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Pseudo t-conorms and interval fuzzy connectives

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Abstract

In this paper, we construct pairs of interval implications, interval pseudo t-norms (pseudo t-conorms) and interval generalized residuated lattice induced by pseudo t-norms. Moreover, we investigate their properties and give examples.

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pairs of negations, pairs of implications, pairs of interval negations, pairs of interval implications

1 Introduction

Bedregal and Takahashi [4] introduced interval fuzzy connectives as an extension for fuzzy connectives. This concept provides tools for approximate reasoning and decision making with a frame work to deal with uncertainty and incompleteness of information [1-3, 10]. Georgescu and Popescue [5-8] introduced pseudo t-norms and generalized residuated lattices in a sense as non-commutative property. Kim [11] introduced pairs of (interval) negations and (interval) implications. which are induced by non-commutative property. Let $(L, \wedge, \vee, \odot, \rightarrow, \Rightarrow, \top, \bot)$ be a complete generalized residuated lattice with the law of double negation defined as $a = n_1(n_2(a)) = n_2(n_1(a))$ where $n_1(a) = a \Rightarrow \bot$ and $n_2(a) = a \rightarrow \bot$ (ref. [5-7,11]).

In this paper, we construct pairs of interval implications, interval pseudo t-norms (pseudo t-conorms) and interval generalized residuated lattice induced by pseudo t-norms. Moreover, we investigate their properties and give examples.

2 Preliminaries

In this paper, we assume that $(L, \vee, \wedge, \perp, \top)$ is a bounded lattice with a bottom element $\perp$ and a top element $\top$. Moreover, we define the following definitions in a sense as non-commutative [5-7] and interval property [1-4].

Definition 2.1 [4,5] A map $T : L \times L \to L$ is called a *pseudo t-norm* if it satisfies the following conditions:
(T1) $T(x, T(y, z)) = T(T(x, y), z)$ for all $x, y, z \in L$,
(T2) If $y \leq z$, $T(x, y) \leq T(x, z)$ and $T(y, x) \leq T(z, x)$,
(T3) $T(x, \top) = T(\top, x) = x$.
A pseudo t-norm is called a *t-norm* if $T(x, y) = T(y, x)$ for $x, y \in L$
A map $S : L \times L \to L$ is called a *pseudo t-conorm* if it satisfies the following conditions:
(S1) $S(x, S(y, z)) = S(S(x, y), z)$ for all $x, y, z \in L$,
(S2) If $y \leq z$, $S(x, y) \leq S(x, z)$ and $S(y, x) \leq S(z, x)$,
(S3) $S(x, \perp) = S(\perp, x) = x$.
A pseudo t-conorm is called a *t-conorm* if $S(x, y) = S(y, x)$ for $x, y \in L$.

Let $(L, \vee, \wedge, \top, \perp)$ be a bounded lattice. Let $L^{[2]} = \{[x_1, x_2] \mid x_1 \leq x_2, x_1, x_2 \in L\}$ where $[x_1, x_2] = \{x \in L \mid x_1 \leq x \leq x_2\}$. We define

$$[x_1, x_2] \leq [y_1, y_2], \text{ iff } x_1 \leq y_1,\ x_2 \leq y_2$$

$$[x_1, x_2] \ll [y_1, y_2], \text{ iff } x_2 \leq y_1,\ x_1 \leq y_2$$

$$[x_1, x_2] \subset [y_1, y_2], \text{ iff } y_1 \leq x_1 \leq x_2 \leq y_2.$$

Definition 2.2 [11] A map $\mathcal{T} : L^{[2]} \times L^{[2]} \to L^{[2]}$ is called an *interval pseudo t-norm* if it satisfies the following conditions:
(IT1) $\mathcal{T}([x_1, x_2], \mathcal{T}([y_1, y_2], [z_1, z_2])) = \mathcal{T}(\mathcal{T}([x_1, x_2], [y_1, y_2]), [z_1, z_2])$ for all $[x_1, x_2], [y_1, y_2], [z_1, z_2] \in L^{[2]}$,
(IT2) $\mathcal{T}([x_1, x_2], [\top, \top]) = \mathcal{T}([\top, \top], [x_1, x_2]) = [x_1, x_2]$.
(IT3) If $[x_1, x_2] \leq [z_1, z_2]$ and $[y_1, y_2] \leq [w_1, w_2]$, then $\mathcal{T}([x_1, x_2], [y_1, y_2]) \leq \mathcal{T}([z_1, z_2], [w_1, w_2])$.
(IT4) If $[x_1, x_2] \subset [z_1, z_2]$ and $[y_1, y_2] \subset [w_1, w_2]$, then $\mathcal{T}([x_1, x_2], [y_1, y_2]) \subset \mathcal{T}([z_1, z_2], [w_1, w_2])$.
A pseudo t-norm is called a *interval t-norm* if $\mathcal{T}([x_1, x_2], [y_1, y_2]) = \mathcal{T}([y_1, y_2], [x_1, x_2])$ for $[x_1, x_2], [y_1, y_2] \in L^{[2]}$

Definition 2.3 [4,5] A map $\mathcal{S} : L^{[2]} \times L^{[2]} \to L^{[2]}$ is called an *interval pseudo t-conorm* if it satisfies the following conditions:

(IS1) $\mathcal{S}([x_1,x_2],\mathcal{S}([y_1,y_2],[z_1,z_2])) = \mathcal{S}(\mathcal{S}([x_1,x_2],[y_1,y_2]),[z_1,z_2])$ for all $[x_1,x_2],[y_1,y_2],[z_1,z_2] \in L^{[2]}$,

(IS2) $\mathcal{S}([x_1,x_2],[\bot,\bot]) = \mathcal{S}([\bot,\bot],[x_1,x_2]) = [x_1,x_2]$.

(IS3) If $[x_1,x_2] \le [z_1,z_2]$ and $[y_1,y_2] \le [w_1,w_2]$, then $\mathcal{S}([x_1,x_2],[y_1,y_2]) \le \mathcal{S}([z_1,z_2],[w_1,w_2])$.

(IS4) If $[x_1,x_2] \subset [z_1,z_2]$ and $[y_1,y_2] \subset [w_1,w_2]$, then $\mathcal{S}([x_1,x_2],[y_1,y_2]) \subset \mathcal{S}([z_1,z_2],[w_1,w_2])$.

An interval pseudo t-conorm is called an *interval t-conorm* if $\mathcal{S}([x_1,x_2],[y_1,y_2]) = \mathcal{S}([y_1,y_2],[x_1,x_2])$ for $[x_1,x_2],[y_1,y_2] \in L^{[2]}$

Definition 2.4 A pair $(\mathcal{N}_1,\mathcal{N}_2)$ with maps $\mathcal{N}_i : L^{[2]} \to L^{[2]}$ is called a *pair of interval negations* if it satisfies the following conditions:

(IN1) $\mathcal{N}_i([\top,\top]) = [\bot,\bot]$, $\mathcal{N}_i([\bot,\bot]) = [\top,\top]$ for all $i \in \{1,2\}$.

(IN2) If $[x_1,x_2] \le [y_1,y_2]$, then $\mathcal{N}([y_1,y_2]) \le \mathcal{N}_i([x_1,x_2])$ for all $i \in \{1,2\}$.

(IN3) If $[x_1,x_2] \subset [y_1,y_2]$, then $\mathcal{N}_i([x_1,x_2]) \subset \mathcal{N}_i([y_1,y_2])$ for all $i \in \{1,2\}$,

(IN4) $\mathcal{N}_1(\mathcal{N}_2([x_1,x_2])) = \mathcal{N}_2(\mathcal{N}_1([x_1,x_2])) = [x_1,x_2]$ for all $[x_1,x_2] \in L^{[2]}$.

Definition 2.5 A pair $(\mathcal{I}_1,\mathcal{I}_2)$ with maps $\mathcal{I}_1,\mathcal{I}_2 : L^{[2]} \times L^{[2]} \to L^{[2]}$ is called a *pair of interval implications* if it satisfies the following conditions:

(II1) $\mathcal{I}_i([\top,\top],[\top,\top]) = \mathcal{I}_i([\bot,\bot],[\top,\top]) = \mathcal{I}_i([\bot,\bot],[\bot,\bot]) = [\top,\top]$, $\mathcal{I}_i([\top,\top],[\bot,\bot]) = [\bot,\bot]$ for all $i \in \{1,2\}$.

(II2) If $[x_1,x_2] \le [y_1,y_2]$, then $\mathcal{I}_i([x_1,x_2],[z_1,z_2]) \ge \mathcal{I}_i([y_1,y_2],[z_1,z_2])$ for all $i \in \{1,2\}$.

(II3) If $[x_1,x_2] \subset [y_1,y_2]$, then $\mathcal{I}_i([x_1,x_2],[z_1,z_2]) \subset \mathcal{I}_i([y_1,y_2],[z_1,z_2])$ for all $i \in \{1,2\}$.

(II4) $\mathcal{I}_i([\top,\top],[x_1,x_2]) = [x_1,x_2]$ for all $i \in \{1,2\}$.

(II5) $\mathcal{I}_1([x_1,x_2],\mathcal{I}_2([y_1,y_2],[z_1,z_2])) = \mathcal{I}_2([y_1,y_2],\mathcal{I}_1([x_1,x_2],[z_1,z_2]))$ for all $[x_1,x_2],[y_1,y_2],[z_1,z_2] \in L^{[2]}$.

(II6) $\mathcal{I}_1(\mathcal{I}_2([x_1,x_2],[\bot,\bot]),[\bot,\bot]) = \mathcal{I}_2(\mathcal{I}_1([x_1,x_2],[\bot,\bot]),[\bot,\bot]) = [x_1,x_2]$.

Definition 2.6 [11] A structure $(L^{[2]},\le,\vee,\wedge,\mathcal{T},\mathcal{I}_1,\mathcal{I}_2,[\top,\top],[\bot,\bot]\}$ is called an *interval generalized residuated lattice* if it satisfies the following conditions:

(G1) $\mathcal{T}$ is an interval pseudo t-norm.

(G2) $\mathcal{T}([x_1,x_2],[y_1,y_2]) \le [z_1,z_2]$ iff $[x_1,x_2] \le \mathcal{I}_1([y_1,y_2],[z_1,z_2])$ iff $[y_1,y_2] \le \mathcal{I}_2([x_1,x_2],[z_1,z_2])$.

3 Pseudo t-conorms and interval fuzzy connectives

Theorem 3.1 *Let $(L, \vee, \wedge, \top, \bot)$ be a bounded lattice, $S : L \times L \to L$ be a pseudo t-conorm and (n_1, n_2) a pair of negations. For $i = \{1, ..., 4\}$, We define $\mathcal{S}, \mathcal{S}^t, \mathcal{T}_{12}, \mathcal{T}_{21}, \mathcal{T}_{12}^t, \mathcal{T}_{21}^t, \mathcal{I}_i : L^{[2]} \times L^{[2]} \to L^{[2]}$*

$$\mathcal{S}([x_1, x_2], [y_1, y_2]) = [S(x_1, y_1), S(x_2, y_2)],$$

$$\mathcal{S}^t([x_1, x_2], [y_1, y_2]) = \mathcal{S}([y_1, y_2], [x_1, x_2]),$$

$$\mathcal{T}_{12}([x_1, x_2], [y_1, y_2]) = \mathcal{N}_1(\mathcal{S}(\mathcal{N}_2([x_1, x_2]), \mathcal{N}_2([y_1, y_2])))$$

$$\mathcal{T}_{21}([x_1, x_2], [y_1, y_2]) = \mathcal{N}_2(\mathcal{S}(\mathcal{N}_1([x_1, x_2]), \mathcal{N}_1([y_1, y_2])))$$

$$\mathcal{T}_k^t([x_1, x_2], [y_1, y_2]) = \mathcal{T}_k([y_1, y_2], [x_1, x_2]),\ k \in \{12, 21\}$$

$$\mathcal{I}_1([x_1, x_2], [y_1, y_2]) = \mathcal{S}(\mathcal{N}_1([x_1, x_2]), [y_1, y_2]),$$

$$\mathcal{I}_2([x_1, x_2], [y_1, y_2]) = \mathcal{S}([y_1, y_2], \mathcal{N}_2([x_1, x_2])),$$

$$\mathcal{I}_3([x_1, x_2], [y_1, y_2]) = \mathcal{S}([y_1, y_2], \mathcal{N}_1([x_1, x_2])),$$

$$\mathcal{I}_4([x_1, x_2], [y_1, y_2]) = \mathcal{S}(\mathcal{N}_2([x_1, x_2]), [y_1, y_2]).$$

The the following properties hold.
(1) $\mathcal{S}$ *and* $\mathcal{S}^t$ *are interval pseudo t-conorms.*
(2) $\mathcal{T}_{12}, \mathcal{T}_{21}, \mathcal{T}_{12}^t, \mathcal{T}_{21}^t$ *are interval pseudo t-norms.*
(3) $\mathcal{T}_{12} = \mathcal{T}_{21}$ *iff* $\mathcal{T}_{12}^t = \mathcal{T}_{21}^t$ *iff*

$$\mathcal{S}([x_1, x_2], [y_1, y_2]) = \mathcal{N}_2\mathcal{N}_2(\mathcal{S}(\mathcal{N}_1(\mathcal{N}_1([x_1, x_2])), \mathcal{N}_1(\mathcal{N}_1([x_1, x_2]))))$$

(4) $(\mathcal{I}_1, \mathcal{I}_2)$ *is a pair of interval implications with*

$$\mathcal{I}_1(\mathcal{T}_{21}([x_1, x_2], [y_1, y_2]), [z_1, z_2]) = \mathcal{I}_1([x_1, x_2], \mathcal{I}_1([y_1, y_2], [z_1, z_2]))$$

$$\mathcal{I}_2(\mathcal{T}_{12}([x_1, x_2], [y_1, y_2]), [z_1, z_2]) = \mathcal{I}_2([y_1, y_2], \mathcal{I}_2([x_1, x_2], [z_1, z_2])).$$

(5) *If* $[x_1, x_2] \le [y_1, y_2]$ *iff* $\mathcal{I}_1([x_1, x_2], [y_1, y_2]) = [\top, \top]$ *iff* $\mathcal{I}_2([x_1, x_2], [y_1, y_2]) = [\top, \top]$, *then*

$$\mathcal{T}_{12}([x_1, x_2], [y_1, y_2]) \le [x_1, x_2] \text{ iff } [y_1, y_2] \le \mathcal{I}_2([x_1, x_2], [z_1, z_2])$$
$$\text{iff } [x_1, x_2] \le \mathcal{I}_1([y_1, y_2], [z_1, z_2]) \text{ iff } \mathcal{T}_{21}([x_1, x_2], [y_1, y_2]) \le [z_1, z_2].$$

Moreover, $(L^{[2]}, \le, \vee, \wedge, \mathcal{T}_{12}, \mathcal{I}_1, \mathcal{I}_2, [\top, \top], [\bot, \bot]\}$ *is an interval generalized residuated lattice with* $\mathcal{T}_{12}([x_1, x_2], [y_1, y_2]) = \mathcal{T}_{21}([x_1, x_2], [y_1, y_2])$.
(6) $(\mathcal{I}_3, \mathcal{I}_4)$ *is a pair of implications with*

$$\mathcal{I}_3(\mathcal{T}_{21}([x_1, x_2], [y_1, y_2]), [z_1, z_2]) = \mathcal{I}_3([y_1, y_2], \mathcal{I}_3([x_1, x_2], [z_1, z_2])),$$

$$\mathcal{I}_4(\mathcal{T}_{12}([x_1,x_2],[y_1,y_2]),[z_1,z_2]) = \mathcal{I}_4([x_1,x_2],\mathcal{I}_4([x_1,x_2],[z_1,z_2])).$$

(7) *If* $[x_1,x_2] \leq [y_1,y_2]$ *iff* $\mathcal{I}_3([x_1,x_2],[y_1,y_2]) = [\top,\top]$ *iff* $\mathcal{I}_4([x_1,x_2],[y_1,y_2]) = [\top,\top]$, *then*

$$\mathcal{T}_{12}([x_1,x_2],[y_1,y_2]) \leq [z_1,z_2] \text{ iff } [y_1,y_2] \leq \mathcal{I}_3([x_1,x_2],[z_1,z_2]) \\ \text{ iff } [x_1,x_2] \leq \mathcal{I}_4([y_1,y_2],[z_1,z_2]) \text{ iff } \mathcal{T}_{21}([x_1,x_2],[y_1,y_2]) \leq [z_1,z_2].$$

Moreover, $(L^{[2]},\leq,\vee,\wedge,\mathcal{T}_{12},\mathcal{I}_4,\mathcal{I}_3,[\top,\top],[\bot,\bot]\}$ *is an interval generalized residuated lattice with* $\mathcal{T}_{12}([x_1,x_2],[y_1,y_2]) = \mathcal{T}_{21}([x_1,x_2],[y_1,y_2])$.

(8) $(\mathcal{I}_1,\mathcal{I}_3)$ *satisfies (II1)-(II5) such that*

$$\mathcal{I}_1(\mathcal{T}_{21}([x_1,x_2],[y_1,y_2]),[z_1,z_2]) = \mathcal{I}_1([x_1,x_2],\mathcal{I}_1([y_1,y_2],[z_1,z_2])),$$

$$\mathcal{I}_3(\mathcal{T}_{21}([x_1,x_2],[y_1,y_2]),[z_1,z_2]) = \mathcal{I}_3([y_1,y_2],\mathcal{I}_3([x_1,x_2],[z_1,z_2])),$$

$$\mathcal{I}_1(\mathcal{I}_3([x_1,x_2],[\bot,\bot]),[\bot,\bot]) = \mathcal{I}_3(\mathcal{I}_1([x_1,x_2],[\bot,\bot]),[\bot,\bot]) = \mathcal{N}_1\mathcal{N}_1([x_1,x_2]).$$

(9) *If* $[x_1,x_2] \leq [y_1,y_2]$ *iff* $\mathcal{I}_1([x_1,x_2],[y_1,y_2]) = [\top,\top]$ *iff* $\mathcal{I}_3([x_1,x_2],[y_1,y_2]) = [\top,\top]$, *then* $(L^{[2]},\leq,\vee,\wedge,\mathcal{T}_{21},\mathcal{I}_1,\mathcal{I}_3,[\top,\top],[\bot,\bot]\}$ *is an interval generalized residuated lattice.*

(10) $(\mathcal{I}_2,\mathcal{I}_4)$ *satisfies (I1)-(I5) such that*

$$\mathcal{I}_2(\mathcal{T}_{12}([x_1,x_2],[y_1,y_2]),[z_1,z_2]) = \mathcal{I}_2([y_1,y_2],\mathcal{I}_2([x_1,x_2],[z_1,z_2])),$$

$$\mathcal{I}_4(\mathcal{T}_{12}([x_1,x_2],[y_1,y_2]),[z_1,z_2]) = \mathcal{I}_4([x_1,x_2],\mathcal{I}_4([y_1,y_2],[z_1,z_2])),$$

$$\mathcal{I}_2(\mathcal{I}_4([x_1,x_2],[\bot,\bot]),[\bot,\bot]) = \mathcal{I}_4(\mathcal{I}_2([x_1,x_2],[\bot,\bot]),[\bot,\bot]) = \mathcal{N}_2\mathcal{N}_2([x_1,x_2]).$$

(11) *If* $[x_1,x_2] \leq [y_1,y_2]$ *iff* $\mathcal{I}_2([x_1,x_2],[y_1,y_2]) = [\top,\top]$ *iff* $\mathcal{I}_4([x_1,x_2],[y_1,y_2]) = [\top,\top]$, *then* $(L^{[2]},\leq,\vee,\wedge,\mathcal{T}_{12},\mathcal{I}_4,\mathcal{I}_2,[\top,\top],[\bot,\bot]\}$ *is an interval generalized residuated lattice.*

(12) $(\mathcal{I}_1,\mathcal{I}_4)$ *satisfies (I1)-(I4) and (I6) such that*

$$\mathcal{I}_1(\mathcal{T}_{21}([x_1,x_2],[y_1,y_2]),[z_1,z_2]) = \mathcal{I}_1([x_1,x_2],\mathcal{I}_1([y_1,y_2],[z_1,z_2])),$$

$$\mathcal{I}_4(\mathcal{T}_{12}([x_1,x_2],[y_1,y_2]),[z_1,z_2]) = \mathcal{I}_4([x_1,x_2],\mathcal{I}_4([y_1,y_2],[z_1,z_2])),$$

(13) *If* $\mathcal{S}(\mathcal{N}_1([x_1,x_2]),\mathcal{S}(\mathcal{N}_2([y_1,y_2]),[z_1,z_2])) = \mathcal{S}(\mathcal{N}_2([y_1,y_2]),\mathcal{S}(\mathcal{N}_1([x_1,x_2]),[z_1,z_2]))$, *then* $(\mathcal{I}_1,\mathcal{I}_4)$ *is an interval implication.*

(14) $(\mathcal{I}_1,\mathcal{I}_4)$ *satisfies (I1)-(I4) and (I6) such that*

$$\mathcal{I}_1(\mathcal{T}_{21}([x_1,x_2],[y_1,y_2]),[z_1,z_2]) = \mathcal{I}_1([x_1,x_2],\mathcal{I}_1([y_1,y_2],[z_1,z_2])),$$

$$\mathcal{I}_4(\mathcal{T}_{12}([x_1,x_2],[y_1,y_2]),[z_1,z_2]) = \mathcal{I}_4([x_1,x_2],\mathcal{I}_4([y_1,y_2],[z_1,z_2])),$$

$$\mathcal{I}_2(\mathcal{I}_4([x_1,x_2],[\bot,\bot]),[\bot,\bot]) = \mathcal{I}_4(\mathcal{I}_2([x_1,x_2],[\bot,\bot]),[\bot,\bot]) = \mathcal{N}_2\mathcal{N}_2([x_1,x_2]).$$

(15) *If* $\mathcal{S}(\mathcal{N}_1([x_1,x_2]),\mathcal{S}(\mathcal{N}_2([y_1,y_2]),[z_1,z_2])) = \mathcal{S}(\mathcal{N}_2([y_1,y_2]),\mathcal{S}(\mathcal{N}_1([x_1,x_2]),[z_1,z_2]))$, *then* $(\mathcal{I}_1,\mathcal{I}_4)$ *is an interval implication.*

Proof (1) are easily proved as a similar method as following (2).

(2) (IS1) $\mathcal{S}^t(\mathcal{S}^t([x_1,x_2],[y_1,y_2]),[z_1,z_2]) = \mathcal{S}^t([x_1,x_2],\mathcal{S}^t([y_1,y_2],[z_1,z_2]))$ from

$$\begin{aligned}&\mathcal{S}^t(\mathcal{S}^t([x_1,x_2],[y_1,y_2]),[z_1,z_2]) = \mathcal{S}^t([S(y_1,x_1),S(y_2,x_2)],[z_1,z_2])\\&= [S(z_1,S(y_1,x_1)),S(z_2,S(y_2,x_2))] = [S(S(z_1,y_1),x_1),S(S(z_2,y_2),x_2)]\\&= \mathcal{S}^t([x_1,x_2],[(S(z_1,y_1),S(z_2,y_2)]) = \mathcal{S}^t([x_1,x_2],\mathcal{S}^t([y_1,y_2],[z_1,z_2]))\end{aligned}$$

(IS2) $\mathcal{S}^t([x_1,x_2],[\perp,\perp]) = [S(\perp,x_1),S(\perp,x_2)] = [x_1,x_2]$. Similarly, $\mathcal{S}^t([\perp,\perp],[x_1,x_2]) = [x_1,x_2]$.

(IS3) If $[x_1,x_2] \le [z_1,z_2]$ and $[y_1,y_2] \le [w_1,w_2]$, then

$$\begin{aligned}&\mathcal{S}^t([x_1,x_2],[y_1,y_2]) = [S(y_1,x_1),S(y_2,x_2)]\\&\le [S(w_1,z_1),S(w_2,z_2)] = \mathcal{S}^t([z_1,z_2],[w_1,w_2]).\end{aligned}$$

(IS4) If $[x_1,x_2] \le [z_1,z_2]$ and $[y_1,y_2] \le [w_1,w_2]$, then

$$z_1 \le x_1 \le x_2 \le z_2, w_1 \le y_1 \le y_2 \le w_2,$$

$$S(w_1,z_1) \le S(y_1,x_1) \le S(y_2,x_2) \le S(w_2,z_2).$$

Thus,

$$\begin{aligned}&\mathcal{S}^t([x_1,x_2],[y_1,y_2]) = [S(y_1,x_1),S(y_2,x_2)]\\&\subset [S(w_1,z_1),S(w_2,z_2)] = \mathcal{S}^t([z_1,z_2],[w_1,w_2]).\end{aligned}$$

Hence $\mathcal{S}^t$ is an interval pseudo t-norm.

(2) (IT1) $\mathcal{T}_{12}(\mathcal{T}_{12}([x_1,x_2],[y_1,y_2]),[z_1,z_2]) = \mathcal{T}_{12}([x_1,x_2],\mathcal{T}_{12}([y_1,y_2],[z_1,z_2]))$ from

$$\begin{aligned}&\mathcal{T}_{12}(\mathcal{T}_{12}([x_1,x_2],[y_1,y_2]),[z_1,z_2])\\&= \mathcal{N}_1(\mathcal{S}(\mathcal{N}_2\mathcal{T}_{12}([x_1,x_2],[y_1,y_2])),\mathcal{N}_2([z_1,z_2])))\\&= \mathcal{N}_1(\mathcal{S}(\mathcal{N}_2\mathcal{N}_1(\mathcal{S}(\mathcal{N}_2([x_1,x_2]),\mathcal{N}_2([y_1,y_2])))),\mathcal{N}_2([z_1,z_2])))\\&= \mathcal{N}_1(\mathcal{S}(\mathcal{S}(\mathcal{N}_2([x_1,x_2]),\mathcal{N}_2([y_1,y_2])),\mathcal{N}_2([z_1,z_2])))\\&= \mathcal{N}_1(\mathcal{S}(\mathcal{N}_2([x_1,x_2]),\mathcal{S}(\mathcal{N}_2([y_1,y_2]),\mathcal{N}_2([z_1,z_2])))),\\&\mathcal{T}_{12}([x_1,x_2],\mathcal{T}_{12}([y_1,y_2],[z_1,z_2]))\\&= \mathcal{N}_1(\mathcal{S}(\mathcal{N}_2([x_1,x_2]),\mathcal{N}_2(\mathcal{T}_{12}([y_1,y_2],[z_1,z_2]))))\\&= \mathcal{N}_1(\mathcal{S}(\mathcal{N}_2([x_1,x_2]),\mathcal{N}_2\mathcal{N}_1(\mathcal{S}(\mathcal{N}_2([y_1,y_2]),\mathcal{N}_2([z_1,z_2])))))\\&= \mathcal{N}_1(\mathcal{S}(\mathcal{N}_2([x_1,x_2]),\mathcal{S}(\mathcal{N}_2([y_1,y_2]),\mathcal{N}_2([z_1,z_2])))).\end{aligned}$$

(IS2) $\mathcal{S}_{12}([x_1,x_2],[\perp,\perp]) = \mathcal{N}_1(\mathcal{T}(\mathcal{N}_2([x_1,x_2]),\mathcal{N}_2([\perp,\perp])) = \mathcal{N}_1(\mathcal{N}_2([x_1,x_2])) = [x_1,x_2]$. Similarly, $\mathcal{S}([\perp,\perp],[x_1,x_2]) = [x_1,x_2]$.

(IS3) If $[x_1,x_2] \le [z_1,z_2]$ and $[y_1,y_2] \le [w_1,w_2]$, then $\mathcal{S}([x_1,x_2],[y_1,y_2]) \le \mathcal{S}([z_1,z_2],[w_1,w_2])$.

(IS4) If $[x_1,x_2] \subset [z_1,z_2]$ and $[y_1,y_2] \subset [w_1,w_2]$, then $\mathcal{S}([x_1,x_2],[y_1,y_2]) \subset \mathcal{S}([z_1,z_2],[w_1,w_2])$.

$$\mathcal{T}_{12}([x_1,x_2],[y_1,y_2]) = \mathcal{N}_1(\mathcal{S}(\mathcal{N}_2([x_1,x_2]),\mathcal{N}_2([y_1,y_2])))$$

$$\begin{aligned}\mathcal{T}_{12}([x_1,x_2],[y_1,y_2]) &= \mathcal{N}_1(\mathcal{S}(\mathcal{N}_2([x_1,x_2]),\mathcal{N}_2([y_1,y_2])))\\ &= \mathcal{N}_1(\mathcal{S}([n_2(x_2),n_2(x_1)],[n_2(y_2),n_2(y_1)]))\\ &= \mathcal{N}_1([S(n_2(x_2),n_2(y_2)),S(n_2(x_1),n_2(y_1))])\\ &= [n_1(S(n_2(x_1),n_2(y_1))),n_1(S(n_2(x_2),n_2(y_2)))]\end{aligned}$$

(3)

$$\begin{aligned}&\mathcal{T}_{12}([x_1,x_2],[y_1,y_2]) = \mathcal{T}_{12}([x_1,x_2],[y_1,y_2])\\ &\text{ iff } \mathcal{N}_1(\mathcal{S}(\mathcal{N}_2([x_1,x_2]),\mathcal{N}_2([y_1,y_2]))) = \mathcal{N}_2(\mathcal{S}(\mathcal{N}_1([x_1,x_2]),\mathcal{N}_1([y_1,y_2])))\\ &\text{ iff } \mathcal{S}(\mathcal{N}_2([x_1,x_2]),\mathcal{N}_2([y_1,y_2])) = \mathcal{N}_2(\mathcal{N}_2(\mathcal{S}(\mathcal{N}_1([x_1,x_2]),\mathcal{N}_1([y_1,y_2]))))\\ &\text{ iff } \mathcal{S}([x_1,x_2],[y_1,y_2]) = \mathcal{N}_2(\mathcal{N}_2(\mathcal{S}(\mathcal{N}_1(\mathcal{N}_1([x_1,x_2])),\mathcal{N}_1(\mathcal{N}_1([y_1,y_2])))))\end{aligned}$$

(4) Since $\mathcal{I}_1([\top,\top],[x_1,x_2]) = \mathcal{S}(\mathcal{N}_1([\top,\top]),[x_1,x_2]) = [x_1,x_2]$ and $\mathcal{I}_2([\top,\top],[x_1,x_2]) = \mathcal{S}([x_1,x_2],\mathcal{N}_2([\top,\top])) = [x_1,x_2]$, then $\mathcal{I}_i([\top,\top],[\bot,\bot]) = [\bot,\bot], \mathcal{I}_i([\top,\top],[\top,\top]) = [\top,\top]$. Moreover, $\mathcal{I}_i([\bot,\bot],[\bot,\bot]) = [\bot,\bot], \mathcal{I}_i([\bot,\bot],[\top,\top]) = [\top,\top]$.

(II2) If $[x_1,x_2] \le [y_1,y_2]$, then $\mathcal{N}_i([x_1,x_2]) \ge \mathcal{N}_i([y_1,y_2])$. Hence $\mathcal{I}_i([x_1,x_2],[z_1,z_2]) \ge \mathcal{I}_i([y_1,y_2],[z_1,z_2])$ for all $i \in \{1,2\}$.

(II3) If $[x_1,x_2] \subset [y_1,y_2]$, then $\mathcal{N}_i([x_1,x_2]) \subset \mathcal{N}_i([y_1,y_2])$. Hence $\mathcal{I}_i([x_1,x_2],[z_1,z_2]) \subset \mathcal{I}_i([y_1,y_2],[z_1,z_2])$ for all $i \in \{1,2\}$.

$$\begin{aligned}&\mathcal{I}_1([x_1,x_2],\mathcal{I}_2([y_1,y_2],[z_1,z_2]))\\ &= \mathcal{S}(\mathcal{N}_1([x_1,x_2]),\mathcal{I}_2([y_1,y_2],[z_1,z_2])))\\ &= \mathcal{S}(\mathcal{N}_1([x_1,x_2]),\mathcal{S}([z_1,z_2],\mathcal{N}_2([y_1,y_2])))\\ &= \mathcal{S}(\mathcal{S}(\mathcal{N}_1([x_1,x_2]),[z_1,z_2]),\mathcal{N}_2([y_1,y_2]))\\ &= \mathcal{S}((\mathcal{I}_1([x_1,x_2],[z_1,z_2])),\mathcal{N}_2([y_1,y_2]))\\ &= \mathcal{I}_2([y_1,y_2],\mathcal{I}_1([x_1,x_2],[z_1,z_2])).\end{aligned}$$

$\mathcal{I}_1([x_1,x_2],[\bot,\bot]) = \mathcal{S}(\mathcal{N}_1([x_1,x_2]),[\bot,\bot]) = \mathcal{N}_1([x_1,x_2])$ and $\mathcal{I}_2([x_1,x_2],[\bot,\bot]) = \mathcal{S}(\mathcal{N}_2([x_1,x_2]),[\bot,\bot]) = \mathcal{N}_2([x_1,x_2])$. Moreover, $\mathcal{I}_2(\mathcal{I}_1([x_1,x_2],[\bot,\bot]),[\bot,\bot]) = \mathcal{N}_2(\mathcal{N}_1([x_1,x_2])) = [x_1,x_2]$ and $\mathcal{I}_1(\mathcal{I}_2([x_1,x_2],[\bot,\bot]),[\bot,\bot]) = \mathcal{N}_1(\mathcal{N}_2([x_1,x_2])) = [x_1,x_2]$. Hence $(\mathcal{I}_1,\mathcal{I}_2)$ is an pair of interval implications.

$$\begin{aligned}&\mathcal{I}_1(\mathcal{T}_{21}([x_1,x_2],[y_1,y_2]),[z_1,z_2])\\ &= \mathcal{S}(\mathcal{N}_1((\mathcal{T}_{21}([x_1,x_2],[y_1,y_2]))),[z_1,z_2])\\ &= \mathcal{S}(\mathcal{N}_1(\mathcal{N}_2\mathcal{S}(\mathcal{N}_1([x_1,x_2]),\mathcal{N}_1([y_1,y_2])))),[z_1,z_2])\\ &= \mathcal{S}(\mathcal{S}(\mathcal{N}_1([x_1,x_2]),\mathcal{N}_1([y_1,y_2])),[z_1,z_2])\\ &= \mathcal{S}(\mathcal{N}_1([x_1,x_2]),\mathcal{S}(\mathcal{N}_1([y_1,y_2]),[z_1,z_2]))\\ &= \mathcal{S}(\mathcal{N}_1([x_1,x_2]),\mathcal{I}_1([y_1,y_2],[z_1,z_2]))\\ &= \mathcal{I}_1([x_1,x_2],\mathcal{I}_1([y_1,y_2],[z_1,z_2])).\end{aligned}$$

$$\begin{array}{l}\mathcal{I}_2(\mathcal{T}_{12}([x_1,x_2],[y_1,y_2]),[z_1,z_2])\\ =\mathcal{S}([z_1,z_2],\mathcal{N}_2(\mathcal{T}_{12}([x_1,x_2],[y_1,y_2])))\\ =\mathcal{S}([z_1,z_2],\mathcal{N}_2(\mathcal{N}_1(\mathcal{S}(\mathcal{N}_2([x_1,x_2]),\mathcal{N}_2([y_1,y_2])))))\\ =\mathcal{S}([z_1,z_2],\mathcal{S}(\mathcal{N}_2([x_1,x_2]),\mathcal{N}_2([y_1,y_2])))\\ =\mathcal{S}(\mathcal{S}([z_1,z_2],\mathcal{N}_2([x_1,x_2])),\mathcal{N}_2([y_1,y_2]))\\ =\mathcal{S}(\mathcal{I}_2([x_1,x_2],[z_1,z_2]),\mathcal{N}_1([y_1,y_2]))\\ =\mathcal{I}_2([y_1,y_2],\mathcal{I}_2([x_1,x_2],[z_1,z_2])).\end{array}$$

(5) Since $[x_1,x_2]\le[y_1,y_2]$ iff $\mathcal{I}_1([x_1,x_2],[y_1,y_2])=[\top,\top]$ iff $\mathcal{I}_2([x_1,x_2],[y_1,y_2])=[\top,\top]$, by (4), then

$$\begin{array}{l}\mathcal{I}_2(\mathcal{T}_{12}([x_1,x_2],[y_1,y_2]),[z_1,z_2])=\mathcal{I}_2([y_1,y_2],\mathcal{I}_2([x_1,x_2],[z_1,z_2]))=[\top,\top]\\ \text{iff } \mathcal{T}_{12}([x_1,x_2],[y_1,y_2])\le[z_1,z_2] \text{ iff } [y_1,y_2]\le\mathcal{I}_2([x_1,x_2],[z_1,z_2])\\ \text{iff } \mathcal{I}_1([x_1,x_2],\mathcal{I}_2([y_1,y_2],[z_1,z_2]))=\mathcal{I}_2([y_1,y_2],\mathcal{I}_1([x_1,x_2],[z_1,z_2]))\\ \quad=\mathcal{I}_1(\mathcal{T}_{21}([x_1,x_2],[y_1,y_2]),[z_1,z_2])\\ \text{iff } [x_1,x_2]\le\mathcal{I}_1([y_1,y_2]),[z_1,z_2]) \text{ iff } \mathcal{T}_{21}([x_1,x_2],[y_1,y_2])\le[z_1,z_2]\end{array}$$

Hence $(L^{[2]},\le,\vee,\wedge,\mathcal{T}_{12},\mathcal{I}_1,\mathcal{I}_2,[\top,\top],[\bot,\bot]\}$ is an interval generalized residuated lattice. Since $\mathcal{T}_{12}([x_1,x_2],[y_1,y_2])\le[z_1,z_2]$ iff $\mathcal{T}_{21}([x_1,x_2],[y_1,y_2])\le[z_1,z_2]$, then $\mathcal{T}_{12}([x_1,x_2],[y_1,y_2])=\mathcal{T}_{21}([x_1,x_2],[y_1,y_2])$.

(6)

$$\begin{array}{l}\mathcal{I}_3([x_1,x_2],\mathcal{I}_4([y_1,y_2],[z_1,z_2]))\\ =\mathcal{S}(\mathcal{I}_4([y_1,y_2],[z_1,z_2])),\mathcal{N}_1([x_1,x_2]))\\ =\mathcal{S}(\mathcal{S}(\mathcal{N}_2([y_1,y_2]),[z_1,z_2]),\mathcal{N}_1([x_1,x_2]))\\ =\mathcal{S}(\mathcal{N}_2([y_1,y_2]),\mathcal{S}([z_1,z_2],\mathcal{N}_1([x_1,x_2])))\\ =\mathcal{S}(\mathcal{N}_2([y_1,y_2]),\mathcal{I}_3([x_1,x_2],[z_1,z_2]))\\ =\mathcal{I}_4([y_1,y_2],\mathcal{I}_3([x_1,x_2],[z_1,z_2])).\end{array}$$

$\mathcal{I}_3([x_1,x_2],[\bot,\bot])=\mathcal{S}([\bot,\bot],\mathcal{N}_1([x_1,x_2]))=\mathcal{N}_1([x_1,x_2])$ and $\mathcal{I}_4([x_1,x_2],[\bot,\bot])=\mathcal{S}(\mathcal{N}_2([x_1,x_2]),[\bot,\bot])=\mathcal{N}_2([x_1,x_2])$. Moreover, $\mathcal{I}_4(\mathcal{I}_3([x_1,x_2],[\bot,\bot]),[\bot,\bot])=\mathcal{N}_2(\mathcal{N}_1([x_1,x_2]))=[x_1,x_2]$ and $\mathcal{I}_3(\mathcal{I}_4([x_1,x_2],\bot),\bot)=\mathcal{N}_1(\mathcal{N}_2([x_1,x_2]))=[x_1,x_2]$.

$$\begin{array}{l}\mathcal{I}_3(\mathcal{T}_{21}([x_1,x_2],[y_1,y_2]),[y_1,y_2])\\ =\mathcal{S}([z_1,z_2],\mathcal{N}_1(\mathcal{T}_{21}([x_1,x_2],[y_1,y_2])))\\ =\mathcal{S}([x_1,x_2],\mathcal{N}_1(\mathcal{N}_2(\mathcal{S}(\mathcal{N}_1([x_1,x_2]),\mathcal{N}_1([y_1,y_2])))))\\ =\mathcal{S}([z_1,z_2],\mathcal{S}(\mathcal{N}_1([x_1,x_2]),\mathcal{N}_1([y_1,y_2])))\\ =\mathcal{S}(\mathcal{S}([z_1,z_2],\mathcal{N}_1([x_1,x_2])),\mathcal{N}_1([y_1,y_2]))\\ =\mathcal{S}(\mathcal{I}_3([x_1,x_2],[z_1,z_2]),\mathcal{N}_1([y_1,y_2]))\\ =\mathcal{I}_3([y_1,y_2],\mathcal{I}_3([x_1,x_2],[z_1,z_2]))\end{array}$$

$$\begin{aligned}
&\mathcal{I}_4(\mathcal{T}_{12}([x_1,x_2],[y_1,y_2]),[z_1,z_2])\\
&=\mathcal{S}(\mathcal{N}_2((\mathcal{T}_{12}([x_1,x_2],[y_1,y_2]))),[z_1,z_2])\\
&=\mathcal{S}(\mathcal{N}_2(\mathcal{N}_1(\mathcal{S}(\mathcal{N}_2([x_1,x_2]),\mathcal{N}_2([y_1,y_2])))),[z_1,z_2])\\
&=\mathcal{S}(\mathcal{S}(\mathcal{N}_2([x_1,x_2]),\mathcal{N}_2([y_1,y_2])),[z_1,z_2])\\
&=\mathcal{S}(\mathcal{N}_2([x_1,x_2]),\mathcal{S}(\mathcal{N}_2([y_1,y_2]),[z_1,z_2]))\\
&=\mathcal{S}(\mathcal{N}_2([x_1,x_2]),\mathcal{I}_4([y_1,y_2],[z_1,z_2]))\\
&=\mathcal{I}_4([x_1,x_2],\mathcal{I}_4([y_1,y_2],[z_1,z_2]))
\end{aligned}$$

(7) It is similarly proved as (5).

(8) We only show the condition (II5) because other cases are easily proved.

$$\begin{aligned}
\mathcal{I}_1([x_1,x_2],\mathcal{I}_3([y_1,y_2],[z_1,z_2])) &=\mathcal{I}_1([x_1,x_2],\mathcal{S}([z_1,z_2],\mathcal{N}_1([y_1,y_2])))\\
&=\mathcal{S}(\mathcal{N}_1([x_1,x_2]),\mathcal{S}([z_1,z_2],\mathcal{N}_1([y_1,y_2])))\\
\mathcal{I}_3([y_1,y_2],\mathcal{I}_1([x_1,x_2],[z_1,z_2])) &=\mathcal{I}_3([x_1,x_2],\mathcal{S}(\mathcal{N}_1([y_1,y_2]),[z_1,z_2]))\\
&=\mathcal{S}(\mathcal{S}(\mathcal{N}_1([x_1,x_2]),[z_1,z_2]),\mathcal{N}_1([y_1,y_2])))
\end{aligned}$$

By (4) and (6),

$$\mathcal{I}_1(\mathcal{T}_{21}([x_1,x_2],[y_1,y_2]),[z_1,z_2])=\mathcal{I}_1([x_1,x_2],\mathcal{I}_1([y_1,y_2],[z_1,z_2])),$$

$$\mathcal{I}_1(\mathcal{T}_{21}([x_1,x_2],[y_1,y_2]),[z_1,z_2])=\mathcal{I}_1([x_1,x_2],\mathcal{I}_1([y_1,y_2],[z_1,z_2])).$$

(9) By (8), $(L^{[2]},\leq,\vee,\wedge,\mathcal{T}_{21},\mathcal{I}_1,\mathcal{I}_3,[\top,\top],[\bot,\bot]\}$ is an interval generalized residuated lattice from:

$$\begin{aligned}
&\mathcal{T}_{21}([x_1,x_2],[y_1,y_2])\leq[z_1,z_2] \text{ iff } [x_1,x_2]\leq\mathcal{I}_1([y_1,y_2],[z_1,z_2])\\
&\text{iff } [y_1,y_2]\leq\mathcal{I}_3([x_1,x_2],[y_1,y_2]).
\end{aligned}$$

(10)

$$\begin{aligned}
\mathcal{I}_2([x_1,x_2],\mathcal{I}_4([y_1,y_2],[z_1,z_2])) &=\mathcal{I}_2([x_1,x_2],\mathcal{S}(\mathcal{N}_2([y_1,y_2],[z_1,z_2])))\\
&=\mathcal{S}(\mathcal{S}(\mathcal{N}_2([y_1,y_2],[z_1,z_2])),\mathcal{N}_2([x_1,x_2]))\\
\mathcal{I}_4([y_1,y_2],\mathcal{I}_2([x_1,x_2],[z_1,z_2])) &=\mathcal{I}_4([y_1,y_2],\mathcal{S}([z_1,z_2],\mathcal{N}_2([x_1,x_2])))\\
&=\mathcal{S}(\mathcal{N}_2([y_1,y_2]),\mathcal{S}([z_1,z_2],\mathcal{N}_2([x_1,x_2])))
\end{aligned}$$

(11) By (10), $(L^{[2]},\leq,\vee,\wedge,\mathcal{T}_{12},\mathcal{I}_4,\mathcal{I}_2,[\top,\top],[\bot,\bot]\}$ is an interval generalized residuated lattice.

$$\begin{aligned}
&\mathcal{T}_{12}([x_1,x_2],[y_1,y_2])\leq[z_1,z_2] \text{ iff } [x_1,x_2]\leq\mathcal{I}_4([y_1,y_2],[z_1,z_2])\\
&\text{iff } [y_1,y_2]\leq\mathcal{I}_2([x_1,x_2],[z_1,z_2]).
\end{aligned}$$

(12)

$$\begin{aligned}
&\mathcal{I}_1(\mathcal{I}_4([x_1,x_2],[\bot,\bot]),[\bot,\bot])=\mathcal{N}_1\mathcal{N}_2([x_1,x_2])=[x_1,x_2]\\
&=\mathcal{I}_4(\mathcal{I}_1([x_1,x_2],[\bot,\bot]),[\bot,\bot]).
\end{aligned}$$

Other cases follows from (8) and (10).

(13)

$$\begin{array}{ll} \mathcal{I}_1([x_1,x_2],\mathcal{I}_4([y_1,y_2],[z_1,z_2])) & = \mathcal{I}_1([x_1,x_2],\mathcal{S}(\mathcal{N}_2([y_1,y_2],[z_1,z_2]))) \\ & = \mathcal{S}(\mathcal{N}_1([x_1,x_2]),\mathcal{S}(\mathcal{N}_2([y_1,y_2],[z_1,z_2]))) \\ \mathcal{I}_4([y_1,y_2],\mathcal{I}_1([x_1,x_2],[z_1,z_2])) & = \mathcal{I}_4([y_1,y_2],\mathcal{S}(\mathcal{N}_1([x_1,x_2]),[z_1,z_2])) \\ & = \mathcal{S}(\mathcal{N}_2([y_1,y_2]),\mathcal{S}(\mathcal{N}_1([x_1,x_2]),[z_1,z_2])) \end{array}$$

If $\mathcal{S}(\mathcal{N}_1([x_1,x_2]),\mathcal{S}(\mathcal{N}_2([y_1,y_2]),[z_1,z_2])) = \mathcal{S}(\mathcal{N}_2([y_1,y_2]),\mathcal{S}(\mathcal{N}_1([x_1,x_2]),[z_1,z_2]))$, then $(\mathcal{I}_1,\mathcal{I}_4)$ is an interval implication.

(14) and (15) are similarly proved as (12) and (13).

Example 3.2 Put $L = \{(x,y) \in R^2 \mid (\frac{1}{2},1) \le (x,y) \le (1,0)\}$ with a bottom element $(\frac{1}{2},1)$ and a top element $(1,0)$ where

$$(x_1,y_1) \le (x_2,y_2) \Leftrightarrow x_1 < x_2 \ \text{ or } \ x_1 = x_2, y_1 \le y_2.$$

(1) Define a map $S : L \times L \to L$ as

$$S((x_1,y_1),(x_2,y_2)) = (2x_1x_2, y_2 - 2x_2 + 2x_2y_1) \wedge (1,0).$$

$$\begin{array}{l} S(S((x_1,y_1),(x_2,y_2)),(x_3,y_3)) \\ = S((2x_1x_2, y_2 - 2x_2 + 2x_2y_1) \wedge (1,0),(x_3,y_3)) \\ = (4x_1x_2x_3, y_3 - 2x_3 + 2x_3y_2 - 4x_2x_3 + 4x_2x_3y_1) \wedge (1,0) \\ = S(((x_1,y_1),S((x_2,y_2),(x_3,y_3))) \end{array}$$

Moreover, $S((x,y),(\frac{1}{2},1)) = S((\frac{1}{2},1),(x,y)) = (x,y)$. Thus S is a t-conorm.

(2) Define $\mathcal{N}_i : L^{[2]} \to L^{[2]}$ as follows:

$$\begin{array}{ll} \mathcal{N}_1([(x_1,y_1),(x_2,y_2)]) & = [(\frac{1}{2x_2}, \frac{1-y_2}{x_2}),(\frac{1}{2x_1}, \frac{1-y_1}{x_1})], \\ \mathcal{N}_2([(x_1,y_1),(x_2,y_2)]) & = [(\frac{1}{2x_2}, 1 - \frac{y_2}{2x_2}),(\frac{1}{2x_1}, 1 - \frac{y_1}{2x_1})]. \end{array}$$

We easily show that $(\mathcal{N}_1,\mathcal{N}_2)$ is a pair of interval negations. Moreover,

$$\begin{array}{ll} \mathcal{N}_1\mathcal{N}_1([(x_1,y_1),(x_2,y_2)]) & = [(x_1, 2x_1 + 2y_1 - 2),(x_2, 2x_2 + 2y_2 - 2)] \\ \mathcal{N}_2\mathcal{N}_2([(x_1,y_1),(x_2,y_2)]) & = [(x_1, 1 - x_1 + \frac{y_1}{2}),(x_2, 1 - x_2 + \frac{y_2}{2})] \end{array}$$

(3)

$$\begin{array}{l} \mathcal{S}([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)]) \\ = [S((x_1,y_1),(z_1,w_1)),S((x_2,y_2),(z_2,w_2))] \\ = [(2x_1z_1, w_1 - 2z_1 + 2z_1y_1) \wedge (1,0),(2x_2z_2, w_2 - 2z_2 + 2z_2y_2) \wedge (1,0)] \end{array}$$

$$
\begin{aligned}
&\mathcal{S}^t([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\
&=\mathcal{S}([(z_1,w_1),(z_2,w_2)],[(x_1,y_1),(x_2,y_2)])\\
&=[(2x_1z_1,y_1-2x_1+2x_1w_1)\wedge(1,0),(2x_2z_2,y_2-2x_2+2x_2w_2)\wedge(1,0)]
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{T}_{12}([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\
&=\mathcal{N}_1\mathcal{S}(\mathcal{N}_2([(x_1,y_1),(x_2,y_2)]),\mathcal{N}_2([(z_1,w_1),(z_2,w_2)]))\\
&=\mathcal{N}_1\mathcal{S}([(\tfrac{1}{2x_2},1-\tfrac{y_2}{2x_2}),(\tfrac{1}{2x_1},1-\tfrac{y_1}{2x_1})],[(\tfrac{1}{2z_2},1-\tfrac{w_2}{2z_2}),(\tfrac{1}{2z_1},1-\tfrac{w_1}{2z_1})])\\
&=\mathcal{N}_1([(\tfrac{1}{2z_2x_2},1-\tfrac{w_2}{2z_2}-\tfrac{y_2}{2z_2x_2})\wedge(1,0),(\tfrac{1}{2z_1x_1},1-\tfrac{w_1}{2z_1}-\tfrac{y_1}{2z_1x_1})\wedge(1,0)])\\
&=[(x_1z_1,x_1w_1+y_1)\vee(\tfrac{1}{2},1),(x_2z_2,x_2w_2+y_2)\vee(\tfrac{1}{2},1)]
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{T}_{21}([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\
&=\mathcal{N}_2\mathcal{S}(\mathcal{N}_1([(x_1,y_1),(x_2,y_2)]),\mathcal{N}_1([(z_1,w_1),(z_2,w_2)]))\\
&=\mathcal{N}_2\mathcal{S}([(\tfrac{1}{2x_2},\tfrac{1-y_2}{x_2}),(\tfrac{1}{2x_1},\tfrac{1-y_1}{x_1})],[(\tfrac{1}{2z_2},\tfrac{1-w_2}{z_2}),(\tfrac{1}{2z_1},\tfrac{1-w_1}{z_1})])\\
&=\mathcal{N}_2([(\tfrac{1}{2z_2x_2},-\tfrac{w_2}{z_2}+\tfrac{1-y_2}{z_2x_2})\wedge(1,0),(\tfrac{1}{2z_1x_1},-\tfrac{w_1}{z_1}-\tfrac{1-y_1}{z_1x_1})\wedge(1,0)])\\
&=[(z_1x_1,x_1w_1+y_1)\vee(\tfrac{1}{2},1),(z_2x_2,x_2w_2+y_2)\vee(\tfrac{1}{2},1)]
\end{aligned}
$$

(4) We have $\mathcal{T}_{12}=\mathcal{T}_{21}$ from

$$
\begin{aligned}
&\mathcal{N}_2\mathcal{N}_2\mathcal{S}(\mathcal{N}_1\mathcal{N}_1[(x_1,y_1),(x_2,y_2)],\mathcal{N}_1\mathcal{N}_1[(z_1,w_1),(z_2,w_2)])\\
&=\mathcal{N}_2\mathcal{N}_2\mathcal{S}([(x_1,2x_1+2y_1-2),(x_2,2x_2+2y_2-2)],\\
&\quad[(z_1,2z_1+2w_1-2),(z_2,2z_2+2w_2-2)])\\
&=\mathcal{N}_2\mathcal{N}_2([(2x_1z_1,2w_1-2+2z_1(2x_1+2y_1-2))\wedge(1,0),\\
&(2x_2z_2,2w_2-2+2z_2(2x_2+2y_2-2))\wedge(1,0)]\\
&=[(2x_1z_1,w_1+2y_1z_1-2z_1)\wedge(1,0),(2x_2z_2,w_2+2y_2z_2-2z_2)\wedge(1,0)]\\
&=\mathcal{S}([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])
\end{aligned}
$$

(5)

$$
\begin{aligned}
&\mathcal{I}_1([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\
&=\mathcal{S}(\mathcal{N}_1([(x_1,y_1),(x_2,y_2)]),[(z_1,w_1),(z_2,w_2)])\\
&=\mathcal{S}([(\tfrac{1}{2x_2},\tfrac{1-y_2}{x_2}),(\tfrac{1}{2x_1},\tfrac{1-y_1}{x_1})],[(z_1,w_1),(z_2,w_2)])\\
&=[(\tfrac{z_1}{x_2},w_1-2z_1+\tfrac{2z_1-2z_1y_2}{x_2})\wedge(1,0),(\tfrac{z_2}{x_1},w_2-2z_2+\tfrac{2z_2-2z_2y_1}{x_1})\wedge(1,0)].
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{I}_2([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\
&=\mathcal{S}([(z_1,w_1),(z_2,w_2)],\mathcal{N}_2([(x_1,y_1),(x_2,y_2)]))\\
&=\mathcal{S}([(z_1,w_1),(z_2,w_2)],[(\tfrac{1}{2x_2},1-\tfrac{y_2}{2x_2}),(\tfrac{1}{2x_1},1-\tfrac{y_1}{2x_1})])\\
&=[(\tfrac{z_1}{x_2},1-\tfrac{y_2}{2x_2}-\tfrac{1}{x_2}+\tfrac{w_1}{x_2})\wedge(1,0),(\tfrac{z_2}{x_1},1-\tfrac{y_1}{2x_1}-\tfrac{1}{x_1}+\tfrac{w_2}{x_1})\wedge(1,0)],
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{I}_3([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\
&=\mathcal{S}([(z_1,w_1),(z_2,w_2)],\mathcal{N}_1([(x_1,y_1),(x_2,y_2)]))\\
&=\mathcal{S}([(z_1,w_1),(z_2,w_2)],[(\tfrac{1}{2x_2},\tfrac{1-y_2}{x_2}),(\tfrac{1}{2x_1},\tfrac{1-y_1}{x_1})])\\
&=[(\tfrac{z_1}{x_2},\tfrac{w_1-y_2}{x_2})\wedge(1,0),(\tfrac{z_2}{x_1},\tfrac{w_2-y_1}{x_1})\wedge(1,0)],
\end{aligned}
$$

$$\begin{aligned}
&\mathcal{I}_4([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\
&=\mathcal{S}(\mathcal{N}_2([(x_1,y_1),(x_2,y_2)]),[(z_1,w_1),(z_2,w_2)])\\
&=\mathcal{S}([(\tfrac{1}{2x_2},1-\tfrac{y_2}{2x_2}),(\tfrac{1}{2x_1},1-\tfrac{y_1}{2x_1})],[(z_1,w_1),(z_2,w_2)])\\
&=[(\tfrac{z_1}{x_2},w_1-\tfrac{z_1y_2}{x_2})\wedge(1,0),(\tfrac{z_2}{x_1},w_2-\tfrac{w_2y_1}{x_1})\wedge(1,0)]\\
&=[S(n_1(x_2,y_2),(z_1,w_1)),S(n_1(x_1,y_1),(z_2,w_2))].
\end{aligned}$$

(6) The converse of Theorem 3.1(5) is not true for which $\mathcal{T}_{12}=\mathcal{T}_{21}$, but

$$\begin{aligned}
&\mathcal{I}_1([(\tfrac{3}{4},0),(\tfrac{3}{4},0)],[(\tfrac{3}{4},-\tfrac{1}{2}),(\tfrac{3}{4},-\tfrac{1}{2})])=[(1,0),(1,0)]\\
&\text{but } [(\tfrac{3}{4},0),(\tfrac{3}{4},0)]\not\leq[(\tfrac{3}{4},-\tfrac{1}{2}),(\tfrac{3}{4},-\tfrac{1}{2})]
\end{aligned}$$

$$\begin{aligned}
&\mathcal{I}_2([(\tfrac{3}{4},1),(\tfrac{3}{4},1)],[(\tfrac{3}{4},\tfrac{3}{4}),(\tfrac{3}{4},\tfrac{3}{4})])=[(1,0),(1,0)]\\
&\text{but } [(\tfrac{3}{4},1),(\tfrac{3}{4},1)]\not\leq[(\tfrac{3}{4},\tfrac{3}{4}),(\tfrac{3}{4},\tfrac{3}{4})]
\end{aligned}$$

(7) We define $[(x_1,y_1),(x_2,y_2)]\ll[(z_1,w_1),(z_2,w_2)]$ iff $(x_1,y_1)\leq(z_2,w_2)$ and $(x_2,y_2)\leq(z_1,w_1)$.

$$\begin{aligned}
&\mathcal{I}_3([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])=[(1,0),(1,0)]\\
&\text{iff } (\tfrac{z_1}{x_2},\tfrac{w_1-y_2}{x_2})\geq(1,0),(\tfrac{z_2}{x_1},\tfrac{w_2-y_1}{x_1})\geq(1,0)\\
&\text{iff } (z_1>x_2 \text{ or } z_1=x_2,w_1\geq y_2) \text{ and } (z_2>x_1 \text{ or } z_2=x_1,w_2\geq y_1)\\
&\text{iff } (x_2,y_2)\leq(z_1,w_1) \text{ and } (x_1,y_1)\leq(z_2,w_2)\\
&[(x_1,y_1),(x_2,y_2)]\ll[(z_1,w_1),(z_2,w_2)]
\end{aligned}$$

$$\begin{aligned}
&\mathcal{I}_4([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])=[(1,0),(1,0)]\\
&\text{iff } (\tfrac{z_1}{x_2},w_1-\tfrac{z_1y_2}{x_2})\geq(1,0),(\tfrac{z_2}{x_1},w_2-\tfrac{w_2y_1}{x_1})\geq(1,0)\\
&\text{iff } (z_1>x_2 \text{ or } z_1=x_2,w_1\geq y_2) \text{ and } (z_2>x_1 \text{ or } z_2=x_1,w_2\geq y_1)\\
&\text{iff } (x_2,y_2)\leq(z_1,w_1) \text{ and } (x_1,y_1)\leq(z_2,w_2)\\
&[(x_1,y_1),(x_2,y_2)]\ll[(z_1,w_1),(z_2,w_2)]
\end{aligned}$$

By Theorem 6, we have

$$\begin{aligned}
&\mathcal{T}_{12}([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\leq[(a_1,b_1),(a_2,b_2)]\\
&\text{iff } [(z_1,w_1),(z_2,w_2)]\ll\mathcal{I}_4([(x_1,y_1),(x_2,y_2)],[(a_1,b_1),(a_2,b_2)])\\
&\text{iff } [(x_1,y_1),(x_2,y_2)]\ll\mathcal{I}_3([(z_1,w_1),(z_2,w_2)]),[(a_1,b_1),(a_2,b_2)])\\
&\text{iff } \mathcal{T}_{21}([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\ll[(a_1,b_1),(a_2,b_2)].
\end{aligned}$$

Since $\mathcal{T}_{12}=\mathcal{T}_{21}$, $(L^{[2]},\ll,\vee,\wedge,\mathcal{T}_{12},\mathcal{I}_4,\mathcal{I}_3,[\top,\top],[\bot,\bot]\}$ is an interval generalized residuated lattice

(8) $(\mathcal{I}_1,\mathcal{I}_3)$ is not a pair of interval implications from:

$$\begin{aligned}
&\mathcal{I}_1(\mathcal{I}_3([(\tfrac{3}{4},1),(\tfrac{2}{3},2)],[(\tfrac{1}{2},1),(\tfrac{1}{2},1)]),[(\tfrac{1}{2},1),(\tfrac{1}{2},1)])\\
&=\mathcal{I}_3(\mathcal{I}_1([(\tfrac{3}{4},1),(\tfrac{2}{3},2)],[(\tfrac{1}{2},1),(\tfrac{1}{2},1)]),[(\tfrac{1}{2},1),(\tfrac{1}{2},1)])\\
&=\mathcal{N}_1\mathcal{N}_1([(\tfrac{3}{4},1),(\tfrac{2}{3},2)])=[(\tfrac{3}{4},\tfrac{3}{2}),(\tfrac{2}{3},\tfrac{10}{3})]\\
&\neq[(\tfrac{3}{4},1),(\tfrac{2}{3},2)].
\end{aligned}$$

(9) $(\mathcal{I}_2, \mathcal{I}_4)$ is not a pair of interval implications from:

$$\begin{aligned}&\mathcal{I}_2(\mathcal{I}_4([(\tfrac{3}{4},1),(\tfrac{2}{3},2)],[(\tfrac{1}{2},1),(\tfrac{1}{2},1)]),[(\tfrac{1}{2},1),(\tfrac{1}{2},1)])\\&=\mathcal{I}_4(\mathcal{I}_2([(\tfrac{3}{4},1),(\tfrac{2}{3},2)],[(\tfrac{1}{2},1),(\tfrac{1}{2},1)]),[(\tfrac{1}{2},1),(\tfrac{1}{2},1)])\\&=\mathcal{N}_2\mathcal{N}_2([(\tfrac{3}{4},1),(\tfrac{2}{3},2)])=[(\tfrac{3}{4},\tfrac{3}{4}),(\tfrac{2}{3},\tfrac{4}{3})]\\&\neq[(\tfrac{3}{4},1),(\tfrac{2}{3},2)].\end{aligned}$$

(10) $(\mathcal{I}_1, \mathcal{I}_4)$ is not a pair of interval implications from:

$$\begin{aligned}&\mathcal{I}_1([(\tfrac{3}{5},0),(\tfrac{3}{4},1)],\mathcal{I}_4([(\tfrac{3}{5},-1),(\tfrac{2}{3},2)],[(\tfrac{1}{2},1),(\tfrac{4}{7},5)]))=[(1,-2),(1,0)]\\&\mathcal{I}_4([(\tfrac{3}{5},-1),(\tfrac{2}{3},2)],\mathcal{I}_1([(\tfrac{3}{5},0),(\tfrac{3}{4},1)],[(\tfrac{1}{2},1),(\tfrac{4}{7},5)]))=[(1,0),(1,0)]\end{aligned}$$

(11) $(\mathcal{I}_2, \mathcal{I}_3)$ is not a pair of interval implications from:

$$\begin{aligned}&\mathcal{I}_2([(\tfrac{3}{5},0),(\tfrac{3}{4},1)],\mathcal{I}_3([(\tfrac{3}{5},-1),(\tfrac{2}{3},2)],[(\tfrac{1}{2},1),(\tfrac{4}{7},5)]))=[(1,-3),(1,0)]\\&\mathcal{I}_3([(\tfrac{3}{5},-1),(\tfrac{2}{3},2)],\mathcal{I}_2([(\tfrac{3}{5},0),(\tfrac{3}{4},1)],[(\tfrac{1}{2},1),(\tfrac{4}{7},5)]))=[(1,-\tfrac{5}{2}),(1,0)].\end{aligned}$$

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