \langle R_G \rangle \rangle^c(A) = G^\partial(A) = \left( \begin{array}{c} 0.8 \odot A(a) \lor 0.4 \odot A(b) \lor 0.5 \odot A(c) \\ 0.7 \odot A(a) \lor 0.9 \odot A(b) \lor 0.4 \odot A(c) \end{array} \right) \]

(7) If \( G \) is a co-sufficiency operator, then \( G(A) = G(\bigwedge_{x \in X} (A^*(x) \rightarrow 1^*_x)) = \bigvee_{x \in X} (A^*(x) \odot G(1^*_x)) \). By Theorem 3.11, we have:

\[
R_G = \left( \begin{array}{cc} 0.8 & 0.7 \\ 0.6 & 0.9 \\ 0.5 & 0.4 \end{array} \right)
\]

\[ \langle \langle R_G \rangle \rangle(A) = G(A) = \left( \begin{array}{c} (0.2 \odot A^*(a)) \lor (0.4 \odot A^*(b)) \lor (0.5 \odot A^*(c)) \\ (0.3 \odot A^*(a)) \lor (0.1 \odot A^*(b)) \lor (0.6 \odot A^*(c)) \end{array} \right) \]

\[ [[R_G]](A) = G^\partial(A) = \left( \begin{array}{c} (A(a) \rightarrow 0.8) \land (A(b) \rightarrow 0.6) \land (A(c) \rightarrow 0.5) \\ (A(a) \rightarrow 0.7) \land (A(b) \rightarrow 0.9) \land (A(c) \rightarrow 0.4) \end{array} \right) \]

(8) If \( G \) is a co-sufficiency operator, then by Theorem 3.13, we have:

\[
R_G = \left( \begin{array}{cc} 0.2 & 0.3 \\ 0.4 & 0.1 \\ 0.5 & 0.6 \end{array} \right)
\]

\[ (R_G)^c(A) = G(A) = \left( \begin{array}{c} (0.2 \odot A^*(a)) \lor (0.4 \odot A^*(b)) \lor (0.5 \odot A^*(c)) \\ (0.3 \odot A^*(a)) \lor (0.1 \odot A^*(b)) \lor (0.6 \odot A^*(c)) \end{array} \right) \]

\[ [R_G]^c(A) = G^\partial(A) = \left( \begin{array}{c} (A(a) \rightarrow 0.8) \land (A(b) \rightarrow 0.6) \land (A(c) \rightarrow 0.5) \\ (A(a) \rightarrow 0.7) \land (A(b) \rightarrow 0.9) \land (A(c) \rightarrow 0.4) \end{array} \right) \]

References


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Received: May, 2014