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On some properties of vectors tangent to a predifferential space

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Abstract

The properties of tangent vectors to a differential space in a sense of Sikorski are well–known. However, recently it has been noted that further generalizations in the spirit of Sikorski differential spaces are interesting. As a result a predifferential space concept has been investigated. Predifferential spaces are constructed by requiring just a slightly less assumptions on the algebra of functions than in the construction of differential spaces. This article presents a few facts about the properties of vectors tangent to a predifferential space. It is based on some previous results from the differential spaces theory.

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1 Introduction

Suppose that there is a family $\mathcal{A}$ of real functions defined on a given set M, i.e.

$$\mathcal{A} := \{f_1, \ldots, f_n, \ldots \mid \forall_n \ f_n : M \to \mathbb{R}\} \quad . \tag{1}$$

Of course, $\mathcal{A}$ is an algebra with pointwise operations of addition and multiplication. By requirement that all functions from $\mathcal{A}$ are continuous, some topology is obtained on M. This topology is denoted by $\tau_{\mathcal{A}}$. Then one can consider two further conditions on $\mathcal{A}$.

The first is called superposition closure and consists of all compositions of functions from $\mathcal{A}$ with arbitrary smooth (i.e. infinitely differentiable) functions from $\mathbb{R}^k$. The superposition closure of $\mathcal{A}$ is usually denoted by sc$\mathcal{A}$. In other words, sc$\mathcal{A} := \{\omega \circ (f_1, \ldots, f_k) \mid \omega \in C^\infty(\mathbb{R}^k), f_1, \ldots, f_k \in \mathcal{A}, k \in \mathbb{N}\}$.

The second condition is called localization closure and it consists of all functions which locally (with respect to the topology $\tau_{\mathcal{A}}$) coincide with at least one function from $\mathcal{A}$ on M. Localization closure of $\mathcal{A}$ on M is usually denoted by $\mathcal{A}_M$. In other words, $\mathcal{A}_M := \{f : M \to \mathbb{R} \mid \forall_{p \in M} \exists_{g_p \in \mathcal{A}, U_p \in \tau_{\mathcal{A}}} g_p|_{U_p} = f|_{U_p}\}$.

If $\mathcal{A} = \mathrm{sc}\mathcal{A}$, then the pair $(M, \mathcal{A})$ is called predifferential space. If $\mathcal{A} = (\mathrm{sc}\mathcal{A})_M$, then the pair $(M, \mathcal{A})$ is called differential space. For example, the differential space $(M, \mathcal{A})$, where $\mathcal{A} := C^\infty(M)$ is a smooth manifold in a classical sense. However, by considering other algebras of function $\mathcal{A}$ one can study more "weird" spaces. Functions $f_1, \ldots, f_n, \ldots$ in Eq. (1) are called generators.

Differential spaces are interesting for investigation, because the differential geometry can be constructed over them [4], [2]. Predifferential spaces are interesting not only because the differential geometry can be constructed over them, but also because of some interesting relationship with differential spaces [1].

2 Main Results

The basics of differential spaces can be found e.g. in [4] or [2]. It is reminded that if $(M, \mathcal{A})$ is a predifferential space and p is a point in M, then the mapping $v : \mathcal{A} \to \mathbb{R}$, which is $\mathbb{R}$–linear and satisfies the Leibniz rule, is called a tangent vector (to $(M, \mathcal{A})$ at the point p).

Of course, every differential space is a predifferential space. The contrary is not true. Suppose that $(M, \mathcal{A})$ is a predifferential space, which is not a differential space, i.e. $\mathcal{A} = \mathrm{sc}\mathcal{A} \neq (\mathrm{sc}\mathcal{A})_M$.

There is a natural mapping $\mathrm{id}_M : (M, \mathcal{A}_M) \to (M, \mathcal{A})$. Indeed $\forall_{f \in \mathcal{A}}\ f \circ \mathrm{id}_M = f \in \mathcal{A}_M$, so the considered mapping is smooth in a sense of differential spaces theory [2]. Therefore it is interesting to find whether $(\mathrm{id}_M)_{*p} : T_p(M, \mathcal{A}_M) \to T_p(M, \mathcal{A})$ is an isomorphism. Indeed

Theorem 2.1. $(\mathrm{id}_M)_{*p} : T_p(M, \mathcal{A}_M) \to T_p(M, \mathcal{A})$ *is an isomorphism.*

Proof. Indeed, in general $(F_{*x} w)(\beta) = w(F^* \beta) = w(\beta \circ F)$, where $F : (M_1, \mathcal{A}_1) \to (M_2, \mathcal{A}_2)$, $x \in M_1$, $w \in T_x(M_1, \mathcal{A}_1)$ and $\beta \in \mathcal{A}_2$.

Let $\widetilde{v} \in T_p(M, \mathcal{A}_M)$, then $\forall_{f \in \mathcal{A}}\ ((\mathrm{id}_M)_{*p} \widetilde{v})(f) = \widetilde{v}(f \circ \mathrm{id}_M) = \widetilde{v}(f)$. Therefore $(\mathrm{id}_M)_{*p} \widetilde{v} = \widetilde{v}|_{\mathcal{A}}$.

Now, it is easy to notice that if $v \in T_p(M, \mathcal{A})$, then there exists the unique $\widetilde{v} \in T_p(M, \mathcal{A}_M)$ such that $\widetilde{v}|_{\mathcal{A}} = v$. □

Of course, tangent vectors can be seen as derivations [2]. All derivations of the algebra $\mathcal{A}$ are denoted by $\mathrm{Der}\mathcal{A}$.

Theorem 2.2. *Every derivation in point p, $v \in T_p(M, \mathcal{A})$, $v : \mathcal{A} \to \mathbb{R}$, is a local operator. In other words $\forall_{f \in \mathcal{A}}\ (f|_U = 0) \Rightarrow (v(f) = 0)$, where U is an arbitrary open neighborhood of $p \in M$.*

Proof. $(\mathrm{id}_M)_{*p} : T_p(M, \mathcal{A}_M) \to T_p(M, \mathcal{A})$ is an isomorphism. Therefore there exists exactly one $\widetilde{v} \in T_p(M, \mathcal{A}_M)$, such that $(\mathrm{id}_M)_{*p}\widetilde{v} = v$ and $\widetilde{v}|_{\mathcal{A}} = v$. If $f \in \mathcal{A}$ and $f|_U = 0$, then $\widetilde{v}(f) = 0$. On the other hand, $v(f) = \widetilde{v}(f)$, so $v(f) = 0$. □

Theorem 2.3. *$X \in \mathrm{Der}\mathcal{A}$ is a local operator. In other words $\forall_{f\in\mathcal{A}}\ (f|_U = 0) \Rightarrow ((Xf)|_U = 0)$ for an arbitrary $U \in \tau_{\mathcal{A}}$.*

Proof. Let $p \in M$. Consider $X_p : \mathcal{A} \to \mathbb{R}$, such that $X_p(f) := (Xf)(p)$ for an arbitrary $f \in \mathcal{A}$. Of course, $X_p \in T_p(M, \mathcal{A})$ and the mapping $p \mapsto X_p$ is a smooth tangent vector field on $(M, \mathcal{A})$ [2]. (In other words X_p is a tangent vector and $Xf \in \mathcal{A}$ for an arbitrary $f \in \mathcal{A}$.)

On the other hand, there exists the unique $\widetilde{X_p} \in T_p(M, \mathcal{A}_M)$, such that $\widetilde{X_p}|_{\mathcal{A}} = X_p$. Therefore consider $\widetilde{X} \in \mathrm{Der}\mathcal{A}_M$, such that $\widetilde{X}(p) = \widetilde{X_p}$.

The smoothness of $\widetilde{X}$ has to be verified. Therefore, let f be an arbitrary function from $\mathcal{A}$. Then $(\widetilde{X}f)(p) = \widetilde{X_p}(f) = X_p(f) = (Xf)(p)$ for an arbitrary $p \in M$. As a result $\widetilde{X}f = Xf \in \mathcal{A}$.

Therefore, if $f|_U = 0$, then $(\widetilde{X}f)|_U = 0$. As a result $(Xf)|_U = 0$, because $(Xf)|_U = (\widetilde{X}f)|_U$. □

Theorem 2.4. *If $\mathcal{A}$–module $\mathfrak{X}(M, \mathcal{A})$ of smooth vector fields on $(M, \mathcal{A})$ is locally free, then $\mathcal{A}_M$–module $\mathfrak{X}(M, \mathcal{A}_M)$ of smooth vector fields on $(M, \mathcal{A}_M)$ is locally free.*

Proof. Let $X \in \mathfrak{X}(M, \mathcal{A})$. If $\mathfrak{X}(M, \mathcal{A})$ is locally free, then there exists the local basis $w_1, \dots, w_n$ with respect to an arbitrary open set $U \subset M$, such that $X(p) = \sum_{i=1}^{n} \varphi_i(p) w_i$, where $\varphi_i \in \mathcal{A}|_U$ for every $i = 1, \dots, n$.

Because of the previous Theorems, for every $i = 1, \dots, n$ there exists the unique $\widetilde{w_i} \in T_p(U, \mathcal{A}_M|_U)$, such that $\widetilde{w_i}|_{\mathcal{A}} = w_i$. Therefore it is possible to construct the unique $\widetilde{X} \in \mathfrak{X}(M, \mathcal{A}_M)$, such that $\widetilde{X}(p) = \sum_{i=1}^{n} \widetilde{\varphi_i}(p)\widetilde{w_i}$, where $\widetilde{\varphi_i} \in \mathcal{A}_M$, $\widetilde{\varphi_i}|_U = \varphi_i$ for every $i = 1, \dots, n$ and $\widetilde{X}|_{\mathcal{A}} = X$. On the other hand, this construction is "onto", because of the isomorphism between $T_p(M, \mathcal{A}_M)$ and $T_p(M, \mathcal{A})$ for every p. □

3 Conclusions

The presented results are in agreement with previous results of investigation on predifferential spaces (for example in [1]). Notice that any differential space emerges by localization closure of some predifferential space. The major conclusion is that most geometrical properties are encoded on a predifferential level. Notice also that the last Theorem, but in the context of passing from $\mathcal{A}$ to $\mathrm{sc}\mathcal{A}$ has been studied in [3].

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