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3-Lie algebra A with $I(A) = 3, 4$ I

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Abstract

The paper introduces the symplectic structure on 3-Lie algebras. And it is proved that: 1) The simple 3-Lie algebra is a metric symplectic 3-Lie algebra. 2). There does not exist metric structure on the non-simple and non-abelian 3-Lie algebra A with the generating index three and four. 3). Except for the cases of abelian and (a_4), there does not exist symplectic structure on the non-simple 3-Lie algebra A with the generating index three and four.

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1 Introduction

The concept of n-Lie algebra or Filippov algebra was introduced in 1985 ([1]). It is a vector space A endowed with an n-ary skew-symmetric multiplication satisfying the n-Jacobi identity $\forall x_1, \cdots, x_n, y_1, \cdots, y_{n-1} \in A$

$$[[x_1, \cdots, x_n], y_2, \cdots, y_n] = \sum_{i=1}^{n} [x_1, \cdots, [x_i, y_2, \cdots, y_n], \cdots, x_n]. \qquad (1)$$

n-Lie algebras, especially, 3-Lie algebras and metric 3-Lie algebras appeared in many fields in mathematics and mathematical physics ([2, 3]). For example, the structure of 3-Lie algebras is applied to the study of the supersymmetry and gauge symmetry transformations of the world-volume theory of multiple coincident M2-branes [2]. The identity (1) for 3-Lie algebras is essential to define an action with $N = 8$ supersymmetry and it can be regarded as a generalized Plücker relation in the physics literature [3].

Since the n-ary multiplication ($n \geq 3$), the structure of n-Lie algebras is more complex than that of Lie algebras. Ling in [4] proved that there exists only one finite dimensional simple n-Lie algebra over the complex field, that is $(n+1)$-dimensional n-Lie algebra A with $A^1 = A$.

Authors in papers [5, 6] studied realizations of n-Lie algebras, obtained methods of constructing n-Lie algebras ($n \geq 3$) from Lie algebras, associative algebras, commutative algebras, pre-Lie algebras and linear functions. So there exists close relation between n-Lie algebras to Lie algebras, pre-Lie algebras and commutative algebras and so on. In paper [7], authors studied the generating index of n-Lie algebras and gave the classifications of 3-Lie algebras with generating indices three and four. In this paper, we pay our main attention to study the metric structures and symplectic structures of 3-Lie algebras with generating indices three and four.

Throughout this paper, all 3-Lie algebras are over the complex field F , and any bracket which is not listed in the multiplication of a basis of a 3-Lie algebra is assumed to be zero.

2 Metric and symplectic structures of 3-Lie algebra A with $I(A) = 3, 4$

2.1 Classification of 3-Lie algebras A with $I(A) = 3, 4$

Definition 2.1.1[7] *Let A be a finite-dimensional 3-Lie algebra. The generating index $I(A)$ of A is the maximum of the dimensions of subalgebras generated by 3 elements of A.*

Lemma 2.1.1[7] *Let A be an m-dimensional 3-Lie algebra with $I(A) = 3$. Then A is abelian or $\dim A^1 = m - 2$. Moreover, for the latter case, there exists a basis $\{e_1, \cdots, e_m\}$ of A whose products are given by*

$$[e_1, e_2, e_i] = e_i, \ 3 \leq i \leq m. \tag{2}$$

Lemma 2.1.2[7] *Let A be a 3-Lie algebra with $I(A) = 4$. Then A is isomorphic to one of the following 3-Lie algebras:*

(a_1) *There is a basis* $\{e_1, e_2, x_1, \cdots, x_k, \cdots, x_m\}$ *of* A *whose products are given as follows: for* $1 \leq i \leq k,\ k+1 \leq j \leq m, 1 \leq k \leq [\frac{m}{2}]$

$$[e_1, e_2, x_i] = x_i + x_{i+k}, [e_1, e_2, x_j] = x_j. \tag{3}$$

(a_2) *There is a basis* $\{e_1, e_2, x_1, \cdots, x_m, e_3, \cdots, e_t\}$ *of* A *with* $t \geq 3$ *with products*

$$[e_1, e_2, x_i] = x_i,\ 1 \leq i \leq m. \tag{4}$$

(a_3) A *is the unique simple 3-Lie algebra, that is, there is a basis* $\{e_1, \cdots, e_4\}$ *of* A *such that*

$$[e_2, e_3, e_4] = e_1,\ \ [e_1, e_3, e_4] = e_2,\ \ [e_1, e_2, e_4] = e_3, [e_1, e_2, e_3] = e_4. \tag{5}$$

(a_4) *There is a basis* $\{e_1, \cdots, e_4\}$ *of* A *whose products are given by:*

$$[e_1, e_2, e_3] = e_3, [e_1, e_2, e_4] = -e_4,\ \ [e_1, e_3, e_4] = e_2. \tag{6}$$

(a_5) *There is a basis* $\{e_1, e_2, x_1, \cdots, x_m, y_1, \cdots, y_t\}$ *of* A *satisfying*

$$[e_1, e_2, x_i] = x_i, [e_1, e_2, y_j] = 3y_j, [x_k, x_i, x_l] \in \mathrm{span}\{y_1, \cdots, y_t\}, \tag{7}$$

where $1 \leq i, k, l \leq m,\ 1 \leq j \leq t.$

(a_6) *There is a basis* $\{e_1, e_2, x_1, \cdots, x_m, y_1, \cdots, y_t\}$ *of* A *satisfying: for* $k = 1, 2, , 1 \leq i, l \leq m; 1 \leq j \leq t,$

$$[e_1, e_2, x_i] = x_i, [e_1, e_2, y_j] = 2y_j, [e_k, x_i, x_l] \in \mathrm{span}\{y_1, \cdots, y_t\}. \tag{8}$$

2.2 Metric structures of 3-Lie algebras A with $I(A) = 3$, 4

Let A be a 3-Lie algebra. If a non-degenerate symmetric bilinear form $B : A \otimes A \to F$ satisfies

$$B([x_1, x_2, x_3], x_4) + B([x_1, x_2, x_4], x_3) = 0, \forall x_1, x_2, x_3, x_4 \in A, \tag{9}$$

then B is called *a metric* on A, and (A, B) is called *a metric* 3*-Lie algebra.* In paper [8], authors studied the structures and the metric dimension of metric n-Lie algebras. In this section we study the metric structures of 3-Lie algebras A with $I(A) = 3$, 4.

Theorem 2.2.1 *The simple* 3*-Lie algebra* A *is a metric* 3*-Lie algebra.*

Proof Defines non-degenerate symmetric bilinear form $B : A \otimes A \to F$ by

$$B(e_1, e_1) = B(e_3, e_3) = -B(e_2, e_2) = -B(e_4, e_4) = 1. \tag{10}$$

Then by the direct computation according to Eq.(5), B satisfies Eq.(9). Therefore, (A, B) is a metric 3-Lie algebra, and B is the unique metric on A up to non-zero multiples.

Theorem 2.2.2 *Except for the cases of simple and abelian 3-Lie algebras, there not exist metric structures on the 3-Lie algebra A with $I(A) = 3$, 4.*

Proof Let $B : A \otimes A \to F$ be a symmetric bilinear form satisfying Eq.(9). If A is a non-simple 3-Lie algebra with $A^1 \neq 0$ and $I(A) = 3$. Then by Lemma 2.1.1, there exists a basis $\{e_1, e_2, \cdots, e_m\}$ such that the multiplication of A is Eq.(5). From Eq.(9), for $3 \leq i \leq j \leq m$,

$$B(e_i, e_1) = B([e_1, e_2, e_i], e_1) = -B([e_1, e_2, e_1], e_i) = 0,$$

$$B(e_i, e_2) = B([e_1, e_2, e_i], e_2) = -B([e_1, e_2, e_2], e_i) = 0,$$

$$B(e_i, e_j) = B([e_1, e_2, e_i], e_j) = -B([e_2, e_i, e_j], e_1) = 0.$$

Therefore, B is degenerate.

If A is the one of the cases (a_1), (a_2), (a_5) and (a_6), by the direct computation according to Eqs.(3), (4), (7) and (8), we have $B(x_i, A) = 0,\ 1 \leq i \leq m$, that is $x_i \in KerB$. Therefore, B is degenerate.

If A is the case (a_4), then by Eqs.(6) and (9), we have $B(e_2, A) = 0$.

Summarizing above discussion, B is degenerate. The result follows.

2.3 Symplectic structures of 3-Lie algebras A with $I(A) = 3$, 4

Let A be a 3-Lie algebra over a field F. If a non-degenerate linear mapping $\omega : A \wedge A \to F$ satisfies

$$\sum_{i=1}^{4} \omega([x_1, \cdots, \hat{x_i}, \cdots, x_4], (-1)^{i-1}x_i) = 0,\ \forall x_i \in A, i = 1, 2, 3, 4, \tag{11}$$

then ω is called *a symplectic structure* on A, and (A, ω) is called a symplectic 3-Lie algebra.

It is clear that if (A, ω) is a symplectic 3-Lie algebra, then the dimension of A is even.

If there exists a metric B and a symplectic structure ω on 3-Lie algebra A, respectively, then (A, B, ω) is called *a metric symplectic 3-Lie algebra.*

Theorem 2.3.1 *Let A be an m-dimensional non-abelian 3-Lie algebra with $I(A) = 3$. Then then there not exist symplectic structures on A except for the case $m = 4$.*

Proof By Lemma 2.1.1, there exists a basis $\{e_1, \cdots, e_m\}$ of A whose products are given by Eq.(2). Let $\omega : A \wedge A \to F$ be a linear mapping satisfying Eq.(11). Then for $3 \leq i, j \leq m$, from

$$\omega([e_1, e_2, e_i], e_j) - \omega([e_1, e_2, e_j], e_i) + \omega([e_2, e_j, e_i], e_1) - \omega([e_1, e_j, e_i], e_2) = 0,$$

we obtain $2\omega(e_i, e_j) = 0$. Therefore, $\omega(e_i, e_j) = 0$ for $3 \leq i, j \leq m$.

If $m = 4$, defines

$$\omega(e_1, e_2) = \omega(e_2, e_3) = \omega(e_1, e_4) = \omega(e_2, e_4) = 1,\ \omega(e_1, e_3) = \omega(e_3, e_4) = 0,$$

then (A, ω) is a symplectic 3-Lie algebra.

If $m > 4$, by the above discussion, ω is degenerate. The result follows.

Theorem 2.3.2 *Let A be an m-dimensional* 3-*Lie algebra with* $I(A) = 4$. *Then, except for the cases* (a_3), (a_4) *and A is abelian, there not exist symplectic structures on A .*

Proof Let $\omega : A \wedge A \to F$ be a linear mapping satisfying Eq.(11). If A is case (a_3), then by the direct computation $\omega(e_1, e_2) = -\omega(e_2, e_1) = \omega(e_3, e_4) = -\omega(e_4, e_3) = 1$ is a symplectic structure on A of the cases (a_3) and (a_4).

If A is the one of the cases (a_1), (a_2), (a_5) and (a_6), then by the similar discussion to Theorem 2.3.1, ω is degenerate. The result follows.

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