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Strongly concave set-valued maps

Odalis Mejía

Universidad Central de Venezuela
Escuela de Matemáticas
Paseo Los Ilustres, VE-1020 A Caracas, Venezuela
email: odalism_18@yahoo.com

Nelson Merentes

Universidad Central de Venezuela
Escuela de Matemáticas
Paseo Los Ilustres, VE-1020 A Caracas, Venezuela
email: nmerucv@gmail.com

Kazimierz Nikodem

University of Bielsko-Biała
Department of Mathematics and Computer Science
ul. Willowa 2, 43-309 Bielsko-Biała, Poland
email: knikodem@ath.bielsko.pl

Abstract

The notion of strongly concave set-valued maps is introduced and some properties of it are presented. In particular, a Kuhn-type result as well as Bernstein-Doetsch and Sierpiński-type theorems for strongly midconcave set-valued maps are obtained. A representation of strongly concave set-valued maps in inner product spaces is given.

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1 Introduction

Let $(X, ||\cdot||)$ be a normed space, D be a convex subset of X and let $c > 0$. A function $f : D \to \mathbf{R}$ is called strongly convex with modulus c if

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2) - ct(1-t)||x_1 - x_2||^2 \qquad (1)$$

for all $x_1, x_2 \in D$ and $t \in [0, 1]$; f is called strongly concave with modulus c if $-f$ is strongly convex with modulus c.

Strongly convex functions were introduced in [16] and many properties and applications of them can be found in the literature (see, for instance [1], [10], [11], [14], [15], [19], [20], [21], and the references therein). Recently, Huang [5], [6] extended the definition (1) of strong convexity to set-valued maps (see also [4], [9]). In this note we introduce the notion of strongly concave (t-concave, midconcave) set-valued maps and present some properties of them. In particular, we prove a Kuhn-type result stating that strongly t-concave set-valued maps are strongly midconcave and give conditions under which strongly midconcave set-valued maps are continuous and strongly concave. We give also some representation of strongly concave set-valued maps in inner product spaces and present a characterization of inner product spaces involving this representation. Our paper is strictly related to [9] where analogous results for strongly convex set-valued maps are presented

For real-valued functions properties of strongly convex and strongly concave functions are quite analogous and, in view of the fact that f is strongly concave if and only if $-f$ is strongly convex, it is not needed to investigate functions of these two kinds individually. However, in the case of set-valued maps the situation is different. If F is strongly convex then $-F$ is also strongly convex and even if some properties of strongly convex and strongly concave set-valued maps are similar, they hold, in general, under different assumptions and have to be proved separately.

2 Preliminary Notes

Let $(X, ||\cdot||)$ and $(Y, ||\cdot||)$ be real normed spaces and D be a convex subset of X. Throughout this paper B denotes the closed unit ball in Y. We denote by $n(Y)$ the family of all nonempty subsets of Y, and by $conv(Y)$ and $cconv(Y)$ the subfamilies of $n(Y)$ of all convex and compact convex sets, respectively.

Definition 2.1 *Let $t \in (0, 1)$ and $c > 0$. We say that a set-valued map $F : D \to n(Y)$ is* strongly t-concave with modulus c *if*

$$F(tx_1 + (1-t)x_2) + ct(1-t)||x_1 - x_2||^2 B \subset tF(x_1) + (1-t)F(x_2), \qquad (2)$$

for all $x_1, x_2 \in D$. F *is called* strongly concave with modulus c *if it satisfies (2) for every* $t \in [0,1]$ *and all* $x_1, x_2 \in D$.

Definition 2.2 *We say that* F *is* strongly midconcave with modulus c *if it satisfies (2) with* $t = 1/2$*, that is*

$$F\left(\frac{x_1+x_2}{2}\right) + \frac{c}{4}||x_1 - x_2||^2 B \subset \frac{1}{2}F(x_1) + \frac{1}{2}F(x_2), \tag{3}$$

for all $x_1, x_2 \in D$.

Clearly, the above definitions are motivated by the definition (1) of strongly convex functions. The standard definition of concave set-valued maps corresponds to (2) with $c = 0$ (cf. e.g. [2], [12], [17]).

Example 2.3 *Let* $f_1, f_2 : D \to \mathbf{R}$ *and* $f_1(x) \leq f_2(x),\ \ x \in D$. *Then the set-valued map* $F : D \to cconv(\mathbf{R})$ *defined by* $F(x) = [f_1(x), f_2(x)],\ x \in D$, *is strongly concave with modulus* c *if and only if* f_1 *is strongly concave with modulus* c *and* f_2 *is strongly convex with modulus* c.

Example 2.4 *Let* $I \subset \mathbf{R}$ *be an interval. The set-valued map* $F : I \to conv(Y)$ *defined by* $F(s) = s^2 B$ *is strongly concave with modulus 1. More general, if* $A \subset Y$ *is convex and* $cB \subset A$ *for some* $c > 0$*, then* $F(s) = s^2 A,\ \ s \in I$*, is strongly concave with modulus* c.

Example 2.5 *If* $G : I \to n(Y)$ *is concave, then* $F(s) = G(s) + cs^2 B,\ s \in I$*, is strongly concave with modulus* c. *In particular, if* $A_1, A_2 \in n(Y)$*, then* $F(s) = A_1 + sA_2 + cs^2 B,\ s \in I$*, is strongly concave with modulus* c.

The following lemma will be used in the sequel.

Lemma 2.6 *If* $F : D \to conv(Y)$ *is strongly midconcave with modulus* c *then*

$$F\left(\frac{k}{2^n}x_1 + \left(1 - \frac{k}{2^n}\right)x_2\right) + c\frac{k}{2^n}\left(1 - \frac{k}{2^n}\right)||x_1 - x_2||^2 B$$
$$\subset \frac{k}{2^n}F(x_1) + \left(1 - \frac{k}{2^n}\right)F(x_2), \tag{4}$$

for all $x_1, x_2 \in D$ *and* $k, n \in \mathbf{N}$ *such that* $k < 2^n$.

The proof of the above lemma is similar to the proof of an analogous result for strongly convex set-valued maps (see [9], Lemma 1), therefore we omit it. Note, however, that in contrast with the case of strongly convex set-valued maps (where $F : D \to n(Y)$), we assume now that F has convex values. The example below shows that without this assumption the assertion of Lemma 2.6 is not true.

Example 2.7 *Let* $G : \mathbf{R} \to n(\mathbf{R}^2)$ *be given by*

$$G(s) = \begin{cases} B & \text{if} \quad s = 0 \\ S & \text{if} \quad s \neq 0, \end{cases}$$

where $B = \{y \in \mathbf{R}^2 : ||y|| \leq 1\}$ *and* $S = \{y \in \mathbf{R}^2 : ||y|| = 1\}$. *Define* $F(s) = G(s) + s^2 B, \quad s \in \mathbf{R}$. *It is clear that* G *is midconcave (note that* $\frac{1}{2}(S + S) = B$*) and, consequently,* F *is strongly midconcave with modulus 1. However, if* $p \in \mathbf{R}$ *is such that* $0 < p < \sqrt{1/6}$ *and* $x_1 = -3p$, $x_2 = p$, *then*

$$F\Big(\frac{1}{4}(-3p) + \frac{3}{4}p\Big) + \frac{1}{4}\frac{3}{4}(4p)^2 B = F(0) + 3p^2 B = (1 + 3p^2)B$$
$$\not\subseteq \frac{1}{4}F(-3p) + \frac{3}{4}F(p) = \frac{1}{4}(S + 9p^2 B) + \frac{3}{4}(S + p^2 B) = \frac{1}{4}S + \frac{3}{4}S + 3p^2 B,$$

(because $0 \in (1 + 3p^2)B \backslash (\frac{1}{4}S + \frac{3}{4}S + 3p^2 B)$ *for* $0 < 3p^2 < \frac{1}{2}$*). Thus (4) does not hold for* F.

Recall also the Rådstrm cancelation law [18] which is a useful tool in our investigations.

Lemma 2.8 *Let* A_1, A_2, C *be subsets of* X *such that* $A_1 + C \subset A_2 + C$. *If* A_2 *is closed convex and* C *is bounded and nonempty, then* $A_1 \subset A_2$.

3 Kuhn-type result

It is known by the Kuhn theorem that t-convex functions (with arbitrarily fixed $t \in (0, 1)$) are midconvex. Similar results hold also for t-convex set-valued maps (see [3]) and strongly t-convex set-valued maps (see [9]). In this section we present a counterpart of those results for strongly t-concave set-valued maps. The idea of the proof is taken from [8], Lemma 1.

Theorem 3.1 *Let* D *be a convex subset of* X *and* $t \in (0, 1)$ *be a fixed number. If a set-valued map* $F : D \to cconv(Y)$ *is strongly* t*-concave with modulus* c*, then it is strongly midconcave with modulus* c.

Proof. Fix $x_1, x_2 \in D$ and put $z := \frac{x_1+x_2}{2}$, $u := (1 - t)x_1 + tz$ and $v := (1 - t)z + tx_2$. Note that $z = tu + (1 - t)v$. Since

$$||x_1 - z|| = ||z - x_2|| = ||u - v|| = \frac{1}{2}||x_1 - x_2||,$$

we have

$$\frac{1}{2}t(1 - t)||x_1 - x_2||^2 B$$
$$= t^2(1 - t)||x_1 - z||^2 B + t(1 - t)^2||z - x_2||^2 B + t(1 - t)||u - v||^2 B.$$

Using this equality and applying three times condition (2) in the definition of strong t-concavity, we obtain

$$
\begin{aligned}
&2t(1-t)F(z) + \frac{1}{2}t(1-t)c||x_1 - x_2||^2 B + tF(u) + (1-t)F(v) + F(z) \\
&= 2t(1-t)F(z) + c[t^2(1-t)||x_1 - z||^2 B + t(1-t)^2||z - x_2||^2 B \\
&\quad +1(1-t)||u - v||^2 B] + tF(u) + (1-t)F(v) + F(z) \\
&= 2t(1-t)F(z) + t[F(u) + ct(1-t)||x_1 - z||^2 B] + (1-t)[F(v) \\
&\quad +c(1-t)t||z - x_2||^2 B] + [F(z) + ct(1-t)||u - v||^2 B] \\
&\subset 2t(1-t)F(z) + t(tF(z) + (1-t)F(x_1)) + (1-t)(tF(x_2) + (1-t)F(z)) \\
&\quad +tF(u) + (1-t)F(v) \\
&= t(1-t)F(x_1) + t(1-t)F(x_2) + 2t(1-t)F(z) + t^2F(z) + (1-t)^2F(z) \\
&\quad +tF(u) + (1-t)F(v) \\
&= t(1-t)F(x_1) + t(1-t)F(x_2) + F(z) + tF(u) + (1-t)F(v).
\end{aligned}
$$

By Lemma 2.8, we get

$$
2t(1-t)F\left(\frac{x_1 + x_2}{2}\right) + \frac{1}{2}t(1-t)c||x_1 - x_2||^2 B \subset t(1-t)F(x_1) + t(1-t)F(x_2).
$$

Hence,

$$
F\left(\frac{x_1 + x_2}{2}\right) + \frac{c}{4}||x_1 - x_2||^2 B \subset \frac{1}{2}F(x_1) + \frac{1}{2}F(x_2),
$$

which finishes the proof.

□

4 Bernstein-Doetsch and Sierpiński-type results

A set-valued function $F : D \to n(Y)$ is said to be continuous (with respect to the Hausdorff topology on $n(Y)$) at a point $x_0 \in D$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$
F(x_0) \subset F(x) + \varepsilon B \tag{5}
$$

and

$$
F(x) \subset F(x_0) + \varepsilon B \tag{6}
$$

for every $x \in D$ such that $||x - x_0|| < \delta$. If we assume only condition (5) (condition (6)) F is said to be lower semicontinuous (upper semicontinuous) at x_0.

The next theorem gives a condition under which strongly midconcave set-valued maps with compact convex values are strongly concave. Analogous result for strongly midconvex set-valued maps with bounded closed values is presented in [9].

Theorem 4.1 *If $F : D \to cconv(Y)$ is strongly midconcave with modulus c and lower semicontinuous on D, then it is strongly concave with modulus c.*

Proof. Let $x_1, x_2 \in D$ and $t \in (0,1)$. Take a sequence (q_n) of dyadic numbers in $(0,1)$ tending to t and fix an $\varepsilon > 0$. Since the set-valued functions of the form $\mathbf{R} \ni s \to sA \in n(Y)$ are continuous provided the set A is bounded (see e.g. [13], Lemma 3.2), we have

$$q_n F(x_1) \subset tF(x_1) + \varepsilon B, \tag{7}$$

$$(1-q_n)F(x_2) \subset (1-t)F(x_2) + \varepsilon B \tag{8}$$

and

$$ct(1-t)||x_1 - x_2||^2 B \subset cq_n(1-q_n)||x_1 - x_2||^2 B + \varepsilon B \tag{9}$$

for all $n \geq n_1$. By the lower semicontinuity of F at the point $tx_1 + (1-t)x_2$, we get

$$F(tx_1 + (1-t)x_2) \subset F(q_n x_1 + (1-q_n)x_2) + \varepsilon B, \tag{10}$$

for all $n \geq n_2$. Hence, using (7), (8), (9), (10) and Lemma 2.6, we obtain

$$\begin{aligned}
&F(tx_1 + (1-t)x_2) + ct(1-t)||x_1 - x_2||^2 B \\
&\subset F(q_n x_1 + (1-q_n)x_2) + cq_n(1-q_n)||x_1 - x_2||^2 B + 2\varepsilon B \\
&\subset q_n F(x_1) + (1-q_n)F(x_2) + 2\varepsilon B \\
&\subset tF(x_1) + \varepsilon B + (1-t)F(x_2) + \varepsilon B + 2\varepsilon B \\
&= tF(x_1) + (1-t)F(x_2) + 4\varepsilon B,
\end{aligned}$$

for all $n \geq max\{n_1, n_2\}$. Since the above inclusions hold for every $\varepsilon > 0$, we have also

$$\begin{aligned}
&F(tx_1 + (1-t)x_2) + ct(1-t)||x_1 - x_2||^2 B \\
&\subset \bigcap_{\varepsilon > 0} (tF(x_1) + (1-t)F(x_2) + 4\varepsilon B) \\
&= cl(tF(x_1) + (1-t)F(x_2)) \\
&= tF(x_1) + (1-t)F(x_2).
\end{aligned}$$

This shows that F is strongly concave with modulus c and completes the proof.

□

It is known that midconcave set-valued maps that satisfy some regularity assumptions, such as upper semicontinuity at a point or boundedness on a set with nonempty interior or measurability are continuous in the interior of their domains (see, for instance, [2], [12], [13], [17]). Therefore, as a consequence of Theorem 3.1, Theorem 4.1 and those results we obtain the following corollaries. Here D is assumed to be an open convex subset of X.

Corollary 4.2 *Let $t \in (0, 1)$. If a set-valued map $F : D \to cconv(Y)$ is strongly t-concave with modulus c and upper semicontinuous at a point of D, then it is continuous and strongly concave with modulus c.*

A set-valued map $F : D \to n(Y)$ is said to be bounded on a set $A \subset D$ if there is a constant $M > 0$ such that $||y|| < M$ for every $y \in F(x)$ and $x \in A$. $F : \mathbf{R}^n \supset D \to n(Y)$ is said to be Lebesgue measurable if for every open set $U \subset Y$ the set $\{x \in D : F(x) \subset U\}$ is measurable in the sense of Lebesgue.

The next two corollaries are counterparts of the celebrated Bernstain-Doetsch and Sierpiński theorems for midconvex real functions (see, e.g. [7], [19]; cf. also [4]).

Corollary 4.3 *Let $t \in (0, 1)$. If a set-valued map $F : D \to cconv(Y)$ is strongly t-concave with modulus c and bounded on a set $A \subset D$ with a nonempty interior, then it is continuous and strongly concave with modulus c.*

Corollary 4.4 *Let $t \in (0, 1)$. If a set-valued map $F : \mathbf{R}^n \supset D \to cconv(Y)$ is strongly t-concave with modulus c and Lebesgue measurable, then it is continuous and strongly concave with modulus c.*

5 A representation Theorem

In the case where $(X, ||\cdot||)$ is a real inner product space (that is the norm $||\cdot||$ is induced by an inner product $||x|| = \sqrt{< x, x >}$), there is a strict relationship between strongly concave and concave set-valued maps. Namely, the following theorem holds.

Theorem 5.1 *Let $(X, ||\cdot||)$ be a real inner product space, D be a convex subset of X and c be a positive number. If $G : D \to n(Y)$ is concave, then the set-valued map $F : D \to n(Y)$ defined by $F(x) = G(x) + c||x||^2 B, x \in D$, is strongly concave with modulus c. Conversely, if $F : D \to cconv(Y)$ defined by $F(x) = G(x) + c||x||^2 B, x \in D$ is strongly concave with modulus c, then G is concave.*

Proof. Assume first that G is concave, that is

$$G(tx_1 + (1-t)x_2) \subset tG(x_1) + (1-t)G(x_2), \quad x_1, x_2 \in D,\ t \in [0,1].$$

Since

$$t||x_1||^2 + (1-t)||x_2||^2 = t(1-t)||x_1 - x_2||^2 + ||tx_1 + (1-t)x_2||^2, \tag{11}$$

we have

$$\begin{aligned}
&F(tx_1 + (1-t)x_2) + ct(1-t)||x_1 - x_2||^2 B \\
&= G(tx_1 + (1-t)x_2) + ct(1-t)||x_1 - x_2||^2 B + c||tx_1 + (1-t)x_2||^2 B \\
&\subset tG(x_1) + (1-t)G(x_2) + c(t||x_1||^2 + (1-t)||x_2||^2)B \\
&= t[G(x_1) + c||x_1||^2 B] + (1-t)[G(x_2) + c||x_2||^2 B] \\
&= tF(x_1) + (1-t)F(x_2),
\end{aligned}$$

which proves that F is strongly concave with modulus c.

Conversely, if F is strongly concave with modulus c, then

$$F(tx_1 + (1-t)x_2) + ct(1-t)||x_1 - x_2||^2 B \subset tF(x_1) + (1-t)F(x_2).$$

By the definition of F we get

$$\begin{aligned}
&G(tx_1 + (1-t)x_2) + c||tx_1 + (1-t)x_2||^2 B + ct(1-t)||x_1 - x_2||^2 B \\
&\subset t[G(x_1) + c||x_1||^2 B] + (1-t)[G(x_2) + c||x_2||^2 B]
\end{aligned}$$

and hence, by (11),

$$\begin{aligned}
&G(tx_1 + (1-t)x_2) + c[t||x_1||^2 + (1-t)||x_2||^2]B \\
&\subset tG(x_1) + (1-t)G(x_2) + c[t||x_1||^2 + (1-t)||x_2||^2]B.
\end{aligned}$$

Using Lemma 2.8 we obtain

$$G(tx_1 + (1-t)x_2) \subset tG(x_1) + (1-t)G(x_2),$$

which shows that G is concave.

□

As a consequence of the above theorem we obtain the following characterization of inner product spaces among normed spaces. Similar characterizations involving strongly convex functions and strongly convex set-valued maps were obtained in [14] and [9].

Theorem 5.2 *Let $(X, ||\cdot||)$ be a real normed space. The following conditions are equivalent:*

1. *$(X, ||\cdot||)$ is an inner product space;*

2. *For every $c > 0$ and for every concave set-valued map $G : D \to n(Y)$ defined on a convex set $D \subset X$, the set-valued map $F(x) = G(x) + c||x||^2 B$ is strongly concave with modulus c.*

3. *The set-valued map $F(x) = ||x||^2 B,\ x \in X$, is strongly concave with modulus 1.*

Proof. (1) $\Rightarrow$ (2) follows from Theorem 5.1.

To show that (2) $\Rightarrow$ (3) it is enough to take $G(x) = \{0\},\ x \in X$.

To prove (3) $\Rightarrow$ (1), observe that by the strong concavity of $F(\cdot) = ||\cdot||^2 B$ with modulus 1, we get

$$\left|\left|\frac{x_1 + x_2}{2}\right|\right|^2 B + \frac{1}{4}||x_1 - x_2||^2 B \subset \frac{1}{2}||x_1||^2 B + \frac{1}{2}||x_2||^2 B$$

for all $x_1, x_2 \in X$. Hence

$$(||x_1 + x_2||^2 + ||x_1 - x_2||^2)B \subset (2||x_1||^2 + 2||x_2||^2)B$$

and, consequently,

$$||x_1 + x_2||^2 + ||x_1 - x_2||^2 \leq 2||x_1||^2 + 2||x_2||^2,\ x_1, x_2 \in X.$$

Simple substitutions show that the converse inequality also holds. Thus, $||\cdot||$ satisfies the parallelogram law and, by the Jordan-von Neumann Theorem, $(X, ||\cdot||)$ is an inner product space.

□

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References

[1] A. Azócar, J. Giménez, K. Nikodem and J. L. Sánchez, *On strongly midconvex functions*, Opuscula Math. 31 (2011), 15–26.

[2] W. W. Breckner, *Continuity of generalized convex and generalized concave set-valued functions*, Rev. Anal. Numér. Théor. Approx. 22 (1993), 39–51.

[3] T. Cardinali, K. Nikodem and F. Papalini, *Some results on stability and on characterization of K-convexity of set-valued functions*, Ann. Polon. Math. 58 (1993), 185–192.

[4] C. Gonzlez, K. Nikodem, Z. Ples, G. Roa, *Bernstein-Doetsch type theorems for set-valed maps of strongly and approximately convex and concave type*, Publ. Math. Debrecen 84/1-2 (2014), 229-252.

[5] H. Huang, *Global error bounds with exponents for multifunctions with set constraints*, Commun. Contemp. Math. 12 (2010), 417–435.

[6] H. Huang, *Inversion theorem for nonconvex multifunctions*, Math. Inequal. Appl. 13 (2010), 841–849.

[7] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality*, PWN – Uniwersytet Śląski, Warszawa–Kraków–Katowice, 1985. Second Edition: Birkhäuser, Basel–Boston–Berlin, 2009.

[8] Z. Daróczy and Zs. Páles, *Convexity with given infinite weight sequences*, Stochastica, 11 (1987), no. 1, 512.

[9] H. Leiva, N. Merentes, K. Nikodem, J. L. Snchez, *Strongly convex set-valued maps*, J. Glob. Optim. 57 (2013), 695-705.

[10] N. Merentes and K. Nikodem, *Remarks on strongly convex functions*, Aequationes Math. 80 (2010), 193–199.

[11] L. Montrucchio, *Lipschitz continuous policy functions for strongly concave optimization problems*, J. Math. Econom. 16 (1987), pp. 259-273.

[12] K. Nikodem, *On concave and midpoint concave set-valued functions*, Glasnik Mat. 22 (1987), 69-76.

[13] K. Nikodem, *K-convex and K-concave set-valued functions*, Zeszyty Nauk. Politech. Łódz. Mat. 559 (Rozprawy Nauk 114), Łódź, 1989, pp. 1–75.

[14] K. Nikodem and Zs. Páles, *Characterizations of inner product spaces by strongly convex functions*, Banach J. Math. Anal. 5 (2011), no.1, 83–87.

[15] E. Polovinkin, *Strongly convex analysis*, Sb. Math. 187 (1996), 259–286.

[16] B. T. Polyak, *Existence theorems and convergence of minimizing sequences in extremum problems with restrictions*, Soviet Math. Dokl. 7 (1966), 72–75.

[17] D. Popa, *Semicontinuity of a class of generalized convex and a class of generalized concave set-valued maps*, Pure Math. Appl. 11 (2000), no.2, 369-374.

[18] H. Rådstrm, *An embedding theorem for spaces of convex sets*, Proc. Amer. Math. Soc. 3, (1952) 165169.

[19] A. W. Roberts and D. E. Varberg, *Convex Functions*, Academic Press, New York–London, 1973.

[20] T. Rajba, Sz. Wąsowicz, *Probabilistic characterization of strong convexity*, Opuscula Math. 31 (2011), 97–103.

[21] J. P. Vial, *Strong convexity of sets and functions*, J. Math. Econom. 9 (1982), 187–205.

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