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3-Lie algebra A with $I(A) = 3, 4$ II

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Abstract

The structure of derivation algebras is very important, specifically, in the representation theory of n-Lie algebras for $n \geq 3$. So in this paper we pay our main attention to the derivation algebras of 3-Lie algebras with generating indices three and four over the complex field F, and give the concrete expression of every derivation.

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1 Introduction

The concept of n-Lie algebra or Filippov algebra was introduced in 1985 ([1]). It is a vector space A endowed with an n-ary skew-symmetric multiplication satisfying the n-Jacobi identity $\forall x_1, \cdots, x_n, y_1, \cdots, y_{n-1} \in A$

$$[[x_1, \cdots, x_n], y_2, \cdots, y_n] = \sum_{i=1}^{n} [x_1, \cdots, [x_i, y_2, \cdots, y_n], \cdots, x_n]. \qquad (1)$$

Since the wide applications, in recent years, lots of mathematicians devote themselves to study n-Lie algebras. Authors in [2-3] studied the inner derivation algebras and derivation algebras of $(n+1)$-dimensional n-Lie algebras over the complex field, and according to the structure of inner derivation algebras of $(n+1)$-dimensional n-Lie algebras, authors in [4] gave the complete classification of $(n+1)$-dimensional n-Lie algebras.

In this paper, we pay our main attention to study the derivation algebras of 3-Lie algebras with the generating indices three and four over the complex field F.

2. Derivations of 3-Lie algebra A with $I(A) = 3, 4$

Definition 2.1[5] *Let A be a finite-dimensional 3-Lie algebra. The generating index $I(A)$ of A is the maximum of the dimensions of subalgebras generated by 3 elements of A.*

Lemma 2.1[5] *Let A be an m-dimensional 3-Lie algebra with $I(A) = 3$. Then A is abelian or $\dim A^1 = m-2$. Moreover, for the latter case, there exists a basis $\{e_1, \cdots, e_m\}$ of A whose products are given by*

$$[e_1, e_2, e_i] = e_i, \ 3 \le i \le m. \tag{2}$$

Lemma 2.2[5] *Let A be a 3-Lie algebra with $I(A) = 4$. Then A is isomorphic to one of the following 3-Lie algebras:*

(a_1) *There is a basis $\{e_1, e_2, x_1, \cdots, x_k, \cdots, x_m\}$ of A whose products are given as follows: for $1 \le i \le k,\ k+1 \le j \le m, 1 \le k \le [\frac{m}{2}]$*

$$[e_1, e_2, x_i] = x_i + x_{i+k}, [e_1, e_2, x_j] = x_j. \tag{3}$$

(a_2) *There is a basis $\{e_1, e_2, x_1, \cdots, x_m, y_1, \cdots, y_t\}$ of A with $t \ge 1$ with products*

$$[e_1, e_2, x_i] = x_i, \ 1 \le i \le m. \tag{4}$$

(a_3) *A is the unique simple 3-Lie algebra, that is, there is a basis $\{e_1, \cdots, e_4\}$ of A such that*

$$[e_2, e_3, e_4] = e_1, \ \ [e_1, e_3, e_4] = e_2, \ \ [e_1, e_2, e_4] = e_3, [e_1, e_2, e_3] = e_4. \tag{5}$$

(a_4) *There is a basis $\{e_1, \cdots, e_4\}$ of A whose products are given by:*

$$[e_1, e_2, e_3] = e_3, [e_1, e_2, e_4] = -e_4, \ [e_1, e_3, e_4] = e_2. \tag{6}$$

(a_5) *There is a basis $\{e_1, e_2, x_1, \cdots, x_m, y_1, \cdots, y_t\}$ of A satisfying*

$$[e_1, e_2, x_i] = x_i, [e_1, e_2, y_j] = 3y_j, [x_i, x_k, x_l] = \sum_{s=1}^{t} c_{ikl}^s y_s, c_{ikl}^s \in F, \tag{7}$$

where $1 \leq i, k, l \leq m,\ 1 \leq j \leq t$.

(a_6) *There is a basis* $\{e_1, e_2, x_1, \cdots, x_m, y_1, \cdots, y_t\}$ *of* A *satisfying: for* $k = 1, 2, \ , 1 \leq i, l \leq m; 1 \leq j \leq t$,

$$[e_1, e_2, x_i] = x_i, [e_1, e_2, y_j] = 2y_j, [e_k, x_i, x_l] = \sum_{s=1}^{t} c_{kil}^s y_s, c_{kil}^s \in F. \tag{8}$$

Let A be a 3-Lie algebra. If a linear mapping $D : A \to A$ satisfies

$$D[x, y, z] = [Dx, y, z] + [x, Dy, z] + [x, y, Dz], \forall x, y, z \in A, \tag{9}$$

then D is called a derivation of A. All the derivations of A, is denoted by $Der(A)$, is a linear Lie algebra.

Theorem 2.3 *Let* A *be an* m*-dimensional* n*-Lie algebra with* $I(A) = 3$. *The derivation algebra is*

$$Der(A) = \sum_{i=1}^{2} \sum_{i \neq j, j=1}^{m} FE_{ij} + F(E_{22} - E_{11}) + \sum_{i,j=3}^{m} FE_{ij}, \tag{11}$$

where E_{ij} *is an* $(m \times m)$*-order matrix with* 1 *at the position of* i^{th}*-row and* j^{th}*-column, and others are zero.*

proof By Lemma 2.1, there exists a basis $\{e_1, \cdots, e_m\}$ of A whose products are given by Eq.(2). Let $D : A \to A$ be a derivation of A and set

$$De_i = \sum_{j=1}^{m} a_{ij} e_j, \quad a_{ij} \in F.$$

Then by Eq.(2), for arbitrary $3 \leq i \leq m$,

$$De_i = [De_1, e_2, e_i] + [e_1, De_2, e_i] + [e_1, e_2, De_i] = (a_{11} + a_{22})e_i + \sum_{j=3}^{m} a_{ij} e_j,$$

we obtain

$$a_{i1} = a_{i2} = 0, 3 \leq i \leq m;\ a_{11} + a_{22} = 0,$$

then the matrix form of D is

$$D = \sum_{i=1}^{2} \sum_{i \neq j, j=1}^{m} a_{ij} E_{ij} + a_{11}(E_{22} - E_{11}) + \sum_{i,j=3}^{m} a_{ij} E_{ij},$$

we obtain Eq.(11). The proof is completed.

Authors in paper [2-3] studied derivation algebras of the four dimensional 3-Lie algebras. Now we discuss derivation algebras of 3-Lie algebras of the cases (a_1), (a_2), (a_5) and (a_6) in Lemma 2.2.

Theorem 2.4 *Let A be a 3-Lie algebra with $I(A) = 4$. If A is the case (a_1), the derivation algebra is as follows*

$$Der(A) = F(E_{11} - E_{22}) + FE_{12} + FE_{21} + \sum_{s=1}^{2}\sum_{j=1}^{m} FE_{sj+2} + \sum_{i=1}^{m}\sum_{j=k+1}^{m} FE_{i+2\ j+2}, \tag{12}$$

where E_{ij} is an $((m+2)\times(m+2))$-order matrix with 1 at the position of i^{th}-row and j^{th}-column, and others are zero.

If A is the case (a_2), then for every derivation D, the matrix form is

$$D = \mu_{11}(E_{11} - E_{22}) + \sum_{i=1}^{2}\sum_{s\neq i,s=1}^{m+t+2} \mu_{is}E_{is} + \sum_{i,j=1}^{m} a_{ij}E_{i+2j+2} + \sum_{k,l=1}^{t} b_{kl}E_{k+m+2l+m+2}. \tag{13}$$

If A is the case (a_5),

$$D = \mu_{11}(E_{11} - E_{22}) + \sum_{i=1}^{2}\sum_{s\neq i,s=1}^{m+t+2} \mu_{is}E_{is} + \sum_{i,j=1}^{m} a_{ij}E_{i+2j+2} + \sum_{k,l=1}^{t} b_{kl}E_{k+m+2l+m+2} \tag{14}$$

where a_{ij}, b_{kl} satisfy, $1 \le i, j \le m$, $1 \le k, l \le t$,

$$\sum_{s=1}^{t} c_{ijl}^{s} b_{sr} = \sum_{k=1}^{m}(a_{ik}c_{kjl}^{r} + a_{jk}c_{ikl}^{r} + a_{lk}c_{ijk}^{r}).$$

If A is the case (a_6),

$$D = \sum_{k=1}^{2}\sum_{u=1}^{m+t+2} \lambda_{ku}E_{ku} + \sum_{i,j=1}^{m} a_{ij}E_{i+2j+2} + \sum_{v,l=1}^{t} b_{vl}E_{v+m+2l+m+2} \tag{15}$$

where $\lambda_{ku}, a_{ij}, b_{vl}$ satisfy, $k = 1, 2$; $1 \le i, l \le m$; $1 \le v \le t$,

$$\mu_{11} + \mu_{22} = 0,\ (\mu_{k1} + \mu_{k2})c_{kil}^{s} = \sum_{v=1}^{m}(a_{iv}c_{kvl}^{s} + a_{lv}c_{kiv}^{s}).$$

Where E_{ij} is an $((m+t+2)\times(m+t+2))$-order matrix with 1 at the position of i^{th}-row and j^{th}-column, and others are zero.

Proof In cases (a_1), let $D : A \to A$ be a derivation of A and

$$De_s = \sum_{j=1}^{2}\lambda_{sj}e_j + \sum_{l=1}^{m}\mu_{sl}x_l, Dx_i = \sum_{j=1}^{2} a_{ij}e_j + \sum_{k=1}^{m} b_{ik}x_k,$$

where $s = 1, 2$; $1 \le i \le m$; $\lambda_{sj}, \mu_{sk}, a_{ij}, b_{ik} \in F$.

Then by Eqs.(3) and (9), for arbitrary $1 \le i \le k$, $k+1 \le j \le m$, we have

$$Dx_i = (\lambda_{11} + \lambda_{22})(x_i + x_{i+k}) + \sum_{l=1}^{k} b_{il}(x_l + x_{l+k}) + \sum_{l=k+1}^{m} b_{il}x_l,$$

$$Dx_j = (\lambda_{11} + \lambda_{22})x_j + \sum_{l=1}^{k} b_{jl}(x_l + x_{l+k}) + \sum_{l=k+1}^{m} b_{jl}x_l,$$

we obtain $\lambda_{11}+\lambda_{22} = 0$, $b_{il} = 0$, $a_{i1} = a_{i2} = 0, 1 \leq i \leq m$, $1 \leq l \leq k$. Therefore,

$$D = \lambda_{11}(E_{11} - E_{22}) + \lambda_{12}E_{12} + \lambda_{21}E_{21} + \sum_{s=1}^{2}\sum_{j=1}^{m} \mu_{sj}E_{sj+2} + \sum_{i=1}^{m}\sum_{j=k+1}^{m} a_{ij}E_{i+2\ j+2}.$$

The Eq.(12) follows.

In the cases (a_2), (a_5) and (a_6), let D be a derivation of A and set

$$De_k = \sum_{j=1}^{2} \mu_{kj}e_j + \sum_{l=1}^{m} \mu_{k+2l+2}x_l + \sum_{l=1}^{t} \mu_{k+m+2l+m+2}y_l,$$

$$Dx_i = \sum_{s=1}^{2} a''_{is}e_s + \sum_{s=1}^{m} a_{is}x_s + \sum_{s=1}^{t} a'_{is}y_s, Dy_j = \sum_{s=1}^{2} b''_{js}e_s + \sum_{s=1}^{m} b'_{js}x_s + \sum_{s=1}^{t} b_{js}y_s,$$

where $k = 1, 2$; $1 \leq i \leq m$; $1 \leq j \leq t$. By the similar discussion to the case (a_1) and Eqs.(3), (7), (8) and (9), we obtain Eqs. (13),(14) and (15), respectively. The proof is completed.

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