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A new approach to study matrix polynomials

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Abstract

In this paper, we introduce a new approach to study classic theory for matrix polynomials. More specifically speaking, we first give a new method to compute multiplication, quotient and remainder of matrix polynomials. Second, we obtain a necessary and sufficient condition for left division and right division, respectively.

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1 Preliminary Notes

Matrix polynomials defined as polynomials whose coefficients are matrices, are usually seen instead as polynomial matrices, and have many good results in [1]. The aim of this paper is to give a new approach to study classic theory for matrix polynomials by means of matrix theory and systems of linear equations theory.

Throughout this paper, we assume that all matrix polynomials have coefficients which are fixed t-square matrices over a number field $\mathbb{F}$. Denote by M^T the transpose of the matrix M. Let $f(x) = A_0 + A_1x + A_2x^2 + \cdots + A_nx^n = (A_0, A_1, A_2, \cdots, A_n)(I_t, I_tx, I_tx^2, \cdots, I_tx^n)^T$ and $g(x) = B_0 + B_1x + B_2x^2 + \cdots + B_mx^m = (B_0, B_1, B_2, \cdots, B_m)(I_t, I_tx, I_tx^2, \cdots, I_tx^m)^T$ be two matrix polynomials over $\mathbb{F}$. We call $A = (A_0, A_1, A_2, \cdots, A_n)$ and $B = (B_0, B_1, B_2, \cdots, B_m)$ the coefficient matrix of $f(x)$ and $g(x)$, respectively.

In this paper, we show that $g(x)$ right divides $f(x)$ if and only if $\text{Rank}(B_g^T) = \text{Rank}(B_g^T \mid A^T)$; $g(x)$ left divides $f(x)$ if and only if $\text{Rank}(B_{g^T}^T) = \text{Rank}(B_{g^T}^T \mid$

M^T) (See Theorem 2.8 and Theorem 2.9), and give a new method to compute multiplication, the quotient and remainder (See Theorems 2.1 and 2.6). As an application of Theorems 2.6, 2.8 and 2.9, we give two new methods to prove the Generalized Bezout Theorem in [P. 81, 2].

2 Main Results

Theorem 2.1 *Let* $f(x) = \sum_{i=0}^{n} A_i x^i$ *be a matrix polynomial of degree* n *and* $g(x) = \sum_{j=0}^{m} B_j x^j$ *be a matrix polynomial of degree* m*, where* A_i, B_j $(0 \leq i \leq n, 0 \leq j \leq m)$ *are* $t \times t$ *matrices. Let* $A = (A_0, A_1, A_2, \cdots, A_n)$ *be the coefficient matrix of* $f(x)$ *and*

$$B_g = \begin{pmatrix} B_0 & B_1 & B_2 & \cdots & B_m & 0 & 0 & 0 & \cdots & 0 \\ 0 & B_0 & B_1 & B_2 & \cdots & B_m & 0 & 0 & \cdots & 0 \\ 0 & 0 & B_0 & B_1 & B_2 & \cdots & B_m & 0 & \cdots & 0 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ 0 & 0 & 0 \cdots & \cdots & 0 & B_0 & B_1 & B_2 & \cdots & B_m \end{pmatrix}_{[t(n+1)] \times [t(m+n+1)]}$$

be the relative coefficient matrix of $g(x)$*, where the number of rows of* B_g *is equal to the number of columns of* A*. Then we have*

$$f(x)g(x) = AB_g(I_t, I_t x, I_t x^2, \cdots, I_t x^s, \cdots, I_t x^{m+n})^T,$$

where I_t *is a* $t \times t$ *identity matrix.*

Proof. Since $A = (A_0, A_1, A_2, \cdots, A_n)$ and

$$B_g = \begin{pmatrix} B_0 & B_1 & B_2 & \cdots & B_m & 0 & 0 & 0 & \cdots & 0 \\ 0 & B_0 & B_1 & B_2 & \cdots & B_m & 0 & 0 & \cdots & 0 \\ 0 & 0 & B_0 & B_1 & B_2 & \cdots & B_m & 0 & \cdots & 0 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ 0 & 0 & 0 \cdots & \cdots & 0 & B_0 & B_1 & B_2 & \cdots & B_m \end{pmatrix}_{[t(n+1)] \times [t(m+n+1)]}$$

$$\begin{aligned} & AB_g(I_t, I_t x, I_t x^2, \cdots, I_t x^s, \cdots, I_t x^{m+n})^T \\ = \; & (A_0B_0, A_0B_1 + A_1B_0, \cdots, A_nB_m)(I_t, I_t x, I_t x^2, \cdots, I_t x^s, \cdots, I_t x^{m+n})^T \\ = \; & A_0B_0 + (A_0B_1 + A_1B_0)x + \cdots + A_nB_m x^{m+n} \\ = \; & (A_0 + A_1 x + A_2 x^2 + \cdots + A_n x^n)(B_0 + B_1 x + B_2 x^2 + \cdots + B_m x^m) \\ = \; & f(x)g(x). \end{aligned}$$

The result follows. □

Corollary 2.2 *Let* $f(x) = \sum_{i=0}^{n} a_i x^i$ *be a polynomial of degree* n *and* $g(x) = \sum_{j=0}^{m} b_j x^j$ *be a polynomial of degree* m*. Let* $A = (a_0, a_1, a_2, \cdots, a_n)$ *be the coefficient matrix of* $f(x)$ *and*

$$B_g = \begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_m & 0 & 0 & 0 & \cdots & 0 \\ 0 & b_0 & b_1 & b_2 & \cdots & b_m & 0 & 0 & \cdots & 0 \\ 0 & 0 & b_0 & b_1 & b_2 & \cdots & b_m & 0 & \cdots & 0 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ 0 & 0 & 0\cdots & \cdots & 0 & b_0 & b_1 & b_2 & \cdots & b_m \end{pmatrix}_{(n+1)\times(m+n+1)}$$

be the relative coefficient matrix of $g(x)$*, where the number of rows of* B_g *is equal to the number of columns of* A*. Then*

$$f(x)g(x) = AB_g(1, x, x^2, \cdots, x^s, \cdots, x^{m+n})^T.$$

Proof. It is clear by virtue of Theorem 2.1 when $t = 1$. □

Let $f(x) = \sum_{i=0}^{n} A_i x^i$ be a matrix polynomial of degree n. Then the derivative of $f(x)$ is denoted by

$$\frac{d}{dx} f(x) = \sum_{i=1}^{n} i A_i x^{i-1},$$

and the integral of $f(x)$ is denoted by

$$\int f(x)dx = \sum_{i=0}^{n} \frac{A_i x^{i+1}}{i+1}.$$

By Theorem 2.1, we have the following two corollaries:

Corollary 2.3 *Let* $f(x) = \sum_{i=0}^{n} A_i x^i$ *be a matrix polynomial of degree* n *and* $g(x) = \sum_{j=0}^{m} B_j x^j$ *be a matrix polynomial of degree* m*. Then*

$$\frac{d}{dx}(f(x)g(x)) = (AB_g)(I_t \frac{d}{dx}1, I_t \frac{d}{dx}x, I_t \frac{d}{dx}x^2, \cdots, I_t \frac{d}{dx}x^s, \cdots, I_t \frac{d}{dx}x^{m+n})^T,$$

$$\int f(x)g(x)dx = (AB_g)(\int I_t dx, \int I_t x dx, \cdots, \int I_t x^s dx, \cdots, \int I_t x^{m+n} dx)^T,$$

where $A = (A_0, A_1, A_2, \cdots, A_n)$ *is the coefficient matrix of* $f(x)$ *and* B_g *is the relative coefficient matrix of* $g(x)$*.*

Corollary 2.4 *Let* $f(x)=\sum_{i=0}^{n}a_ix^i$ *be a polynomial of degree* n *and* $g(x)=\sum_{j=0}^{m}b_jx^j$ *be a polynomial of degree* m*. Then*

$$\frac{d}{dx}(f(x)g(x))=(AB_g)(\frac{d}{dx}1,\frac{d}{dx}x,\frac{d}{dx}x^2,\cdots,\frac{d}{dx}x^s,\cdots,\frac{d}{dx}x^{m+n})^T,$$

$$\int f(x)g(x)dx=(AB_g)(\int 1dx,\int xdx,\cdots,\int x^sdx,\cdots,\int x^{m+n}dx)^T,$$

$$\int_a^b(f(x)g(x))dx=(AB_g)(\int_a^b 1dx,\int_a^b xdx,\cdots,\int_a^b x^sdx,\cdots,\int_a^b x^{m+n}dx)^T,$$

where $A=(a_0,a_1,a_2,\cdots,a_n)$ *is the coefficient matrix of* $f(x)$ *and* B_g *is the relative coefficient matrix of* $g(x)$*.*

Example 2.5 *Let* $f(x)=\begin{pmatrix}1&2\\-1&1\end{pmatrix}+\begin{pmatrix}0&1\\2&-1\end{pmatrix}x+\begin{pmatrix}1&0\\2&1\end{pmatrix}x^2$ *and* $g(x)=\begin{pmatrix}0&2\\1&1\end{pmatrix}+\begin{pmatrix}1&2\\-1&-1\end{pmatrix}x$*. We can compute* $f(x)g(x)$, $\frac{d}{dx}(f(x)g(x))$ *and* $\int f(x)g(x)dx$ *by Theorem 2.1 and Corollary 2.3.*

From the above, it is easy to see that $A=\begin{pmatrix}1&2&0&1&1&0\\-1&1&2&-1&2&1\end{pmatrix}$ and

$$B_g=\begin{pmatrix}0&2&1&2&0&0&0&0\\1&1&-1&-1&0&0&0&0\\0&0&0&2&1&2&0&0\\0&0&1&1&-1&-1&0&0\\0&0&0&0&0&2&1&2\\0&0&0&0&1&1&-1&-1\end{pmatrix}_{6\times 8}.$$

Then

$$\begin{aligned}&f(x)g(x)=AB_g(I_2,I_2x,I_2x^2,I_2x^3)^T\\=&\begin{pmatrix}2-x^2+x^3&4+x+x^2+2x^3\\1-3x+4x^2+x^3&-1+10x^2+3x^3\end{pmatrix},\\&\frac{d}{dx}(f(x)g(x))=AB_g(I_2\frac{d}{dx}1,I_2\frac{d}{dx}x,I_2\frac{d}{dx}x^2,I_2\frac{d}{dx}x^3)^T\\=&\begin{pmatrix}-2x+3x^2&1+2x+6x^2\\-3+8x+3x^2&20x+9x^2\end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}&\int f(x)g(x)dx=(AB_g)(I_2\int 1dx,I_2\int xdx,I_2\int x^2dx,I_2\int x^3dx)^T\\=&\begin{pmatrix}2x-\frac{x^3}{3}+\frac{x^4}{4}+c&4x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{2}+c\\x-\frac{3x^2}{2}+\frac{4x^3}{3}+\frac{x^4}{4}+c&-x+\frac{10x^3}{3}+\frac{3x^4}{4}+c\end{pmatrix},\end{aligned}$$

where c is a constant.

In order to obtain more important results, we need to do some preparations.

Let $f(x)=\sum_{i=0}^{n}A_ix^i$ be a matrix polynomial of degree n and $A=(A_0,A_1,\cdots,A_n)$ the coefficient matrix of $f(x)$. Then the transpose of the matrix polynomial $f(x)$ is denoted by $f^T(x):=[(A_0,A_1,A_2,\cdots,A_n)(I_t,I_tx,I_tx^2,\cdots,I_tx^s,\cdots,I_tx^n)^T]^T$, i.e., the coefficient matrix of $f^T(x)$ is $(A_0^T,A_1^T,A_2^T,\cdots,A_n^T)$.

Theorem 2.6 can be found in page 78 of [2], where we will give a new proof as an application of Theorem 2.1.

Theorem 2.6 *If* $f(x)=\sum_{i=0}^{n}A_ix^i$ *is a matrix polynomial of degree* n *and* $g(x)=\sum_{j=0}^{m}B_jx^j$ *is a matrix polynomial of degree* m *such that* B_m *is an invertible matrix, then the following statements hold:*

(1) *There are unique matrix polynomial* $q_R(x)$ *and* $r_R(x)$ *such that*

$$f(x)=q_R(x)g(x)+r_R(x),$$

where $r_R(x)=0$ *or* $\deg(r_R(x))<\deg(g(x))$. *If* $r_R(x)=0$, *we say that* $g(x)$ *right divides* $f(x)$, *denoted by* $g(x)\mid_R f(x)$.

(2) *There are unique matrix polynomials* $q_L(x)$ *and* $r_L(x)$ *such that*

$$f(x)=g(x)q_L(x)+r_L(x),$$

where $r_L(x)=0$ *or* $\deg(r_L(x))<\deg(g(x))$. *If* $r_L(x)=0$, *we say that* $g(x)$ *left divides* $f(x)$, *denoted by* $g(x)\mid_L f(x)$.

Proof. (1) If $m>n$, we can take $q_R(x)=0$ and $r_R(x)=f(x)$.

If $m\le n$, then we can let $q_R(x)=(C_0,C_1,C_2,\cdots,C_{n-m})(I_t,I_tx,\cdots,I_tx^{n-m})^T$, $r_R(x)=(R_0,R_1,R_2,\cdots,R_{m-1})(I_t,I_tx,\cdots,I_tx^{m-1})^T$, where C_i,R_j $(0\le i\le n-m, 0\le j\le m-1)$ are $t\times t$ matrices. Suppose that $A=(A_0,A_1,A_2,\cdots,A_n)$ is the coefficient matrix of $f(x)$ and

$$B_g=\begin{pmatrix} B_0 & B_1 & B_2 & \cdots & B_m & 0 & 0 & 0 & \cdots & 0\\ 0 & B_0 & B_1 & B_2 & \cdots & B_m & 0 & 0 & \cdots & 0\\ 0 & 0 & B_0 & B_1 & B_2 & \cdots & B_m & 0 & \cdots & 0\\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ 0 & 0 & 0\cdots & \cdots & 0 & B_0 & B_1 & B_2 & \cdots & B_m \end{pmatrix}_{[t(n-m+1)]\times[t(n+1)]}$$

the relative coefficient matrix of $g(x)$. Since $f(x)=q_R(x)g(x)+r_R(x)$, $g(x)\mid_R (f(x)-r_R(x))$. By Theorem 2.1, we have

$$(C_0,C_1,C_2,\cdots,C_{n-m})B_g=(A_0-R_0,\cdots,A_{m-1}-R_{m-1},A_m,\cdots,A_n),$$

i.e.,

$$\left\{\begin{array}{l} C_0B_0 = A_0 - R_0 \\ C_0B_1 + C_1B_0 = A_1 - R_1 \\ C_0B_2 + C_1B_1 + C_2B_0 = A_2 - R_2 \\ \cdots\cdots \\ C_{n-m-2}B_m + C_{n-m-1}B_{m-1} + C_{n-m}B_{m-2} = A_{n-2} \\ C_{n-m-1}B_m + C_{n-m}B_{m-1} = A_{n-1} \\ C_{n-m}B_m = A_n \end{array}\right. . \tag{2.1}$$

Since B_m is invertible, the system (2.1) can be solved in a step by step manner beginning with C_{n-m}. Then we can obtain $q_R(x)$ by the last $n-m+1$ equations of (2.1) and $r_R(x)$ by the first m equations of (2.1). Hence the existence and uniqueness have been proved.

(2) If $g(x) = \sum_{j=0}^{m} B_j x^j$, then $g^T(x) = \sum_{j=0}^{m} B_j^T x^j$. B_m is invertible if and only if B_m^T is invertible and $f(x) = g(x)q_L(x) + r_L(x)$ if and only if $f^T(x) = q_L^T(x)g^T(x) + r_L^T(x)$. Thus, the result follows from (1) and the definition of $f^T(x)$. □

Remark 2.7 *As an application of Theorems 2.6, we give a new method to prove the Generalized Bezout Theorem in [P. 81, 2].*

Let $f(x) = \sum_{i=0}^{n} A_i x^i$ be a matrix polynomial of degree n and $g(x) = xI_t - B_0$. Suppose $q_R(x) = (C_0, C_1, \cdots, C_{n-1})$ and $r_R(x) = R_0$ such that $f(x) = q_R(x)g(x) + r_R(x)$. From above we have

$$(C_0, C_1, C_2, \cdots, C_{n-1})B_g = (A_0 - R_0, A_1, A_2, \cdots, A_n),$$

i.e.,

$$\left\{\begin{array}{l} -B_0C_0 = A_0 - R_0 \\ C_0 - B_0C_1 = A_1 \\ C_1 - B_0C_2 = A_2 \\ \cdots\cdots \\ C_{n-2} - B_0C_{n-1} = A_{n-1} \\ C_{n-1} = A_n \end{array}\right. .$$

Then we have

$$\left\{\begin{array}{l} C_{n-1} = A_n \\ C_{n-2} = B_0A_n + A_{n-1} \\ C_{n-3} = B_0^2A_n + B_0A_{n-1} + A_{n-2} \\ \cdots\cdots \\ C_0 = B_0^{n-1}A_n + B_0^{n-2}A_{n-1} + \cdots + A_1 \\ R_0 = A_0 + B_0^nA_n + B_0^{n-1}A_{n-1} + \cdots + B_0A_1 \end{array}\right. .$$

Hence we obtain $q_R(x)$ and $r_R(x)$. If $R_0 = A_0 + B_0^n A_n + B_0^{n-1}A_{n-1} + \cdots + B_0A_1 = 0$, then $xI_t - B_0$ right divides $f(x)$. In the case $f(x) = g(x)q_L(x) + r_L(x)$, we have $q_L(x) = q_R^T(x)$ and $r_L(x) = r_R^T(x)$ in [2]. Therefore we can compute the $q_L(x)$ and $r_L(x)$. □

In the following two theorems, we will give a necessary and sufficient condition to left division and right division, respectively.

Theorem 2.8 *Suppose that $f(x) = \sum_{i=0}^{n} A_i x^i$ is a matrix polynomial of degree n and $g(x) = \sum_{j=0}^{m} B_j x^j$ is a matrix polynomial of degree m such that $m \le n$ and B_m is an invertible matrix. Let $A = (A_0, A_1, A_2, \cdots, A_n)$ be the coefficient matrix of $f(x)$ and*

$$B_g = \begin{pmatrix} B_0 & B_1 & B_2 & \cdots & B_m & 0 & 0 & 0 & \cdots & 0 \\ 0 & B_0 & B_1 & B_2 & \cdots & B_m & 0 & 0 & \cdots & 0 \\ 0 & 0 & B_0 & B_1 & B_2 & \cdots & B_m & 0 & \cdots & 0 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ 0 & 0 & 0\cdots & \cdots & 0 & B_0 & B_1 & B_2 & \cdots & B_m \end{pmatrix}_{[t(n-m+1)]\times[t(n+1)]}$$

be the relative coefficient matrix of $g(x)$, where the number of columns of B_g is equal to the number of columns of A. Then $g(x) \mid_R f(x)$ if and only if

$$\mathrm{Rank}{B_g}^T = \mathrm{Rank}({B_g}^T \mid A^T).$$

Proof. Clearly, we have $\mathrm{Rank}B_g = t(n-m+1)$ since B_m is invertible. It follows from Theorem 2.6 that $g(x) \mid_R f(x)$ if and only if there exists a unique

$$q_R(x) = (C_0, C_1, \cdots, C_{n-m})(I_t, I_t x, \cdots, I_t x^{n-m})^T$$

such that $f(x) = q_R(x)g(x)$.

Let $C = (C_0, C_1, \cdots, C_{n-m})$, $c_{ij}^{(k)} \in C_k$, $a_{ij}^{(s)} \in A_s (0 \le k \le n-m, 0 \le s \le n, 1 \le i, j \le t)$. Then we have

$$f(x) = q_R(x)g(x) \Leftrightarrow (A_0, A_1, \cdots, A_n) =$$

$$C \begin{pmatrix} B_0 & B_1 & B_2 & \cdots & B_m & 0 & 0 & 0 & \cdots & 0 \\ 0 & B_0 & B_1 & B_2 & \cdots & B_m & 0 & 0 & \cdots & 0 \\ 0 & 0 & B_0 & B_1 & B_2 & \cdots & B_m & 0 & \cdots & 0 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ 0 & 0 & 0\cdots & \cdots & 0 & B_0 & B_1 & B_2 & \cdots & B_m \end{pmatrix}_{[t(n-m+1)]\times[t(n+1)]}$$

$$\Leftrightarrow B_g^T \begin{pmatrix} C_0^T \\ C_1^T \\ \vdots \\ C_{n-m}^T \end{pmatrix} = \begin{pmatrix} A_0^T \\ A_1^T \\ \vdots \\ A_n^T \end{pmatrix}$$

$$\Leftrightarrow B_g^T \begin{pmatrix} c_{11}^{(0)} & \cdots & c_{t1}^{(0)} \\ \vdots & & \vdots \\ c_{1t}^{(0)} & \cdots & c_{tt}^{(0)} \\ c_{11}^{(1)} & \cdots & c_{t1}^{(1)} \\ \vdots & & \vdots \\ c_{1t}^{(1)} & \cdots & c_{tt}^{(1)} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ c_{11}^{(n-m)} & \cdots & c_{t1}^{(n-m)} \\ \vdots & & \vdots \\ c_{1t}^{(n-m)} & \cdots & c_{tt}^{(n-m)} \end{pmatrix} = \begin{pmatrix} a_{11}^{(0)} & \cdots & a_{t1}^{(0)} \\ \vdots & & \vdots \\ a_{1t}^{(0)} & \cdots & a_{tt}^{(0)} \\ a_{11}^{(1)} & \cdots & a_{t1}^{(1)} \\ \vdots & & \vdots \\ a_{1t}^{(1)} & \cdots & a_{tt}^{(1)} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ a_{11}^{(n)} & \cdots & a_{t1}^{(n)} \\ \vdots & & \vdots \\ a_{1t}^{(n)} & \cdots & a_{tt}^{(n)} \end{pmatrix}$$

$$\Leftrightarrow B_g^T \begin{pmatrix} c_{i1}^{(0)} \\ \vdots \\ c_{it}^{(0)} \\ c_{i1}^{(1)} \\ \vdots \\ c_{it}^{(1)} \\ \vdots \\ \vdots \\ c_{i1}^{(n-m)} \\ \vdots \\ c_{it}^{(n-m)} \end{pmatrix} = \begin{pmatrix} a_{i1}^{(0)} \\ \vdots \\ a_{it}^{(0)} \\ a_{i1}^{(1)} \\ \vdots \\ a_{it}^{(1)} \\ \vdots \\ \vdots \\ a_{i1}^{(n)} \\ \vdots \\ a_{it}^{(n)} \end{pmatrix} (i = 1, 2, \cdots, t)$$

has a unique solution if and only if

$$\text{Rank}{B_g}^T = \text{Rank}({B_g}^T \mid {A_{\tilde{i}}}^T)(\tilde{i} = 1, 2, \cdots, t),$$

where $A_{\tilde{i}}$ is the i-th row of A. Thus $g(x) \mid_R f(x)$ if and only if $\text{Rank}{B_g}^T = \text{Rank}({B_g}^T \mid A^T)$. □

Theorem 2.9 *Suppose that* $f(x) = \sum\limits_{i=0}^{n} A_i x^i$ *is a matrix polynomial of degree* n *and* $g(x) = \sum\limits_{j=0}^{m} B_j x^j$ *is a matrix polynomial of degree* m *such that* $m \leq n$ *and* B_m *is an invertible matrix. Let* $M = (A_0^T, A_1^T, A_2^T, \cdots, A_n^T)$ *be the coefficient matrix of* $f^T(x)$ *and*

$$B_g = \begin{pmatrix} B_0 & B_1 & B_2 & \cdots & B_m & 0 & 0 & 0 & \cdots & 0 \\ 0 & B_0 & B_1 & B_2 & \cdots & B_m & 0 & 0 & \cdots & 0 \\ 0 & 0 & B_0 & B_1 & B_2 & \cdots & B_m & 0 & \cdots & 0 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ 0 & 0 & 0\cdots & \cdots & 0 & B_0 & B_1 & B_2 & \cdots & B_m \end{pmatrix}_{[t(n-m+1)]\times[t(n+1)]}$$

be the relative coefficient matrix of $g^T(x)$, where the number of columns of B_{g^T} is equal to the number of columns of M. Then $g(x) \mid_L f(x)$ if and only if

$$\mathrm{Rank}(B_{g^T})^T = \mathrm{Rank}((B_{g^T})^T \mid M^T).$$

Proof. Since B_m is invertible if and only if B_m^T is invertible, we have $\mathrm{Rank}B_{g^T} = t(n-m+1)$. Then it follows from Theorem 2.6 that $g(x) \mid_L f(x)$ if and only if there exists a unique $q_L(x) = (C_0, C_1, \cdots, C_{n-m})(I_t, I_t x, \cdots, I_t x^{n-m})^T$ such that $f(x) = g(x)q_L(x)$.

Let $D = (C_0^T, C_1^T, \cdots, C_{n-m}^T)$, $c_{ij}^{(k)} \in C_k$, $a_{ij}^{(s)} \in A_s$ $(0 \le k \le n-m, 0 \le s \le n, 1 \le i, j \le t)$. Then we have

$$f(x) = g(x)q_L(x) \Leftrightarrow f^T(x) = q_L^T(x)g^T(x) \Leftrightarrow (A_0^T, A_1^T, \cdots, A_n^T) =$$

$$D\begin{pmatrix} B_0 & B_1 & B_2 & \cdots & B_m & 0 & 0 & 0 & \cdots & 0 \\ 0 & B_0 & B_1 & B_2 & \cdots & B_m & 0 & 0 & \cdots & 0 \\ 0 & 0 & B_0 & B_1 & B_2 & \cdots & B_m & 0 & \cdots & 0 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ 0 & 0 & 0\cdots & \cdots & 0 & B_0 & B_1 & B_2 & \cdots & B_m \end{pmatrix}_{[t(n-m+1)]\times[t(n+1)]}$$

$$\Leftrightarrow (B_{g^T})^T \begin{pmatrix} C_0 \\ C_1 \\ \vdots \\ C_{n-m} \end{pmatrix} = \begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_n \end{pmatrix}$$

$$\Leftrightarrow (B_{g^T})^T \begin{pmatrix} c_{11}^{(0)} & \cdots & c_{1t}^{(0)} \\ \vdots & & \vdots \\ c_{t1}^{(0)} & \cdots & c_{tt}^{(0)} \\ c_{11}^{(1)} & \cdots & c_{1t}^{(1)} \\ \vdots & & \vdots \\ c_{t1}^{(1)} & \cdots & c_{tt}^{(1)} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ c_{11}^{(n-m)} & \cdots & c_{1t}^{(n-m)} \\ \vdots & & \vdots \\ c_{t1}^{(n-m)} & \cdots & c_{tt}^{(n-m)} \end{pmatrix} = \begin{pmatrix} a_{11}^{(0)} & \cdots & a_{1t}^{(0)} \\ \vdots & & \vdots \\ a_{t1}^{(0)} & \cdots & a_{tt}^{(0)} \\ a_{11}^{(1)} & \cdots & a_{1t}^{(1)} \\ \vdots & & \vdots \\ a_{t1}^{(1)} & \cdots & a_{tt}^{(1)} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ a_{11}^{(n)} & \cdots & a_{1t}^{(n)} \\ \vdots & & \vdots \\ a_{t1}^{(n)} & \cdots & a_{tt}^{(n)} \end{pmatrix}$$

$$\Leftrightarrow (B_{g^T})^T \begin{pmatrix} c_{1i}^{(0)} \\ \vdots \\ c_{ti}^{(0)} \\ c_{1i}^{(1)} \\ \vdots \\ c_{ti}^{(1)} \\ \vdots \\ \vdots \\ c_{1i}^{(n-m)} \\ \vdots \\ c_{ti}^{(n-m)} \end{pmatrix} = \begin{pmatrix} m_{i1}^{(0)} \\ \vdots \\ m_{it}^{(0)} \\ m_{i1}^{(1)} \\ \vdots \\ m_{it}^{(1)} \\ \vdots \\ \vdots \\ m_{i1}^{(n)} \\ \vdots \\ m_{it}^{(n)} \end{pmatrix} (i = 1, 2, \cdots, t)$$

has a unique solution if and only if $\mathrm{Rank}(B_{g^T})^T = \mathrm{Rank}((B_{g^T})^T \mid M_{\tilde{i}}^T)(\tilde{i} = 1, 2, \cdots, t)$, where $M_{\tilde{i}}$ is the i-th row of M. Thus, $g(x) \mid_L f(x)$ if and only if $\mathrm{Rank}(B_{g^T})^T = \mathrm{Rank}((B_{g^T})^T \mid M^T)$. □

Although, the following corollary can be found in many books on theory of matrix and we give a new proof in Remark 2.7, we will give another proof as an application of Theorems 2.8 and 2.9.

Corollary 2.10 *Suppose that $f(x) = \sum\limits_{i=0}^{n} A_i x^i$ is a matrix polynomial of degree n and $g(x) = I_t x - B_0$ is a matrix polynomial. Then the following statements hold.*

(1) *$g(x) \mid_L f(x)$ if and only if*

$$B_0^n A_n + \cdots + B_0^j A_j \cdots + B_0 A_1 + A_0 = 0.$$

(2) *$g(x) \mid_R f(x)$ if and only if*

$$A_n B_0^n + \cdots + A_j B_0^j \cdots + A_1 B_0 + A_0 = 0.$$

Proof. (1) Let $A = (A_0, A_1, A_2, \cdots, A_n)$ be the coefficient matrix of $f(x)$ and

$$B_g = \begin{pmatrix} -B_0 & I_t & 0 & \cdots & 0 & 0 & 0 \\ 0 & -B_0 & I_t & 0 & \cdots & 0 & 0 \\ 0 & 0 & -B_0 & I_t & 0 & \cdots & 0 \\ & & \ddots & \ddots & \ddots & & \\ 0 & 0 & 0 & \cdots & 0 & -B_0 & I_t \end{pmatrix}_{tn\times[t(n+1)]}$$

the relative coefficient matrix of $g(x)$, where the number of columns of B_g is equal to the number of columns of A. Then it follows from Theorem 2.9 that $g(x) \mid_L f(x)$ if and only if

$$\mathrm{Rank}(B_{g^T})^T = \mathrm{Rank}((B_{g^T})^T \mid M^T),$$

i.e.,

$$\mathrm{Rank}\begin{pmatrix} -B_0 & 0 & 0 & 0 & \cdots & 0 \\ I_t & -B_0 & 0 & 0 & \cdots & 0 \\ 0 & I_t & -B_0 & 0 & \cdots & 0 \\ 0 & 0 & I_t & -B_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I_t & -B_0 \\ 0 & 0 & 0 & \cdots & 0 & I_t \end{pmatrix} =$$

$$\mathrm{Rank}\begin{pmatrix} -B_0 & 0 & 0 & 0 & \cdots & 0 & A_0 \\ I_t & -B_0 & 0 & 0 & \cdots & 0 & A_1 \\ 0 & I_t & -B_0 & 0 & \cdots & 0 & A_2 \\ 0 & 0 & I_t & -B_0 & \cdots & 0 & A_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I_t & -B_0 & A_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & I_t & A_n \end{pmatrix}.$$

Hence, there are t-square matrices $X_0, X_1, \cdots, X_n$ such that

$$\begin{cases} -B_0X_0 = A_0 & \cdots\cdots\cdots\cdots\cdots\cdots (0) \\ I_tX_0 - B_0X_1 = A_1 & \cdots\cdots\cdots\cdots\cdots (1) \\ I_tX_1 - B_0X_2 = A_2 & \cdots\cdots\cdots\cdots\cdots (2) \\ \cdots\cdots\cdots & \\ I_tX_{n-2} - B_0X_{n-1} = A_{n-1} & \cdots\cdots\cdots (n-1) \\ I_tX_n = A_n & \cdots\cdots\cdots\cdots\cdots\cdots (n) \end{cases}.$$

Using $(0) + B_0 \cdot (1) + B_0^2 \cdot (2) + \cdots B_0^j \cdot (j) + \cdots B_0^n \cdot (n)$, we have

$$B_0^nA_n + \cdots + B_0^jA_j \cdots + B_0A_1 + A_0 = 0.$$

(2) Similarly, $g(x) \mid_R f(x)$ if and only if $A_nB_0^n + \cdots + A_jB_0^j + \cdots + A_1B_0 + A_0 = 0$. □

Remark 2.11 *For ordinary polynomials, the left division and right division is the same. Let $t = 1$, it follows from Theorem 2.8 or Theorem 2.9 that $g(x) \mid f(x)$ if and only if $\mathrm{Rank}{B_g}^T = \mathrm{Rank}({B_g}^T \mid A^T)$, where $f(x)$ and $g(x)$ are polynomials.*

References

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