

# On Hyers-Ulam stability of dynamic integral equation on time scales

**Liubin Hua**

Guangzhou Sontan Polytechnic College,  
Guangzhou 511370, P. R. China  
E-mail: hualiubin@163.com

**Yongjin Li**

Department of Mathematics, Sun Yat-Sen University,  
Guangzhou, 510275, P. R. China  
E-mail: stslyj@mail.sysu.edu.cn

**Jiarong Feng**

Department of Mathematics, Sun Yat-Sen University,  
Guangzhou, 510275, P. R. China  
E-mail: fengjiar@foxmail.com

## Abstract

We investigate the stability of nonlinear dynamic integral equation of the form  $x(t) = f(t, x(t), \int_{t_0}^t g(t, \tau, x(\tau))\Delta\tau)$  on time scales by using a fixed point method.

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## 1 Introduction and preliminaries

### 1.1 The Hyers-Ulam stability

In 1940, S.M. Ulam gave a wide range of talks at the Mathematics Club of the University of Wisconsin, in which discussed a number of important unsolved problems. He [52] posed the following question concerning the stability

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<sup>0</sup>Corresponding author: Yongjin Li, E-mail address: stslyj@mail.sysu.edu.cn

of group homomorphisms before a Mathematical Colloquium: *When can we assert that the solutions of an inequality are close to one of the exact solutions of the corresponding equation?*

A year later, D.H. Hyers [19] dealt with  $\varepsilon$ -additive mapping by direct method, which gave a partial solution to the above question. The result was extended by T. Aoki [2], D.G. Bourgin [7] and Th.M. Rassias [44]. We mention here that the interest of this topic has been increasing since it came into being, some other results concerning functional equations one can find, e.g., in [13, 46, 47, 14, 12, 23, 41] and some related information (e.g.,  $\varepsilon$ -isometries, superstability of functional equations and the stability of differential expressions) we refer to [20, 21, 22, 4, 11, 37, 18, 50, 8, 9].

To the best of our knowledge, the first one who pay attention to the stability of differential equations is M. Obłozza [38, 39]. Thereafter, C. Alsina and R. Ger [1] proved that the stability holds true for differential equation  $y'(x) = y(x)$ . Then, a generalized result was given by S.-E. Takahasi, T. Miura and S. Miyajima [51], in which they investigated the stability of the Banach space valued linear differential equation of first order (see also [34, 36]). A more general result on the linear differential equations of first order of the form  $y'(t) + \alpha(t)y(t) + \beta(t) = 0$  was given by S.-M. Jung [26] and the stability of linear differential equations of second order was established by Y. Li et al. (see [30, 32, 15, 29]).

In the near past many research papers have been published about the Ulam-Hyers stability of functional, differential and difference equations. The main tool used by the authors for obtaining stability results was the direct method. Recently Rus developed a unified approach based on Gronwall type inequalities and Picard operators. This approach can be applied to a wide range of problems.

**Definition 1.1** *Let  $(X, d)$  be a metric space and  $T : X \rightarrow X$  be an operator. The fixed point equation  $Tx = x$  is said to be Ulam-Hyers stable if there exists a real number  $C_T > 0$  such that: for each real number  $\varepsilon > 0$  and each solution  $y^*$  of the equation  $d(y, Ty) < \varepsilon$ , there exists a solution  $x^*$  of the  $Tx = x$  such that  $d(y^*, x^*) < C_T \cdot \varepsilon$ .*

## 1.2 Time scale analysis

Stefan Hilger in his doctoral dissertation, that resulted in his seminal paper [17] in 1990, initiated the study of time scales in order to unify continuous and discrete analysis. In recent years, the theory of dynamic equations on time scales, which provides powerful new tools for exploring connections between the traditionally separated fields, has been developing rapidly and has received much attention. We refer the reader to the book by Bohner and Peterson [5]

and to the papers cited therein. The time scales calculus has a tremendous potential for applications in mathematical models of real processes, for instance, in biotechnology, chemical technology, economic, neural networks, physics, social sciences and so on, see the monographs of Aulbach and Hilger [3], Bohner and Perterson [5] and the references therein.

For convenience, we will provide without proof several foundational definitions and results from the calculus on time scales so that the paper is self-contained.

Let  $\mathbb{T}$  be a nonempty closed subset (time scale) of  $\mathbb{R}$ . The forward and backward jump operators  $\sigma, \rho : \mathbb{T} \rightarrow \mathbb{T}$  and the graininess  $\mu : \mathbb{T} \rightarrow \mathbb{R}^+$  are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \quad \mu(t) = \sigma(t) - t.$$

A point  $t \in \mathbb{T}$  is called left-dense if  $t > \inf \mathbb{T}$  and  $\rho(t) = t$ , left-scattered if  $\rho(t) < t$ , right-dense if  $t < \sup \mathbb{T}$  and  $\sigma(t) = t$ , and right-scattered if  $\sigma(t) > t$ . If  $\mathbb{T}$  has a left-scattered maximum  $m$ , then  $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$ ; otherwise  $\mathbb{T}^k = \mathbb{T}$ .

**Definition 1.2** Fix  $t \in \mathbb{T}$ . Let  $f : \mathbb{T} \rightarrow \mathbb{R}$ . The delta derivative of  $f$  at the point  $t$  is defined to be the number  $f^\Delta(t)$  (provided it exists), with the property that, for each  $\epsilon > 0$ , there is a neighborhood  $U$  of  $t$  such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s|,$$

for all  $s \in U$ . Define  $f^{\Delta^n}(t)$  to be the delta derivative of  $f^{\Delta^{n-1}}(t)$ ; i.e.,  $f^{\Delta^n}(t) = (f^{\Delta^{n-1}}(t))^\Delta$ .

**Theorem 1.3** Assume  $f : \mathbb{T} \rightarrow \mathbb{R}$  is a function and let  $t \in \mathbb{T}^k$ . Then we have the following:

- (1) If  $f$  is differentiable at  $t$ , then  $f$  is continuous at  $t$ .
- (2) If  $f$  is continuous at  $t$  and  $t$  is right-scattered, then  $f$  is differentiable at  $t$  with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

- (3) If  $t$  is right-dense, then  $f$  is differentiable at  $t$  if and only if the limit

$$\lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s}$$

exists and is a finite number. In this case

$$f^\Delta(t) = \lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s}.$$

- (4) If  $f$  is differentiable at  $t$ , then

$$f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t).$$

**Definition 1.4** A function  $f$  is left-dense continuous (i.e.  $ld$ -continuous), if  $f$  is continuous at each left-dense point in  $\mathbb{T}$  and its right-sided limit exists at each right-dense point in  $\mathbb{T}$ . If  $F^\Delta(t) = f(t)$ , then define the delta integral by

$$\int_a^t f(s)\Delta s = F(t) - F(a).$$

**Definition 1.5** The function  $p : \mathbb{T} \rightarrow R$  is said to be regressive if  $1 + \mu(t)p(t) \neq 0$ , for all  $t \in \mathbb{T}^k$ . We denote by  $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$  the set of all regressive and rd-continuous functions and define

$$\mathcal{R}^+ = \{p \in \mathcal{R} \mid 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}.$$

**Definition 1.6** For  $p \in \mathcal{R}$  we define (see [5]) the exponential function  $e_p(\cdot, t_0)$  on the time scale  $\mathbb{T}$  as the unique solution to the scalar initial value problem

$$x^\Delta(t) = p(t)x(t), \quad x(t_0) = 1.$$

if  $p \in \mathcal{R}^+$ , then  $e_p(\cdot, t_0) > 0$ , for all  $t \in \mathbb{T}$ . We note that, if  $\mathbb{T} = \mathbb{R}$ , the exponential function is given by

$$e_p(t, s) = \exp\left(\int_s^t p(\tau)d\tau\right), \quad e_\alpha(t, s) = \exp(\alpha(t-s)), \quad e_\alpha(t, 0) = \exp(\alpha t)$$

for  $s, t \in \mathbb{R}$ , where  $\alpha \in \mathbb{R}$  is a constant and  $p : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function. To compare with the discrete case, if  $\mathbb{T} \rightarrow \mathbb{Z}$  (the set of integers), the exponential function is given by

$$e_p(t, s) = \prod_{\tau=s}^{t-1} [1 + p(\tau)], \quad e_\alpha(t, s) = (1 + \beta)^{t-s}, \quad e_\alpha(t, 0) = (1 + \alpha)^t$$

for  $s, t \in \mathbb{Z}$  with  $s < t$ , where  $\alpha \neq -1$  is a constant and  $p : \mathbb{Z} \rightarrow \mathbb{R}$  is a sequence satisfying  $p(t) \neq -1$  for all  $t \in \mathbb{Z}$ .

**Theorem 1.7** (*Properties of the exponential function*). If  $p, q \in \mathcal{R}$ , then

- (1)  $e_0(t, s) \equiv 1$  and  $e_p(t, t) \equiv 1$ ;
- (2)  $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$ ;
- (3)  $e_p(t, s) = \frac{1}{e_p(s, t)} = e_{\ominus p}(s, t)$ ;
- (4)  $e_p(t, s)e_p(s, r) = e_p(t, r)$ ;
- (5)  $e_p(t, s)e_q(t, s) = e_{p \oplus q}(t, s)$ ;
- (6)  $\frac{e_p(t, s)}{e_q(t, s)} = e_{p \ominus q}(t, s)$ ;
- (7)  $\left(\frac{1}{e_p(\cdot, s)}\right)^\Delta = -\frac{p(t)}{e_p^\sigma(\cdot, s)}$ , where for all  $p, q \in \mathcal{R}$  we define

$$(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t)$$

and

$$(\ominus p)(t) := -\frac{p(t)}{1 + \mu(t)p(t)}$$

for all  $t \in \mathbb{T}^k$ .

We remark, that  $(\mathcal{R}, \oplus)$  is an Abelian group, called the regressive group.

For more details about calculus on time scales, one can see [3], [5].

## 2 Nonlinear dynamic integral equation on time scale

S. András, A.R. Mészáros discussed the Ulam-Hyers stability of dynamic equations on time scales via Picard operators [53]. D.R. Anderson, B. Gates and D. Heuer studied the Hyers-Ulam stability of second-order linear dynamic equations on time scales [54]. In the study of dynamic equations on time scales, most often the analysis turns to that of a related integral equation on time scales. It seems integral equations on time scales have an enormous potential for rich and diverse applications and thus they are most worthy of attention. In this paper we consider the Hyers-Ulam stability nonlinear dynamic integral equation

$$x(t) = f(t, x(t), \int_{t_0}^t g(t, \tau, x(\tau))\Delta\tau), \quad (2.1)$$

where  $x$  is the unknown function to be found,  $g : I_{\mathbb{T}}^2 \times \mathbb{R}^n \rightarrow \mathbb{R}^n, f : I_{\mathbb{T}} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n, t$  is from a time scale  $\mathbb{T}, \tau \leq t$  and  $I_{\mathbb{T}} = I \cap \mathbb{T}, I = [t_0, \infty)$  be the given subset of  $R, \mathbb{R}^n$  the real  $n$ -dimensional Euclidean space with appropriate norm defined by  $\|\cdot\|, x_0$  is a given constant in  $\mathbb{R}^n$  and the integral sign represents a very general type of operation, known as the delta integral. The existence and uniqueness of solutions of above nonlinear dynamic integral equation was proved by D.B. Pachpatte in [40].

Let  $I_{\mathbb{T}} := [t_0, \infty)_{\mathbb{T}}, \beta > 0$  be a constant and consider the space of continuous functions  $C([t_0, \infty)_{\mathbb{T}}; \mathbb{R}^n)$  such that  $\sup_{t \in [t_0, \infty)_{\mathbb{T}}} \frac{x(t)}{e_{\beta}(t, t_0)} < \infty$  and denote this special space by  $C_{\beta}([t_0, \infty)_{\mathbb{T}}; \mathbb{R}^n)$ . The norm of linear space  $C_{\beta}([t_0, \infty)_{\mathbb{T}}; \mathbb{R}^n)$  can be defined as following:

$$\|x\|_{\beta}^{\infty} = \sup_{t \in [t_0, \infty)_{\mathbb{T}}} \frac{|x(t)|}{e_{\beta}(t, t_0)},$$

thus

$$d(x, y)_{\beta}^{\infty} = \sup_{t \in [t_0, \infty)_{\mathbb{T}}} \frac{|x(t) - y(t)|}{e_{\beta}(t, t_0)},$$

is the metric of  $C_\beta([t_0, \infty)_{\mathbb{T}}; \mathbb{R}^n)$ . T. Kulik and C. C. Tisdell show that  $(C_\beta([t_0, \infty)_{\mathbb{T}}; \mathbb{R}^n), \|\cdot\|_\beta^\infty)$  is Banach spaces and  $(C_\beta([t_0, \infty)_{\mathbb{T}}; \mathbb{R}^n), d_\beta^\infty)$  is complete linear metric space in [28].

We need the following lemma proved in Bohner and Peterson [5].

**Lemma 2.1** [5] *Let  $t_0 \in \mathbb{T}^k$  and assume that  $k : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{R}$  is continuous at  $(t, t)$ , where  $t \in \mathbb{T}$  with  $t > t_0$ . Also assume that  $k(t, \cdot)$  is rd-continuous on  $[t_0, \sigma(t)]$ . Suppose that for each  $\varepsilon > 0$  there exists a neighborhood  $N$  of  $t$  independent of  $[t_0, \sigma(t)]$  such that*

$$|k(\sigma(t), \tau) - k(s, \tau) - k^\Delta(t, \tau)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|,$$

for  $s \in N$ , where  $k^\Delta$  denotes the  $\Delta$  derivative of  $k$  with respect to the first variable. Then

$$g(t) = \int_{t_0}^t k(t, \tau) \Delta\tau,$$

for all  $t \in I_{\mathbb{T}}$ , implies

$$g^\Delta(t) = \int_{t_0}^t k^\Delta(t, \tau) \Delta\tau + k(\sigma(t), t)$$

for  $t \in I_{\mathbb{T}}$ .

The following lemma proved in [6] is useful to prove our main results.

**Lemma 2.2** [6] *Assume that  $\mu \in c, d$  and  $\mu \geq 0$  and  $c \geq 0$  is a real constant. Let  $k(t, s)$  be defined as in Lemma 2.1 such that  $k(\sigma(t), t) \geq 0$  and  $k^\Delta(t, s) \geq 0$  for  $s, t \in \mathbb{T}$  with  $s \leq t$ . Then*

$$\mu(t) \leq c + \int_{t_0}^t k(t, \tau) \mu(\tau) \Delta\tau$$

implies

$$\mu(t) \leq ce_A(t, t_0),$$

for all  $t \in \mathbb{T}$ , where

$$A(t) = k(\sigma(t), t) + \int_{t_0}^t k^\Delta(t, \tau) \Delta\tau.$$

Next, we present our result on the Hyers-Ulam stability of (2.1).

**Theorem 2.3** *Let  $L > 0, \beta > 0, M \geq 0, \gamma > 1$  be constants with  $\beta = L\gamma$ . Suppose that the functions  $f, g$  in (2.1) are rd-continuous and satisfy the conditions*

$$\begin{aligned} |f(t, u, v) - f(t, \bar{u}, \bar{v})| &\leq M[|u - \bar{u}| + |v - \bar{v}|], \\ |g(t, s, u) - g(t, s, v)| &\leq L|u - v|, \\ d_1 = \sup_{t \in [t_0, \infty)_{\mathbb{T}}} \frac{1}{e_\beta(t, t_0)} &|f(t, 0, \int_{t_0}^t g(t, \tau, \Delta\tau)| < \infty. \end{aligned}$$

*If  $M(1 + \frac{1}{\gamma}) < 1$ , then nonlinear dynamic integral equation (2.1) has Hyers-Ulam stability.*

**Proof.** Let  $x \in C_\beta([t_0, \infty)_{\mathbb{T}}; \mathbb{R}^n)$  and define the operator  $T$  by

$$\begin{aligned} (Tx)(t) &= f\left(t, x(t), \int_{t_0}^t g(t, \tau, x(\tau))\Delta\tau\right) - f\left(t, 0, \int_{t_0}^t g(t, \tau, 0)\Delta\tau\right) \\ &\quad + f\left(t, 0, \int_{t_0}^t g(t, \tau, 0)\Delta\tau\right). \end{aligned}$$

(1) Firstly, we show that  $T$  maps  $C_\beta([t_0, \infty)_{\mathbb{T}}; \mathbb{R}^n)$  into itself. Let  $x \in C_\beta([t_0, \infty)_{\mathbb{T}}; \mathbb{R}^n)$ . Using the hypotheses, we have

$$\begin{aligned} \|Tx\|_\beta^\infty &= \sup_{t \in [t_0, \infty)_{\mathbb{T}}} \frac{|(Tx)(t)|}{e_\beta(t, t_0)} \\ &= \sup_{t \in [t_0, \infty)_{\mathbb{T}}} \frac{1}{e_\beta(t, t_0)} \left| f\left(t, x(t), \int_{t_0}^t g(t, \tau, x(\tau))\Delta\tau\right) - f\left(t, 0, \int_{t_0}^t g(t, \tau, 0)\Delta\tau\right) \right. \\ &\quad \left. + f\left(t, 0, \int_{t_0}^t g(t, \tau, 0)\Delta\tau\right) \right| \\ &\leq \sup_{t \in [t_0, \infty)_{\mathbb{T}}} \frac{1}{e_\beta(t, t_0)} \left| f\left(t, x(t), \int_{t_0}^t g(t, \tau, x(\tau))\Delta\tau\right) - f\left(t, 0, \int_{t_0}^t g(t, \tau, 0)\Delta\tau\right) \right| \\ &\quad + \sup_{t \in [t_0, \infty)_{\mathbb{T}}} \frac{1}{e_\beta(t, t_0)} \left| f\left(t, 0, \int_{t_0}^t g(t, \tau, 0)\Delta\tau\right) \right| \\ &\leq d_1 + \sup_{t \in [t_0, \infty)_{\mathbb{T}}} \frac{1}{e_\beta(t, t_0)} M[|x(t)| + \int_{t_0}^t L|x(\tau)|\Delta\tau] \\ &\leq d_1 + M \left[ \sup_{t \in [t_0, \infty)_{\mathbb{T}}} \frac{|x(t)|}{e_\beta(t, t_0)} + L \sup_{t \in [t_0, \infty)_{\mathbb{T}}} \frac{1}{e_\beta(t, t_0)} \int_{t_0}^t e_\beta(\tau, t_0) \frac{|x(\tau)|}{e_\beta(\tau, t_0)} \Delta\tau \right] \\ &\leq d_1 + M \left[ \|x\|_\beta^\infty + L \|x\|_\beta^\infty \sup_{t \in [t_0, \infty)_{\mathbb{T}}} \frac{1}{e_\beta(t, t_0)} \int_{t_0}^t e_\beta(\tau, t_0) \Delta\tau \right] \\ &= d_1 + M \|x\|_\beta^\infty \left[ 1 + L \sup_{t \in [t_0, \infty)_{\mathbb{T}}} \frac{1}{e_\beta(t, t_0)} \left( \frac{e_\beta(t, t_0) - 1}{\beta} \right) \right] \\ &= d_1 + M \|x\|_\beta^\infty \left( 1 + \frac{L}{\beta} \right) \\ &= d_1 + \|x\|_\beta^\infty M \left( 1 + \frac{1}{\gamma} \right) \\ &< \infty. \end{aligned}$$

Thus operator  $T$  maps  $C_\beta([t_0, \infty)_{\mathbb{T}}; \mathbb{R}^n)$  into itself.

(2) Next, we verify that  $T$  is a contraction mapping, so dynamic integral equation (2.1) has a unique solution. Let  $u, v \in C_\beta([t_0, \infty)_{\mathbb{T}}; \mathbb{R}^n), u \neq v$ . We have

$$\begin{aligned} d_\beta^\infty(Tu, Tv) &= \sup_{t \in [t_0, \infty)_{\mathbb{T}}} \frac{|(Tu)(t) - (Tv)(t)|}{e_\beta(t, t_0)} \\ &= \sup_{t \in [t_0, \infty)_{\mathbb{T}}} \frac{1}{e_\beta(t, t_0)} \left| f(t, u(t), \int_{t_0}^t g(t, \tau, u(\tau)) \Delta \tau) - f(t, v(t), \int_{t_0}^t g(t, \tau, v(\tau)) \Delta \tau) \right| \\ &\leq \sup_{t \in [t_0, \infty)_{\mathbb{T}}} \frac{1}{e_\beta(t, t_0)} M \left[ |u(t) - v(t)| + \int_{t_0}^t L |u(\tau) - v(\tau)| \Delta \tau \right] \\ &= M \left[ \sup_{t \in [t_0, \infty)_{\mathbb{T}}} \frac{|u(t) - v(t)|}{e_\beta(t, t_0)} + \sup_{t \in [t_0, \infty)_{\mathbb{T}}} \frac{1}{e_\beta(t, t_0)} L \int_{t_0}^t e_\beta(\tau, t_0) \frac{|u(\tau) - v(\tau)|}{e_\beta(\tau, t_0)} \Delta \tau \right] \\ &\leq M \left[ d_\beta^\infty(u, v) + L d_\beta^\infty(u, v) \sup_{t \in [t_0, \infty)_{\mathbb{T}}} \frac{1}{e_\beta(t, t_0)} \int_{t_0}^t e_\beta(\tau, t_0) \Delta \tau \right] \\ &= M d_\beta^\infty(u, v) \left[ 1 + L \sup_{t \in [t_0, \infty)_{\mathbb{T}}} \frac{1}{e_\beta(t, t_0)} \frac{e_\beta(t, t_0) - 1}{\beta} \right] \\ &= M d_\beta^\infty(u, v) \left( 1 + \frac{L}{\beta} \right) \\ &= M \left( 1 + \frac{1}{\gamma} \right) d_\beta^\infty(u, v). \end{aligned}$$

Let  $\alpha = M(1 + \frac{1}{\gamma})$ , then  $M(1 + \frac{1}{\gamma}) < 1$ , and  $d_\beta^\infty(Tu, Tv) \leq \alpha d_\beta^\infty(u, v)$ , by Banach's fixed point theorem  $T$  has unique fixed point  $v^*$  in  $C_\beta([t_0, \infty)_{\mathbb{T}}; \mathbb{R}^n)$ . Thus  $v^*$  is the unique solution of nonlinear dynamic integral equation (2.1).

(3) Finally, we will show that dynamic integral equation (2.1) has Hyers-Ulam stability.

For any  $\varepsilon > 0$ , since  $(C_\beta([t_0, \infty)_{\mathbb{T}}; \mathbb{R}^n), \|\cdot\|_\beta^\infty)$  is Banach spaces and  $(C_\beta([t_0, \infty)_{\mathbb{T}}; \mathbb{R}^n), d_\beta^\infty)$  is complete linear metric space, if  $d_\beta^\infty(Tu, u) = d_\beta^\infty(Tu - u, 0) \leq \varepsilon$ , by  $Tv^* = v^*$ , we have

$$\begin{aligned} d_\beta^\infty(u, v^*) &= d_\beta^\infty(u - Tu + Tu - Tv^*, 0) \\ &= d_\beta^\infty(Tu - u, Tu - Tv^*) \\ &\leq d_\beta^\infty(Tu - u, 0) + d_\beta^\infty(Tu - Tv^*, 0) \\ &\leq \varepsilon + \alpha d_\beta^\infty(u, v^*). \end{aligned}$$

Thus

$$d_\beta^\infty(u, v^*) \leq \frac{1}{(1 - \alpha)} \cdot \varepsilon$$

Hence, nonlinear dynamic integral equation (2.1) has Hyers-Ulam stability.



By the same technique, we can show that the following theorem holds.

**Theorem 2.4** *For the integral equation*

$$x(t) = f(t) + \int_a^t k(t, s, x^\sigma(s))\Delta s, \quad t \in I_{\mathbb{T}}. \quad (2.2)$$

where  $x : I_{\mathbb{T}} \rightarrow \mathbb{R}^n$  is the unknown function,  $f : I_{\mathbb{T}} \rightarrow \mathbb{R}^n$  is continuous and  $k : I_{\mathbb{T}} \times I_{\mathbb{T}} \times I_{\mathbb{T}} \rightarrow \mathbb{R}^n$  is continuous in its first and third variables and rd-continuous in its second variable. Let  $L > 0, \beta > 0, \gamma > 1$  be constants with  $\beta = L\gamma$ . If for any  $(t, s) \in [a, \infty)_{\mathbb{T}}^2, (u, v) \in \mathbb{R}^{2n}$ .

$$|k(t, s, u) - k(t, s, v)| \leq L|u - v|,$$

$$L \sup_{t \in [a, \infty)_{\mathbb{T}}} \mu(t) < 1 - \frac{1}{\gamma},$$

$$\sup_{t \in [a, \infty)_{\mathbb{T}}} \frac{1}{e_{\beta}(t, a)} \|f(t) + \int_a^t k(t, s, 0)\Delta\tau\| < \infty.$$

Then nonlinear dynamic integral equation (2.2) has Hyers-Ulam stability.

In fact, it is easy to see the following is an equivalent formulation of (2.2).

$$x(t) = \left( f(t) + \int_a^t k(t, s, 0)\Delta s \right) + \int_a^t (k(t, s, x^\sigma(s)) - k(t, s, 0))\Delta s, \quad t \in [a, \infty)_{\mathbb{T}}. \quad (2.3)$$

Since  $C_{\beta}([a, \infty)_{\mathbb{T}}; \mathbb{R}^n)$  is complete metric space, and the operator is defined by

$$(Tx)(t) := \left( f(t) + \int_a^t k(t, s, 0)\Delta s \right) + \int_a^t (k(t, s, x^\sigma(s)) - k(t, s, 0))\Delta s, \quad t \in [a, \infty)_{\mathbb{T}}.$$

is a contractive map and Banach fixed point theorem will then apply. By the same technique in the proof of theorem 2.3, we can show that equation (2.2) has Hyers-Ulam stability.

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