Mathematica Aeterna, Vol. 4, 2014, no. 6, 671 - 682

Lie triple derivations and Jordan derivations of Hom-Lie algebra

Bing Sun

School of Mathematics and Statistics
Northeast Normal University

Liangyun Chen

School of Mathematics and Statistics
Northeast Normal University

Abstract

The paper studies generalized derivations of Hom-Lie algebras. More specifically speaking, (generalized) Lie triple (θ, φ)-derivations and (generalized) Jordan triple (θ, φ)-derivations on a Hom-Lie algebra are investigated. It is proved that Jordan triple (θ, φ)-derivations (resp. generalized Jordan triple (θ, φ)-derivations) are Lie triple (θ, φ)-derivations (resp. generalized Lie triple (θ, φ)-derivations) on a Hom-Lie algebra under some conditions. In particular, θ-Jordan derivations are Lie triple θ-derivations on a Hom-Lie algebra.

Mathematics Subject Classification: 17B40, 16W25

Keywords: Lie triple derivation, Jordan derivation, generalized derivation, Hom-Lie algebra

1 Introduction

As is well known, derivation and generalized derivation algebras are very important subjects in the research of Lie algebras. Leger and Luks investigated the structure of the generalized derivations of Lie algebras systematically (cf. [6]). Generalized derivations on rings was studied (cf.[4] [7]). Generalized derivations also play a key role in Benoist's study of Levi factors in derivation algebras of nilpotent Lie algebras(cf.[2]). Brešar, M. together with Vukman, J. generalized Herstein's result to Jordan (Θ, φ)-derivations(cf. [3]). In [1], the authors proved that in a 2-torsion free non-commutative prime ring R, a generalized Jordan (θ, φ)-derivation is a generalized (θ, φ)-derivation when θ is an automorphism of R.

In recent years, there has been an increasing interest in investigating Lie triple derivation (cf. [5] [8][9][11][12]). People studied Lie triple derivation over a field of characteristic zero of Borel subalgebra (cf. [5]), parabolic subalgebras (cf. [12]), triangular matrices (cf. [11]). Hom-Lie algebras are the natural generalization of Lie algebras and have important applications both in mathematics and physics (cf.[10]). The aim of this paper is to characterize generalized derivations for Hom-Lie algebras. More specifically speaking, (generalized) Lie triple (θ, φ)-derivations and (generalized) Jordan triple (θ, φ)-derivations on a Hom-Lie algebra are investigated. It is proved that Jordan triple (θ, φ)-derivations (resp. generalized Jordan triple (θ, φ)-derivations) are Lie triple (θ, φ)-derivations (resp. generalized Lie triple (θ, φ)-derivations) on a Hom-Lie algebra under some conditions. In particular, θ-Jordan derivations are Lie triple θ-derivations on a Hom-Lie algebra.

Throughout this paper, the base field $\mathbf{F}$ is assumed to be of characteristic not equal to 3. We now recall some elementary definitions.

2 Preliminary Notes

Definition 2.1 *[11] A Lie triple derivation of a Lie algebra L is a linear mapping $D : L \to L$ such that*

$$D([[x, y], z]) = [[D(x), y], z] + [[x, D(y)], z] + [[x, y], D(z)], \forall x, y, z \in L.$$

Definition 2.2 *[4] A Jordan triple derivation of a Lie algebra L is a linear mapping $D' : L \to L$ such that*

$$D'([[x, y], x]) = [[D'(x), y], x] + [[x, D'(y)], x] + [[x, y], D'(x)], \forall x, y, z \in L.$$

Definition 2.3 *[10] A Hom-Lie algebra is a triple $(L, [\cdot, \cdot], \alpha)$ consisting of a vector space L, a skew-symmetric bilinear map $[\cdot, \cdot] : \wedge^2 L \to L$ and a linear map $\alpha : L \to L$ satisfying the following hom-Jacobi identity:*

$$[\alpha(u), [v, w]] + [\alpha(v), [w, u]] + [\alpha(w), [u, v]] = 0.$$

3 Main Results

Definition 3.1 *A Lie triple derivation of a Hom-Lie algebra $(L, [\cdot, \cdot], \alpha)$ is a linear map $D : L \to L$ such that*

$$D \circ \alpha = \alpha \circ D$$

and

$$D([[x,y],z]=[[D(x),\alpha^k(y)],\alpha^k(z)]+[[\alpha^k(x),D(y)],\alpha^k(z)]+[[\alpha^k(x),\alpha^k(y)],D(z)],$$

for all $x, y \in L$.

Definition 3.2 *A Jordan Lie triple derivation of a Hom-Lie algebra* $(L,[\cdot,\cdot],\alpha)$ *is a linear map* $D' : L \to L$ *such that*

$$D' \circ \alpha = \alpha \circ D'$$

and

$$D'([[x,y],x]=[[D'(x),\alpha^k(y)],\alpha^k(x)]+[[\alpha^k(x),D'(y)],\alpha^k(x)]+[[\alpha^k(x),\alpha^k(y)],D'(x)],$$

for all $x, y \in L$.

Definition 3.3 *Let* $(L,[\cdot,\cdot],\alpha)$ *be a Hom-Lie algebra and let* $D,\theta,\varphi : L \to L$ *be linear maps satisfying* $D \circ \alpha = \alpha \circ D$.

1. *D is called a* ***Lie triple*** $(\theta,\varphi)_1$***-derivation*** *if*

$$\begin{aligned} D([[x,y],z]) &= [[D(x),\theta(\alpha^k(y))],\varphi(\alpha^k(z))] + [[\theta(\alpha^k(x)),D(y)],\varphi(\alpha^k(z))] \\ &\quad +[[\theta(\alpha^k(x)),\varphi(\alpha^k(y))],D(z)], \qquad \forall x,y,z \in L. \end{aligned}$$

2. *D is called a* ***Lie triple*** $(\theta,\varphi)_2$***-derivation*** *if*

$$\begin{aligned} D([[x,y],z]) &= [[D(x),\theta(\alpha^k(y))],\theta(\alpha^k(z))] + [[\theta(\alpha^k(x)),D(y)],\varphi(\alpha^k(z))] \\ &\quad +[[\varphi(\alpha^k(x)),\varphi(\alpha^k(y))],D(z)], \qquad \forall x,y,z \in L. \end{aligned}$$

3. *D is called a* ***Lie triple*** $(\theta,\varphi)_3$***-derivation*** *if*

$$\begin{aligned} D([[x,y],z]) &= [[D(x),\theta(\alpha^k(y))],\theta(\alpha^k(z))] + [[\varphi(\alpha^k(x)),D(y)],\theta(\alpha^k(z))] \\ &\quad +[[\varphi(\alpha^k(x)),\varphi(\alpha^k(y))],D(z)], \qquad \forall x,y,z \in L. \end{aligned}$$

In particular, $\forall i = 1,2,3$*, a* $(\theta,\varphi)_i$*-Lie triple derivation D is called a* ***Lie triple*** θ***-derivation*** *if* $\theta = \varphi$*. It is clear that a* $(\theta,\varphi)_i$*-Lie triple derivation is a Lie triple derivation when* $\theta = \varphi = 1_T$.

Definition 3.4 *Let* $(L,[\cdot,\cdot],\alpha)$ *be a Hom-Lie algebra and let* $D,\theta,\varphi : L \to L$ *be linear maps satisfying* $D \circ \alpha = \alpha \circ D$.

1. *D is called a* ***Jordan triple*** $(\theta,\varphi)_1$***-derivation*** *if*

$$\begin{aligned} D([[x,y],x]) &= [[D(x),\theta(\alpha^k(y))],\varphi(\alpha^k(x))]+[[\theta(\alpha^k(x)),D(y)],\varphi(\alpha^k(x))] \\ &\quad +[[\theta(\alpha^k(x)),\varphi(\alpha^k(y))],D(x)], \qquad \forall x,y \in L. \end{aligned}$$

2. *D is called a* ***Jordan triple*** $(\theta,\varphi)_2$***-derivation*** *if*

$$\begin{aligned} D([[x,y],z]) &= [[D(x),\theta(\alpha^k(y))],\theta(\alpha^k(x))]+[[\theta(\alpha^k(x)),D(y)],\varphi(\alpha^k(x))] \\ &\quad +[[\varphi(\alpha^k(x)),\varphi(\alpha^k(y))],D(x)], \qquad \forall x,y\in L. \end{aligned}$$

3. *D is called a* ***Jordan triple*** $(\theta,\varphi)_3$***-derivation*** *if*

$$\begin{aligned} D([[x,y],z]) &= [[D(x),\theta(\alpha^k(y))],\theta(\alpha^k(x))]+[[\varphi(\alpha^k(x)),D(y)],\theta(\alpha^k(x))] \\ &\quad +[[\varphi(\alpha^k(x)),\varphi(\alpha^k(y))],D(x)], \qquad \forall x,y\in L. \end{aligned}$$

In particular, $\forall i=1,2,3$*, a* $(\theta,\varphi)_i$*-Jordan triple derivation D is called a* ***Jordan triple*** θ***-derivation*** *if* $\theta=\varphi$*. It is clear that a* $(\theta,\varphi)_i$*-Jordan triple derivation is a Jordan triple derivation when* $\theta=\varphi=1_T$*.*

It is clear that if D_i is a Lie triple $(\theta,\varphi)_i$-derivation of $(L,[\cdot,\cdot],\alpha)$, then D_i is a Jordan triple $(\theta,\varphi)_i$-derivation of $(L,[\cdot,\cdot],\alpha)$, where $i=1,2,3$.

In this section, $(L,[\cdot,\cdot],\alpha)$ is a Hom-Lie algebra and $\theta,\varphi: L\to L$ are defined to be linear maps, where α is a injection and $\alpha[x,y]=[x,\alpha(y)]$.

Theorem 3.5 *D is a Lie triple* $(\theta,\varphi)_1$*-derivation of* $(L,[\cdot,\cdot],\alpha)$ *if and only if D is a Jordan triple* $(\theta,\varphi)_1$*-derivation of* $(L,[\cdot,\cdot],\alpha)$ *such that*

1. $[[\theta(\alpha^k(x)),\varphi(\alpha^k(y))],D(z)]=[[\varphi(\alpha^k(x)),\theta(\alpha^k(y))],D(z)]$,
2. $A(x,y,z)+A(y,z,x)+A(z,x,y)=0$,

where $x,y,z\in L$ *and*

$$\begin{aligned} A(x,y,z) &= \alpha([[D(x),\theta(\alpha^k(y))],\varphi(\alpha^k(z))]+[[\theta(\alpha^k(x)),D(y)],\varphi(\alpha^k(z))] \\ &\quad +[[\theta(\alpha^k(x)),\varphi(\alpha^k(y))],D(z)]). \end{aligned}$$

Proof. Assume that D is a Lie triple $(\theta,\varphi)_1$-derivation of $(L,[\cdot,\cdot],\alpha)$. Clearly, D is a Jordan triple $(\theta,\varphi)_1$-derivation of$(L,[\cdot,\cdot],\alpha)$ and $D([[x,y],z])=-D([[y,x],z])$; note that

$$\begin{aligned} -D([[y,x],z]) &= -[[D(y),\theta(\alpha^k(x))],\varphi(\alpha^k(z))]-[[\theta(\alpha^k(y)),D(x)],\varphi(\alpha^k(z))] \\ &\quad -[[\theta(\alpha^k(y)),\varphi(\alpha^k(x))],D(z)] \\ &= [D(x),\theta(\alpha^k(y)),\varphi(\alpha^k(z))]+[[\theta(\alpha^k(x)),D(y)],\varphi(\alpha^k(z))] \\ &\quad +[[\varphi(\alpha^k(x)),\theta(\alpha^k(y))],D(z)] \end{aligned}$$

and

$$\begin{aligned} D([[x,y],z]) &= [[D(x),\theta(\alpha^k(y))],\varphi(\alpha^k(z))]+[[\theta(\alpha^k(x)),D(y)],\varphi(\alpha^k(z))] \\ &\quad + [[\theta(\alpha^k(x)),\varphi(\alpha^k(y))],D(z)], \end{aligned}$$

then 1 follows. Since D is a Lie triple $(\theta, \varphi)_1$-derivation of $(L, [\cdot,\cdot], \alpha)$, we have

$$\alpha(D([[x, y], z])) = A(x, y, z);$$

hence

$$\begin{aligned} & A(x, y, z) + A(y, z, x) + A(z, x, y) \\ = \ & \alpha(D([[x, y], z]) + D([[y, z], x]) + D([[z, x], y])) \\ = \ & D[[x, y], \alpha(z)] + [[y, z], \alpha(x)] + [[z, x], \alpha(y)] \\ = \ & 0. \end{aligned}$$

Conversely, let D be a Jordan triple $(\theta, \varphi)_1$-derivation of $(L, [\cdot,\cdot], \alpha)$ for which 1 and 2 hold. Then $\alpha(D([[x, y], x])) = A(x, y, x)$. It follows that

$$\begin{aligned} & \alpha(D([[x + z, y], x + z])) \\ = \ & \alpha(D([[x, y], x]) + D([[x, y], z]) + D([[z, y], x]) + D([[z, y], z])) \\ = \ & A(x, y, x) + A(z, y, z) + \alpha(D([[x, y], z])) + \alpha(D([[z, y], x])) \end{aligned}$$

and

$$\begin{aligned} \alpha(D([[x + z, y], x + z])) &= A(x + z, y, x + z) \\ &= A(x, y, x) + A(x, y, z) + A(z, y, x) + A(z, y, z). \end{aligned}$$

Thus we obtain

$$\alpha(D([[x, y], z])) + \alpha(D([[z, y], x])) = A(x, y, z) + A(z, y, x). \tag{1}$$

By 1,

$$\begin{aligned} & A(y, x, z) \\ = \ & \alpha([[D(y), \theta(\alpha^k(x))], \varphi(\alpha^k(z))] + [[\theta(\alpha^k(y)), D(x)], \varphi(\alpha^k(z))] \\ & +[[\theta(\alpha^k(y)), \varphi(\alpha^k(x))], D(z)]) \\ = \ & -\alpha([[D(x), \theta(\alpha^k(y))], \varphi(\alpha^k(z))] + [[\theta(\alpha^k(x)), D(y)], \varphi(\alpha^k(z))] \\ & +[[\varphi(\alpha^k(x)), \theta(\alpha^k(y))], D(z)]) \\ = \ & -\alpha([[D(x), \theta(\alpha^k(y))], \varphi(\alpha^k(z))] + [[\theta(\alpha^k(x)), D(y)], \varphi(\alpha^k(z))] \\ & +[[\theta(\alpha^k(x)), \varphi(\alpha^k(y))], D(z)]) \\ = \ & -A(x, y, z). \end{aligned}$$

This implies that

$$\alpha(D([[x, y], y])) = -\alpha(D([[y, x], y])) = -A(y, x, y) = A(x, y, y).$$

A similar argument proves

$$\alpha(D([[x, y], z])) + \alpha(D([[x, z], y])) = A(x, y, z) + A(x, z, y). \tag{2}$$

By(1)+(2), we have

$$\begin{aligned}&\alpha(D([[x,y],z]))+\alpha(D([[z,y],x]))+\alpha(D([[x,y],z]))+\alpha(D([[x,z],y]))\\=\;&A(x,y,z)+A(z,y,x)+A(x,y,z)+A(x,z,y),\end{aligned}$$

then

$$\begin{aligned}&\alpha(D([[x,y],z]))+\alpha(D([[x,y],z]))-\alpha(D([[y,x],z]))\\=\;&(A(x,y,z)+A(z,y,x)+A(x,y,z)+A(x,z,y),\end{aligned}$$

that is,

$$3\alpha(D([[x,y],z]))=3A(x,y,z)+A(z,y,x)+A(x,z,y)+A(y,x,z)=3A(x,y,z),$$

where the last equality uses (ii). Since ch$\mathbf{F}\neq\mathbf{3}$, we have $\alpha(D([[x,y],z]))=A(x,y,z)$, and so $D([[x,y],z])=[[D(x),\theta(\alpha^k(y))],\varphi(\alpha^k(z))]+[[\theta(\alpha^k(x)),D(y)],\varphi(\alpha^k(z))]+[[\theta(\alpha^k(x)),\varphi(\alpha^k(y))],D(z)]$, i.e., D is a Lie triple $(\theta,\varphi)_1$-derivation of $(L,[\cdot,\cdot],\alpha)$. □

Corollary 3.6 *D is a Lie triple θ-derivation of $(L,[\cdot,\cdot],\alpha)$ if and only if D is a Jordan triple θ-derivation of $(L,[\cdot,\cdot],\alpha)$.*

Proof. If D is a Jordan triple θ-derivation of $(L,[\cdot,\cdot],\alpha)$, then 1 follows immediately. 2 holds because

$$\begin{aligned}&A(x,y,z)+A(y,z,x)+A(z,x,y)\\=\;&\alpha([[D(x),\theta(\alpha^k(y))],\theta(\alpha^k(z))]+[[\theta(\alpha^k(x)),D(y)],\theta(\alpha^k(z))]\\&+[[\theta(\alpha^k(x)),\theta(\alpha^k(y))],D(z)])+\alpha([[D(y),\theta(\alpha^k(z))],\theta(\alpha^k(x))]\\&+[[\theta(\alpha^k(y)),D(z)],\theta(\alpha^k(x))]+[[\theta(\alpha^k(y)),\theta(\alpha^k(z))],D(x)])\\&+\alpha([[D(z),\theta(\alpha^k(x))],\theta(\alpha^k(y))]+[[\theta(\alpha^k(z)),D(x)],\theta(\alpha^k(y))]\\&+[[\theta(\alpha^k(z)),\theta(\alpha^k(x))],D(y)])\\=\;&\alpha([[D(x),\theta(\alpha^k(y))],\theta(\alpha^k(z))]+[[\theta(\alpha^k(y)),\theta(\alpha^k(z))],D(x)]\\&+[[\theta(\alpha^k(z)),D(x)],\theta(\alpha^k(y))])+\alpha([[\theta(\alpha^k(x)),D(y)],\theta(\alpha^k(z))]\\&+[[D(y),\theta(\alpha^k(z))],\theta(\alpha^k(x))]+[[\theta(\alpha^k(z)),\theta(\alpha^k(x))],D(y)])\\&+\alpha([[\theta(\alpha^k(x)),\theta(\alpha^k(y))],D(z)]+[[\theta(\alpha^k(y)),D(z)],\theta(\alpha^k(x))]\\&+[[D(z),\theta(\alpha^k(x))],\theta(\alpha^k(y))])\\=\;&0.\end{aligned}$$

Therefore, D is a Lie triple θ-derivation of $(L,[\cdot,\cdot],\alpha)$ by Theorem 3.5. □

Theorem 3.7 *D is a Lie triple $(\theta,\varphi)_2$-derivation of $(L,[\cdot,\cdot],\alpha)$ if and only if D is a Jordan triple $(\theta,\varphi)_2$-derivation of $(L,[\cdot,\cdot],\alpha)$ such that*

1. $[[D(x), \theta(\alpha^k(y))], (\varphi - \theta)(\alpha^k(z))] = [[\theta(\alpha^k(x)), D(y)], (\varphi - \theta)(\alpha^k(z))]$,

2. $A'(x, y, z) + A'(y, z, x) + A'(z, x, y) = 0$,

where $x, y, z \in L$ *and* $A'(x, y, z) = \alpha([[D(x), \theta(\alpha^k(y))], \theta(\alpha^k(z))] + [[\theta(\alpha^k(x)), D(y)], \varphi(\alpha^k(z))] + [[\varphi(\alpha^k(x)), \varphi(\alpha^k(y))], D(z)])$.

Proof. Let D be a Lie triple $(\theta, \varphi)_2$-derivation of $(L, [\cdot, \cdot], \alpha)$. Use the fact that

$$D([[x, y], z]) = -D([[y, x], z]),$$

as well as the fact that

$$\begin{aligned} & -D([[y, x], z]) \\ = & -[[D(y), \theta(\alpha^k(x))], \theta(\alpha^k(z))] - [[\theta(\alpha^k(y)), D(x)], \varphi(\alpha^k(z))] \\ & -[[\varphi(\alpha^k(y)), \varphi(\alpha^k(x))], D(z)] \\ = & [[D(x), \theta(\alpha^k(y))], \varphi(\alpha^k(z))] + [[\theta(\alpha^k(x)), D(y)], \theta(\alpha^k(z))] \\ & +[[\varphi(\alpha^k(x)), \varphi(\alpha^k(y))], D(z)], \end{aligned}$$

then we have

$$\begin{aligned} & [[D(x), \theta(\alpha^k(y))], \theta(\alpha^k(z))] + [[\theta(\alpha^k(x)), D(y)], \varphi(\alpha^k(z))] \\ = & [[D(x), \theta(\alpha^k(y))], \varphi(\alpha^k(z))] + [[\theta(\alpha^k(x)), D(y)], \theta(\alpha^k(z))], \end{aligned}$$

that is, $[[D(x), \theta(\alpha^k(y))], (\varphi - \theta)(\alpha^k(z))] = [[\theta(\alpha^k(x)), D(y)], (\varphi - \theta)(\alpha^k(z))]$. It is routine to prove 2.

Suppose, conversely, that D is a Jordan triple $(\theta, \varphi)_2$-derivation of $(L, [\cdot,\cdot], \alpha)$ satisfying 1 and 2. Note that

$$\begin{aligned} & A'(y, x, z) \\ = & \alpha([[D(y), \theta(\alpha^k(x))], \theta(\alpha^k(z))] + [[\theta(\alpha^k(y)), D(x)], \varphi(\alpha^k(z))] \\ & +[[\varphi(\alpha^k(y)), \varphi(\alpha^k(x))], D(z)]) \\ = & -\alpha([[D(x), \theta(\alpha^k(y))], \varphi(\alpha^k(z))] - [[\theta(\alpha^k(x)), D(y)], \theta(\alpha^k(z))] \\ & -[[\varphi(\alpha^k(x)), \varphi(\alpha^k(y))], D(z)]) \\ = & -\alpha([[D(x), \theta(\alpha^k(y))], \theta(\alpha^k(z))] - [[\theta(\alpha^k(x)), D(y)], \varphi(\alpha^k(z))] \\ & -[[\varphi(\alpha^k(x)), \varphi(\alpha^k(y))], D(z)]) \\ = & -A'(x, y, z). \end{aligned}$$

In the same way, we can get equalities (1) and (2). The rest proof is the same as the corresponding proof of Theorem 3.5. □

A similar argument proves the following result.

Theorem 3.8 *D is a Lie triple $(\theta, \varphi)_3$-derivation of $(L, [\cdot, \cdot], \alpha)$ if and only if D is a Jordan triple $(\theta, \varphi)_3$-derivation of $(L, [\cdot, \cdot], \alpha)$ such that*

1. $[[D(x),(\theta-\varphi)(\alpha^k(y))],\theta(\alpha^k(z))]=[[(\theta-\varphi)(\alpha^k(x)),D(y)],\theta(\alpha^k(z))]$,

2. $A''(x,y,z)+A''(y,z,x)+A''(z,x,y)=0$,

where $x,y,z\in L$ *and* $A''(x,y,z)=\alpha([[D(x),\theta(\alpha^k(y))],\theta(\alpha^k(z))]+[[\varphi(\alpha^k(x)),D(y)],\theta(\alpha^k(z))]+[[\varphi(\alpha^k(x)),\varphi(\alpha^k(y))],D(z)])$.

Remark 3.9 *Corollary 3.6 can also be concluded from Theorem 3.7 or Theorem 3.8 since for any* $x,y,z\in L$, $A(x,y,z)=A'(x,y,z)=A''(x,y,z)$ *when* D *is a Jordan triple* θ*-derivation.*

Definition 3.10 *Let* $(L,[\cdot,\cdot],\alpha)$ *be a Hom-Lie algebra and let* $D_i,\theta,\varphi:L\to L$ *be linear maps satisfying* $D_i\circ\alpha=\alpha\circ D_i,\forall i=1,2,3$.

1. *A* ***generalized Lie triple*** $(\theta,\varphi)_1$***-derivation*** *with respect to a Lie triple* $(\theta,\varphi)_1$*-derivation* δ_1 *is a linear map* $D_1:L\to L$ *such that*

$$\begin{aligned}D_1([[x,y],z]) &= [[\delta_1(x),\theta(\alpha^k(y))],\varphi(\alpha^k(z))]+[[\theta(\alpha^k(x)),\delta_1(y)],\varphi(\alpha^k(z))]\\ &\quad+[[\theta(\alpha^k(x)),\varphi(\alpha^k(y))],D_1(z)], \qquad \forall x,y,z\in L.\end{aligned}$$

2. *A* ***generalized Lie triple*** $(\theta,\varphi)_2$***-derivation*** *with respect to a Lie triple* $(\theta,\varphi)_2$*-derivation* δ_2 *is a linear map* $D_2:L\to L$ *such that*

$$\begin{aligned}D_2([[x,y],z]) &= [[\delta_2(x),\theta(\alpha^k(y))],\theta(\alpha^k(z))]+[[\theta(\alpha^k(x)),\delta_2(y)],\varphi(\alpha^k(z))]\\ &\quad+[[\varphi(\alpha^k(x)),\varphi(\alpha^k(y))],D_2(z)], \qquad \forall x,y,z\in L.\end{aligned}$$

3. *A* ***generalized Lie triple*** $(\theta,\varphi)_3$***-derivation*** *with respect to a Lie triple* $(\theta,\varphi)_3$*-derivation* δ_3 *is a linear map* $D_3:L\to L$ *such that*

$$\begin{aligned}D_3([[x,y],z]) &= [[\delta_3(x),\theta(\alpha^k(y))],\theta(\alpha^k(z))]+[[\varphi(\alpha^k(x)),\delta_3(y)],\theta(\alpha^k(z))]\\ &\quad+[[\varphi(\alpha^k(x)),\varphi(\alpha^k(y))],D_3(z)], \qquad \forall x,y,z\in L.\end{aligned}$$

In particular, $\forall i=1,2,3$, *a generalized Lie triple* $(\theta,\varphi)_i$*-derivation* D *is called a* ***generalized Lie triple*** θ***-derivation*** *with respect to a Lie triple* θ*-derivation* δ *if* $\theta=\varphi$. *It is clear that* D *is a generalized Lie triple derivation when* $\theta=\varphi=1_T$ *and* δ *is a Lie triple derivation.*

Definition 3.11 *Let* $(L,[\cdot,\cdot],\alpha)$ *be a Hom-Lie algebra and let* $D_i,\theta,\varphi:L\to L$ *be linear maps satisfying* $D_i\circ\alpha=\alpha\circ D_i,\forall i=1,2,3$.

1. *A* ***generalized Jordan triple*** $(\theta,\varphi)_1$***-derivation*** *with respect to a Lie triple* $(\theta,\varphi)_1$*-derivation* δ_1 *is a linear map* $D_1:L\to L$ *such that*

$$\begin{aligned}D_1([[x,y],x]) &= [[\delta_1(x),\theta(\alpha^k(y))],\varphi(\alpha^k(x))]+[[\theta(\alpha^k(x)),\delta_1(y)],\varphi(\alpha^k(x))]\\ &\quad+[[\theta(\alpha^k(x)),\varphi(\alpha^k(y))],D_1(x)], \qquad \forall x,y\in L.\end{aligned}$$

2. *A* ***generalized Jordan triple*** $(\theta,\varphi)_2$***-derivation*** *with respect to a Lie triple* $(\theta,\varphi)_2$*-derivation* δ_2 *is a linear map* $D_2 : L \to L$ *such that*

$$\begin{aligned} D_2([[x,y],z]) &= [[\delta_2(x),\theta(\alpha^k(y))],\theta(\alpha^k(x))]+[[\theta(\alpha^k(x)),\delta_2(y)],\varphi(\alpha^k(x))] \\ &\quad +[[\varphi(\alpha^k(x)),\varphi(\alpha^k(y))],D_2(x)], \qquad \forall x,y\in L. \end{aligned}$$

3. *A* ***generalized Jordan triple*** $(\theta,\varphi)_3$***-derivation*** *with respect to a Lie triple* $(\theta,\varphi)_3$*-derivation* δ_3 *is a linear map* $D_3 : L \to L$ *such that*

$$\begin{aligned} D_3([[x,y],z]) &= [[\delta_3(x),\theta(\alpha^k(y))],\theta(\alpha^k(x))]+[[\varphi(\alpha^k(x)),\delta_3(y)],\theta(\alpha^k(x))] \\ &\quad +[[\varphi(\alpha^k(x)),\varphi(\alpha^k(y))],D_3(x)], \qquad \forall x,y\in L. \end{aligned}$$

In particular, $\forall i = 1,2,3$*, a generalized Jordan Lie triple* $(\theta,\varphi)_i$*-derivation* D *is called a* ***generalized Jordan triple*** θ***-derivation*** *with respect to a Jordan triple* θ*-derivation* δ *if* $\theta=\varphi$*. It is clear that* D *is a generalized Jordan triple derivation when* $\theta=\varphi=1_T$ *and* δ *is a Jordan triple derivation.*

Theorem 3.12 *D is a generalized Lie triple $(\theta,\varphi)_1$-derivation of $(L,[\cdot,\cdot],\alpha)$ with respect to a Lie triple $(\theta,\varphi)_1$-derivation δ if and only if D is a generalized Jordan triple $(\theta,\varphi)_1$-derivation of $(L,[\cdot,\cdot],\alpha)$ with respect to a Lie triple $(\theta,\varphi)_1$-derivation δ such that*

1. $[[\theta(\alpha^k(x)),\varphi(\alpha^k(y))],D(z)] = [[\varphi(\alpha^k(x)),\theta(\alpha^k(y))],D(z)]$,
2. $B(x,y,z)+B(y,z,x)+B(z,x,y)=0$,

where $x,y,z\in L$ *and* $B(x,y,z)=\alpha([[\delta(x),\theta(\alpha^k(y))],\varphi(\alpha^k(z))]+[[\theta(\alpha^k(x)),\delta(y)],\varphi(\alpha^k(z))]+[[\theta(\alpha^k(x)),\varphi(\alpha^k(y))],D(z)])$.

Proof. Suppose that D is a generalized Lie triple $(\theta,\varphi)_1$-derivation of $(L,[\cdot,\cdot],\alpha)$ with respect to a Lie triple $(\theta,\varphi)_1$-derivation δ. Clearly, D is a generalized Jordan triple $(\theta,\varphi)_1$-derivation of $(L,[\cdot,\cdot],\alpha)$ and $D([x,y,z])=-D([y,x,z])$. 1 follows from the fact

$$\begin{aligned} -D([[y,x],z]) &= -[[\delta(y),\theta(\alpha^k(x))],\varphi(\alpha^k(z))]-[[\theta(\alpha^k(y)),\delta(x)],\varphi(\alpha^k(z))] \\ &\quad -[[\theta(\alpha^k(y)),\varphi(\alpha^k(x))],D(z)]) \\ &= [[\delta(x),\theta(\alpha^k(y))],\varphi(\alpha^k(z))]+[[\theta(\alpha^k(x)),\delta(y)],\varphi(\alpha^k(z))] \\ &\quad +[[\varphi(\alpha^k(x)),\theta(\alpha^k(y))],D(z)]. \end{aligned}$$

Since D is a generalized Lie triple $(\theta,\varphi)_1$-derivation of $(L,[\cdot,\cdot],\alpha)$ with respect to δ, we have $\alpha(D([[x,y],z]))=B(x,y,z)$; hence

$$\begin{aligned} & B(x,y,z)+B(y,z,x)+B(z,x,y) \\ = {} & \alpha(D([[x,y],z]))+\alpha(D([[y,z],x]))+\alpha(D([[z,x],y])) \\ = {} & D\left([[x,y],\alpha(z)]+[[y,z],\alpha(x)]+[[z,x],\alpha(y)]\right) \\ = {} & 0. \end{aligned}$$

Conversely, if D is a generalized Jordan triple $(\theta,\varphi)_1$-derivation of $(L,[\cdot,\cdot],\alpha)$ with respect to a Lie triple $(\theta,\varphi)_1$-derivation δ satisfying 1 and 2, then, refer to the proof of Theorem 3.5, it suffices to prove $B(x,y,z)=-B(y,x,z)$. In fact,

$$\begin{aligned}
& B(y,x,z) \\
= \ & \alpha([[\delta(y),\theta(\alpha^k(x))],\varphi(\alpha^k(z))]+[[\theta(\alpha^k(y)),\delta(x)],\varphi(\alpha^k(z))] \\
& +[[\theta(\alpha^k(y)),\varphi(\alpha^k(x))],D(z)]) \\
= \ & -\alpha([[\delta(x),\theta(\alpha^k(y))],\varphi(\alpha^k(z))]+[[\theta(\alpha^k(x)),\delta(y)],\varphi(\alpha^k(z))] \\
& +[[\varphi(\alpha^k(x)),\theta(\alpha^k(y))],D(z)]) \\
= \ & -\alpha([[\delta(x),\theta(\alpha^k(y))],\varphi(\alpha^k(z))]+[[\theta(\alpha^k(x)),\delta(y)],\varphi(\alpha^k(z))] \\
& +[[\theta(\alpha^k(x)),\varphi(\alpha^k(y))],D(z)]) \\
= \ & -B(x,y,z).
\end{aligned}$$

This completes the proof. □

Corollary 3.13 *D is a generalized Lie triple θ-derivation of $(L,[\cdot,\cdot],\alpha)$ with respect to a Lie triple θ-derivation δ if and only if D is a generalized Jordan triple θ-derivation of $(L,[\cdot,\cdot],\alpha)$ with respect to a Jordan triple θ-derivation δ such that*

$$\begin{aligned}
[[\theta(\alpha^k(x)),\theta(\alpha^k(y))],(D-\delta)(z)]+[[\theta(\alpha^k(y)),\theta(\alpha^k(z))],(D-\delta)(x)] & \\
+[[\theta(\alpha^k(z)),\theta(\alpha^k(x))],(D-\delta)(y)] &= 0.
\end{aligned}$$

Proof. If D is a generalized Jordan triple θ-derivation of $(L,[\cdot,\cdot],\alpha)$, then (i) follows immediately. (ii) holds because

$$\begin{aligned}
& B(x,y,z)+B(y,z,x)+B(z,x,y) \\
= \ & \alpha([[\delta(x),\theta(\alpha^k(y))],\theta(\alpha^k(z))]+[[\theta(\alpha^k(x)),\delta(y)],\theta(\alpha^k(z))] \\
& +[[\theta(\alpha^k(x)),\theta(\alpha^k(y))],D(z)])+\alpha([[\delta(y),\theta(\alpha^k(z))],\theta(\alpha^k(x))] \\
& +[[\theta(\alpha^k(y)),\delta(z)],\theta(\alpha^k(x))]+[[\theta(\alpha^k(y)),\theta(\alpha^k(z))],D(x)]) \\
& +\alpha([[\delta(z),\theta(\alpha^k(x))],\theta(\alpha^k(y))]+[[\theta(\alpha^k(z)),\delta(x)],\theta(\alpha^k(y)] \\
& +[[\theta(\alpha^k(z)),\theta(\alpha^k(x))],D(y)]) \\
= \ & \alpha([[\delta(x),\theta(\alpha^k(y))],\theta(\alpha^k(z))]+[[\theta(\alpha^k(x)),\delta(y)],\theta(\alpha^k(z))] \\
& +[[\theta(\alpha^k(x)),\theta(\alpha^k(y))],\delta(z)])+\alpha([[\delta(y),\theta(\alpha^k(z))],\theta(\alpha^k(x))] \\
& +[[\theta(\alpha^k(y)),\delta(z)],\theta(\alpha^k(x))]+[[\theta(\alpha^k(y)),\theta(\alpha^k(z))],\delta(x)]) \\
& +\alpha([[\delta(z),\theta(\alpha^k(x))],\theta(\alpha^k(y))]+[[\theta(\alpha^k(z)),\delta(x)],\theta(\alpha^k(y)] \\
& +[[\theta(\alpha^k(z)),\theta(\alpha^k(x))],\delta(y)]) \\
= \ & \alpha([[\delta(x),\theta(\alpha^k(y))],\theta(\alpha^k(z))]+[[\theta(\alpha^k(y)),\theta(\alpha^k(z))],\delta(x)] \\
& +[[\theta(\alpha^k(z)),\delta(x)],\theta(\alpha^k(y)])+\alpha([[\theta(\alpha^k(x)),\delta(y)],\theta(\alpha^k(z))]
\end{aligned}$$

$$
\begin{aligned}
& +[[\delta(y),\theta(\alpha^k(z))],\theta(\alpha^k(x))]+[[\theta(\alpha^k(z)),\theta(\alpha^k(x))],\delta(y)]) \\
& +\alpha([[\theta(\alpha^k(x)),\theta(\alpha^k(y))],\delta(z)]+[[\theta(\alpha^k(y)),\delta(z)],\theta(\alpha^k(x))] \\
& +[[\delta(z),\theta(\alpha^k(x))],\theta(\alpha^k(y))]) \\
= {} & 0.
\end{aligned}
$$

Note that δ is a Lie triple θ-derivation by Corollary 3.6. Therefore, D is a generalized Lie triple θ-derivation of $(L,[\cdot,\cdot],\alpha)$ with respect to a Lie triple θ-derivation δ by Theorem 3.12. □

As before, one can prove the following theorems.

Theorem 3.14 *D is a generalized Lie triple $(\theta,\varphi)_2$-derivation of $(L,[\cdot,\cdot],\alpha)$ with respect to a Lie triple $(\theta,\varphi)_2$-derivation δ if and only if D is a generalized Jordan triple $(\theta,\varphi)_2$-derivation of $(L,[\cdot,\cdot],\alpha)$ with respect to a Lie triple $(\theta,\varphi)_2$-derivation δ such that*

1. $[[\delta(x),\theta(\alpha^k(y))],(\varphi-\theta)(\alpha^k(z)))]=[[\theta(\alpha^k(x)),\delta(y)],(\varphi-\theta)(\alpha^k(z))]$,

2. $B'(x,y,z)+B'(y,z,x)+B'(z,x,y)=0$,

where $x,y,z\in L$ and $B'(x,y,z)=\alpha([[\delta(x),\theta(\alpha^k(y))],\theta(\alpha^k(z))]+[[\theta(\alpha^k(x)),\delta(y)],\varphi(\alpha^k(z))]+[[\varphi(\alpha^k(x)),\varphi(\alpha^k(y))],D(z)])$.

Theorem 3.15 *D is a generalized Lie triple $(\theta,\varphi)_3$-derivation of $(L,[\cdot,\cdot],\alpha)$ with respect to a Lie triple $(\theta,\varphi)_3$-derivation δ if and only if D is a generalized Jordan triple $(\theta,\varphi)_3$-derivation of $(L,[\cdot,\cdot],\alpha)$ with respect to a Lie triple $(\theta,\varphi)_3$-derivation δ such that*

1. $[[\delta(x),(\theta-\varphi)(\alpha^k(y))],\theta(\alpha^k(z))]=[[(\theta-\varphi)(\alpha^k(x)),\delta(y)],\theta(\alpha^k(z))]$,

2. $B''(x,y,z)+B''(y,z,x)+B''(z,x,y)=0$,

where $x,y,z\in L$ and $B''(x,y,z)=\alpha([[\delta(x),\theta(\alpha^k(y))],\theta(\alpha^k(z))]+[[\varphi(\alpha^k(x)),\delta(y)],\theta(\alpha^k(z))]+[[\varphi(\alpha^k(x)),\varphi(\alpha^k(y))],D(z)])$.

Remark 3.16 *Corollary 3.13 can also be concluded from Theorem 3.14 or Theorem 3.15 since for any $x,y,z\in L$, $B(x,y,z)=B'(x,y,z)=B''(x,y,z)$ when D is a generalized Jordan triple θ-derivation.*

ACKNOWLEDGEMENTS. The authors thank Professor Liangyun Chen for their helpful com- ments and suggestions. We also give our special thanks to referees for many helpful suggestions.

References

[1] Ashraf, M.; Ali, A.; Ali, S., On Lie ideals and generalized (θ, φ)-derivations in prime rings, Comm. Algebra, 32 (2004), 2977-2985.

[2] Benoist, Y., La partie semi-simple de l'algbre des drivations d'une algbre de Lie nilpotente. (French) C. R. Acad. Sci. Paris Sér. I Math., 307 (1988), 901-904.

[3] Brešar, M.; Vukman, J., Jordan (Θ, φ)-derivations, Glas. Mat. Ser. III, 26(46) (1991), 13-17.

[4] Brešar, M.; Vukman, J., Jordan mappings of semiprime rings, J. Algebra, (1989), 218-228.

[5] Chen, Z.; Xiao, Z., Nonlinear Lie triple derivations on parabolic subalgebras of finite-dimensional simple Lie algebras, Linear Multilinear Algebra, 60 (2012), 645-656.

[6] Leger, G.; Luks, E., Generalized derivations of Lie algebras, J. Algebra, 228 (2000), 165-203.

[7] Liu, C-K; Shiue, W-K., Generalized Jordan triple (θ, ϕ)-derivations on semiprime rings, Taiwanese J. Math., 11 (2007), 1397-1406.

[8] Lu, F., Lie triple derivations on nest algebras, Math. Nachr., 280 (2007), 882-887.

[9] Miers C., Lie triple derivations of von Neumann algebras, Proc. Amer. Math. Soc., (1978), 57-61.

[10] Sheng, Y., Representations of hom-Lie algebras, Algebr. Represent. Theory, 15 (2012), 1081-1098.

[11] Wang, H.; Li, Q., Lie triple derivation of the Lie algebra of strictly upper triangular matrix over a commutative ring, Linear Algebra Appl., (2009), 66-77.

[12] Wang, D.; Yu, X., Lie triple derivations on the parabolic subalgebras of simple Lie algebras, Linear Multilinear Algebra, 59 (2011), 837-840.

Received: June, 2014