Mathematica Aeterna, Vol. 4, 2014, no. 6, 613 - 619

ON SOME PROPERTIES OF GENERALIZED q-MITTAG LEFFLER FUNCTION

S.K. Sharma

School of Physical Sciences,
ITM University, Gwalior – 474 001, INDIA
e-mail : sksharma_itm@rediffmail.com

R. Jain

School of Mathematics and Allied Science,
Jiwaji University, Gwalior – 474 001, INDIA
e-mail : renujain3@rediffmail.com

Abstract

In the present paper, we make an attempt to introduce q-analogue of generalized Mittag Leffler function $E_{\alpha,\beta}(z;q)$ and its q-recurrence relations with q-derivative. Also, we present q-fractional operators and properties of $E^{\gamma}_{\alpha,\beta}(z;q)$ by using fractional q-calculus.

2000 Mathematics Subject Classification: 33E12, 26A33, 33D05.

Key Words and Phrases: Mittag Leffler function, q-beta function, fractional q-derivative.

1. Introduction

In 1903, the Swedish mathematician Gosta Mittag Leffler [5] introduced the function $E_\alpha(z)$ by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad (\alpha \in \mathbb{C}, \operatorname{Re}(\alpha) > 0) \tag{1.1}$$

The generalization of $E_\alpha(z)$ was studied by Wiman [12], who defined the function $E_{\alpha,\beta}(z)$ as below

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}, \qquad (\alpha,\beta,\gamma\in\mathbb{C},\ \mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0,\ \mathrm{Re}(\gamma)>0) \quad (1.2)$$

In 1971, Prabhakar [6] introduced the function $E^{\gamma}_{\alpha,\beta}(z)$, $\alpha,\beta,\gamma\in\mathbb{C}$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\gamma) > 0$ which is defined by

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}.\frac{z^n}{n!} \quad (1.3)$$

where $(\lambda)_n$ is the Pochammer symbol [7] defined by

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}=\begin{cases}1, & n=0,\lambda\neq 0\\ \lambda(\lambda+1)\ldots(\lambda+n-1), & n\in N,\lambda\in\mathbb{C}\end{cases} \quad (1.4)$$

N being the set of positive integers.

In the sequel to this study, we define the q-analogue of generalized Mittag – Leffler function $E^{\gamma}_{\alpha,\beta}(z;q)$ as follows

Definition 1 : For $\alpha,\beta,\gamma\in\mathbb{C}$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\gamma) > 0$ and $|q| < 1$ the function $E^{\gamma}_{\alpha,\beta}(z;q)$ is defined as

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(q^{\gamma};q)_n}{(q;q)_n}.\frac{z^n}{\Gamma_q(\alpha n+\beta)} \quad (1.5)$$

where $\Gamma_q(\lambda)$ is the q-gamma function.

The q-analogue of the Pochhammer symbol (q-shifted factorial) is defined by

$$(\lambda;q)_n=\prod_{k=0}^{n-1}(1-\lambda q^k)=\frac{(\lambda;q)_\infty}{(\lambda q^n;q)_\infty} \quad (1.6)$$

and the q-analogue of the power $(a-b)^n$ is

$$(a-b)^0=1,\ (a-b)^n=\prod_{k=0}^{n-1}(a-bq^k) \quad (1.7)$$

There is following relationship between them :

$$(a-b)^n=a^n\left(b/a;q\right)_n,\ (a\neq 0)$$

$$= a^n \frac{\left(b/a;q\right)_\infty}{\left(q^n\, b/a;q\right)_\infty} \tag{1.8}$$

Also, Predrag M. Rajkovic, et. al. [8], define a q-derivative of a function f(z) by

$$D_q f(z) = \frac{f(z) - f(qz)}{z - qz} \quad (z \neq 0) \tag{1.9}$$

Further, the $\Gamma_q(z)$ satisfies the functional equation,

$$\Gamma_q(z+1) = \frac{1-q^z}{1-q}\Gamma_q(z) \tag{1.10}$$

Again, the q-analogue of the beta function is defined by

$$B_q(x,y) = \int_0^1 t^{x-1} \frac{(tq;q)_\infty}{(tq^y;q)_\infty}\, d_q(t) \tag{1.11}$$

The relation between q-beta function and q-gamma function is

$$B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)} \text{, (Re(x) > 0, Re(y) > 0)}$$

(1.12)

The detailed account of generalized Mittag-Leffler function can be found in research monographs due to Agrawal [1], Kilbas, et. al. [3], Gupta and Debnath [2], and Shukla and Prajapati [9, 10, 11].

In this paper, the motive is to evaluate the recurrence relation with q-derivative and in the last section, properties of $E^{\gamma}_{\alpha,\beta}(z;q)$ by using fractional q-calculus.

2. Recurrence Relations

Theorem 1 : If α,β,γ∈ℂ, Re(α) > 0, Re(β) > 0, Re(γ) > 0 then

$$E^{\gamma}_{\alpha,\beta}(z;q) = E^{\gamma+1}_{\alpha,\beta}(z;q) - zq^{\gamma} E^{\gamma}_{\alpha,\alpha+\beta}(z;q) \tag{2.1}$$

Proof: From (1.5), we write

$$E^{\gamma}_{\alpha,\beta}(z;q) = \sum_{n=0}^{\infty} \frac{(q^{\gamma};q)_n}{(q;q)_n} . \frac{z^n}{\Gamma_q(\alpha n + \beta)} = \frac{1}{\Gamma_q(\beta)} + \sum_{n=1}^{\infty} \frac{(q^{\gamma};q)_n}{(q;q)_n} . \frac{z^n}{\Gamma_q(\alpha n + \beta)}$$

$$= \frac{1}{\Gamma_q(\beta)} + \sum_{n=1}^{\infty} \frac{(1-q^{\gamma})(q^{\gamma+1};q)_{n-1}}{(q;q)_n} . \frac{z^n}{\Gamma_q(\alpha n+\beta)}$$

Since $(1-q^{\gamma}) = (1-q^{\gamma+n}) - q^{\gamma}(1-q^n)$ then, the above equation becomes equal to

$$\frac{1}{\Gamma_q(\beta)} + \sum_{n=1}^{\infty} \frac{(q^{\gamma+1};q)_{n-1}}{(q;q)_n} . \frac{[(1-q^{\gamma+n}) - q^{\gamma}(1-q^n)]z^n}{\Gamma_q(\alpha n+\beta)}$$

$$= \frac{1}{\Gamma_q(\beta)} + \sum_{n=1}^{\infty} \frac{(q^{\gamma+1};q)_n}{(q;q)_n} . \frac{z^n}{\Gamma_q(\alpha n+\beta)} - q^{\gamma} \sum_{n=1}^{\infty} \frac{(q^{\gamma+1};q)_{n-1}}{(q;q)_{n-1}} . \frac{z^n}{\Gamma_q(\alpha n+\beta)}$$

$$= \sum_{n=0}^{\infty} \frac{(q^{\gamma+1};q)_n}{(q;q)_n} . \frac{z^n}{\Gamma_q(\alpha n+\beta)} - q^{\gamma} \sum_{n=1}^{\infty} \frac{(q^{\gamma+1};q)_{n-1}}{(q;q)_{n-1}} . \frac{z^n}{\Gamma_q(\alpha n+\beta)}$$

On replacing n by m+1 in second summation, the RHS of above equation becomes

$$\sum_{n=0}^{\infty} \frac{(q^{\gamma+1};q)_n}{(q;q)_n} . \frac{z^n}{\Gamma_q(\alpha n+\beta)} - q^{\gamma} z \sum_{m=0}^{\infty} \frac{(q^{\gamma+1};q)_m}{(q;q)_m} . \frac{z^m}{\Gamma_q[\alpha m+(\alpha+\beta)]}$$

In view of definition (1.5), the above expression becomes

$$E_{\alpha,\beta}^{\gamma+1}(z;q) - \gamma z E_{\alpha,\alpha+\beta}^{\gamma+1}(z;q)$$

This completes the proof of the result (2.1).

Theorem 2 : Let α,β,γ,ω∈ℂ, then for any n = 1, 2, 3, ...

$$D_q^n[z^{\beta-1}E_{\alpha,\beta}^{\gamma}(\omega z^{\alpha};q)] = z^{\beta-n-1}E_{\alpha,\beta-n}^{\gamma}(\omega z^{\alpha};q) \tag{2.2}$$

where Re(β) > n.

Proof : Consider the function

$f(z) = z^{\beta-1}E_{\alpha,\beta}^{\gamma}(\omega z^{\alpha};q)$ in (1.9) and applying the definition (1.5)

$D_q[z^{\beta-1}E_{\alpha,\beta}^{\gamma}(\omega z^{\alpha};q)]$ becomes

$$\sum_{n=0}^{\infty} \frac{(q^{\gamma};q)_n}{(q;q)_n} \frac{(1-q^{\alpha n+\beta-1})}{(1-q)} \frac{\omega^n z^{\alpha n+\beta-2}}{\Gamma_q(\alpha n+\beta)}$$

According to the functional equation (1.10) the above expression becomes

$$\sum_{n=0}^{\infty} \frac{(q^{\gamma};q)_n}{(q;q)_n} \frac{\omega^n z^{\alpha n+\beta-2}}{\Gamma_q(\alpha n+\beta-1)}$$

which equals $z^{\beta-2}E^{\gamma}_{\alpha,\beta-1}(\omega z^{\alpha};q)$

Finally, we obtain

$$D_q[z^{\beta-1}E^{\gamma}_{\alpha,\beta}(\omega z^{\alpha};q)] = z^{\beta-2}E^{\gamma}_{\alpha,\beta-1}(\omega z^{\alpha};q)$$

Iterating this result, upto n–1 times, we obtain the required formula.

3. Fractional q-Calculus

We define the fractional q-integral of operator and the fractional q-derivative of Miller and Ross [4] type by

Definition 2: The fractional q-integral operator of order ν defined as for Re(ν) > 0

$$I_q^{\nu} f(t) = \frac{1}{\Gamma_q(\nu)}\int_0^t (t-q\xi)^{(\nu-1)} f(\xi)d_q\xi \tag{3.1}$$

Definition 3 : The fractional q-differential operator of order μ defined as

$$D_q^{\mu} f(t) = D_q^k\{I^{k-\mu} f(t)\} \tag{3.2}$$

where Re(μ) > 0 and if k is the smallest integer with the property that $k \geq \text{Re}(\mu)$.

Theorem 3 : Let $\gamma \in \mathbb{C}$, Re(γ) > 0 and c is any arbitrary constant, then

$$I_q^{\nu} E^{\gamma}_{1,1}(ct;q) = t^{\nu} E^{\gamma}_{1,\nu+1}(ct;q) \tag{3.3}$$

Proof : Consider the function f(t) = $E^{\gamma}_{1,1}(ct;q)$ in (3.1) and applying the definition (1.5), the LHS of above expression becomes.

$$\frac{1}{\Gamma_q(\nu)}\int_0^t (t-q\xi)^{(\nu-1)} \sum_{n=0}^{\infty} \frac{(q^{\gamma};q)_n (c\xi)^n}{(q;q)_n \Gamma_q(n+1)} d_q\xi$$

Now, using relation (1.8) above expression reduce to

$$\frac{1}{\Gamma_q(\nu)}\sum_{n=0}^{\infty}\frac{(q^{\gamma};q)_n c^n}{(q;q)_n}\int_0^t \frac{t^{\nu-1}\left(\frac{q\xi}{t};q\right)_{\infty}}{\left(\frac{q^{\nu}\xi}{t};q\right)_{\infty}} \xi^n d_q\xi$$

On simplification, we have

$$\frac{1}{\Gamma_q(\nu)}\sum_{n=0}^{\infty}\frac{(q^{\gamma};q)_n c^n}{(q;q)_n} t^{\nu+n-1}\int_0^t \left(\frac{\xi}{t}\right)^n \frac{\left(\frac{q\xi}{t};q\right)_{\infty}}{\left(\frac{q^{\nu}\xi}{t};q\right)_{\infty}} d_q\xi$$

substituting ξ=xt, which yields

$$= \frac{1}{\Gamma_q(\nu)} \sum_{n=0}^{\infty} \frac{(q^\gamma;q)_n c^n}{(q;q)_n} t^{\nu+n} \int_0^1 x^n \frac{(qx;q)_\infty}{(q^\nu x;q)_\infty} d_q x$$

Using the definition of beta function (1.11), the above expression becomes

$$\frac{1}{\Gamma_q(\nu)} \sum_{n=0}^{\infty} \frac{(q^\gamma;q)_n c^n}{(q;q)_n} t^{\nu+n} B_q(n+1,\nu)$$

Also, using the relation (1.12) and on simplification, the RHS of above equation reduce to

$$t^\nu \sum_{n=0}^{\infty} \frac{(q^\gamma;q)_n}{(q;q)_n} \frac{(ct)^n}{\Gamma_q(\nu+n+1)} = t^\nu E_{1,\nu+1}^{\gamma}(ct;q)$$

This completes the proof of the result (3.3).

Theorem 4 : Let γ∈ℂ, Re(γ) > 0 and c is any arbitrary constant, then

$$D_q^\mu E_{1,1}^{\gamma}(ct;q) = t^{-\mu} E_{1,1-\mu}^{\gamma}(ct;q) \qquad (3.4)$$

Proof : Consider the function f(t) = $E_{1,1}^{\gamma}(ct;q)$ in (3.2)

The LHS of above expression reduce to $D_q^k \left\{ I_q^{k-\mu} E_{1,1}^{\gamma}(ct;q) \right\}$

Now using theorem (3) it becomes $D_q^k \left\{ t^{k-\mu} E_{1,k-\mu+1}^{\gamma}(ct;q) \right\}$

Using definition (1.5) and on simplification we have

$$\sum_{n=0}^{\infty} \frac{(q^\gamma;q)_n c^n}{(q;q)_n} D_q^k \left\{ \frac{t^{k-\mu+n}}{\Gamma_q(k-\mu+n+1)} \right\}$$

On applying (1.9) and (1.10) upto k times the above equation reduce to

$$\sum_{n=0}^{\infty} \frac{(q^\gamma;q)_n c^n}{(q;q)_n} \frac{t^{-\mu+n}}{\Gamma_q(n+1-\mu)}$$

It can be written as $t^{-\mu} E_{1,1-\mu}^{\gamma}(ct;q)$

This completes the proof of the result (3.4).

References

1. R.P. Agrawal, A propos d’une note de M. Pierre Humbert. C.R. Acad. Sci. Paris. **296**, (1953), 2031 – 2032.

2. I.S. Gupta and L. Debnath, Some properties of the Mittag – Leffler functions. Integral Trans. Special Function. **18**(5), (2007), 329 – 336.

3. A.A. Kilbas Saigo, M. Saigo and R.K. Saxena, Generalized Mittag – Leffler function and generalized fractional calculus operators. Integral Transform Special Function. **15**, (2004), 31 – 49.

4. K.S. Miller and B. Ross, An Introduction to Fractional Calculus and Fractional Differential Equations. Wiley–New York. (1993).

5. G.M. Mittag – Leffler, Sur la nouvelle function $E_\alpha(x)$. C.R. Acad. Sci. Paris. (ser. II) **137**, (1903), 554 – 558.

6. T.R. Prabhakar, A singular integral equation with a generalized Mittag – Leffler function in the kernel. Yokohoma Math. J. **19**, (1971), 7 – 15.

7. E.D. Rainville, Special Functions, Macmillan, New York. (1960).

8. P.M. Rajkovic, S.D. Marinkovic and M.S. Stankovic, Fractional integrals and derivatives in q-calculus. Applicable Analysis and Discrete Mathematics. **1**, (2007), 311 – 323.

9. A.K. Shukla and J.C. Prajapati, On a generalization Mittag – Leffler of function and its properties. J. Math. Anal. Appl. **336**, (2007), 797 – 811.

10. A.K. Shukla and J.C. Prajapati, On recurrence relation of generalized Mittag – Leffler function. Survey in Mathematics and its applications **4**, (2009), 132 – 138.

11. A.K. Shukla and J.C. Prajapati, Some remarks on generalized Mittage–Leffler function Proyecciones Universided Catolica del Norte. Antofagasta. Chile. **28**, (2009), 27 – 34.

12. A. Wiman, Uber den Fundamental satz in der Theorie de Funktionen $E_\alpha(x)$, Acta Math. **29**, (1905), 191 – 201.

Received: June, 2014