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Ruin Probability in a Generalized Risk Process under Rates of Interest with Homogenous Markov Chain Claims

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ABSTRACT

The aim of this paper is to give recursive and integral equations for ruin probabilities of generalized risk processes under interest force with homogenous Markov chain claims. Generalized Lundberg inequalities for ruin probabilities of these processes are derived by using recursive technique. We first give recursive equations for finite – time probability and an integral equation for ultimate ruin probability in Theorem 2.1 and Theorem 2.2. Using these equations, we can derive probability inequalities for finite – time probabilities and ultimate ruin probability in Theorem 3.1 and Theorem 3.2. The above results give upper bounds for finite – time probability and ultimate ruin probability.

KEYWORDS: Integral equation, Recursive equation, Ruin probability, Homogeneous Markov chain.

Mathematics Subject Classifications: 62P05, 60G40, 12E05

1. Introduction

For over a century, there has been a major interest in actuarial science. Since a large portion of the surplus of insurance business from investment income, actuaries have been studying ruin problems under risk models with rates of interest. For example, Teugels and Sundt (1995,1997) studied the effects of constant rate on the ruin probability under the compound Poisson risk model. Yang (1999) established both exponential and non – exponential upper bounds for ruin probabilities in a risk model with constant interest force and independent premiums and claims. Cai (2002a, 2002b) investigated the ruin probabilities in two risk models, with independent premiums and claims and used a first – order autoregressive process to model the rates of in interest. Cai and Dickson (2004) obtained Lundberg inequalities for ruin probabilities in two discrete- time risk process with a Markov chain interest model and independent premiums and claims.

In this paper, we study the models considered by Cai and Dickson (2004) to the case homogenous markov chain claims, independent rates of interest and independent premiums. The main difference between the model in our paper and the one in Cai and

Dickson (2004) is that claims in our model are assumed to follow homogeneous Markov chains.
In this paper, we study two style of premium collections. On one hand of the premiums are collected at the beging of each period then the surplus process $\left\{U_n^{(1)}\right\}_{n\geq 1}$ with initial surplus u can be written as

$$U_n^{(1)} = U_{n-1}^{(1)}(1+I_n) + X_n - Y_n, \tag{1}$$

which can be rearranged as

$$U_n^{(1)} = u.\prod_{k=1}^{n}(1+I_k) + \sum_{k=1}^{n}\left(X_k - Y_k\right)\prod_{j=k+1}^{n}(1+I_j). \tag{2}$$

On the other hand, if the premiums are collected at the end of each period, then the surplus process $\left\{U_n^{(2)}\right\}_{n\geq 1}$ with initial surplus u can be written as

$$U_n^{(2)} = (U_{n-1}^{(2)} + X_n)(1+I_n) - Y_n, \tag{3}$$

which is equivalent to

$$U_n^{(2)} = u.\prod_{k=1}^{n}(1+I_k) + \sum_{k=1}^{n}\left[X_k(1+I_k) - Y_k\right]\prod_{j=k+1}^{n}(1+I_j).$$

where throughout this paper, we denote $\prod_{t=a}^{b} x_t = 1$ and $\sum_{t=a}^{b} x_t = 0$ if $a > b$.

We assume that:
Assumption 1.1 $U_o^{(1)} = U_o^{(2)} = u > 0$.
Assumption 1.2 $X = \left\{X_n\right\}_{n\geq 0}$ is sequence of independent and identically distributed non – negative continuous random variables with the same distribution function $F(x) = P(X_0 \leq x)$.
Assumption 1.3 $I = \left\{I_n\right\}_{n\geq 0}$ is sequence of independent and identically distributed non – negative continuous random variables with the same distribution function $G(t) = P(I_0 \leq t)$.
Assumption 1.4 $Y = \left\{Y_n\right\}_{n\geq 0}$ is a homogeneous Markov chain such that for any n, Y_n takes values in a finite set of non - negative numbers $E = \left\{y_1, y_2, ..., y_M\right\}$ with $Y_o = y_i$ and

$$p_{ij} = P\left[Y_{m+1} = y_j \middle| Y_m = y_i\right], (m \in N); y_i, y_j \in E \text{ where } 0 \leq p_{ij} \leq 1, \sum_{j=1}^{M} p_{ij} = 1.$$

Assumption 1.5 X, Y and I are assumed to be independent.
We define the finite time and ultimate ruin probabilities in model (1) with assumption 1.1 to assumption 1.5, respectively, by

$$\psi_n^{(1)}(u, y_i) = P\left(\bigcup_{k=1}^{n}(U_k^{(1)} < 0) \middle| U_o^{(1)} = u, Y_o = y_i\right), \tag{5}$$

$$\psi^{(1)}(u, y_i) = \lim_{n\to\infty}\psi_n^{(1)}(u, y_i) = P\left(\bigcup_{k=1}^{\infty}(U_k^{(1)} < 0) \middle| U_o^{(1)} = u, Y_o = y_i\right). \tag{6}$$

Similarly, we define the finite time and ultimate ruin probabilities in model (3) with assumption 1.1 to assumption 1.5, respectively, by

$$\psi_n^{(2)}(u,y_i)=P\left(\bigcup_{k=1}^{n}(U_k^{(2)}<0)\middle|U_o^{(2)}=u,Y_o=y_i\right), \tag{7}$$

$$\psi^{(2)}(u,y_i)=\lim_{n\to\infty}\psi_n^{(2)}(u,y_i)=P\left(\bigcup_{k=1}^{\infty}(U_k^{(2)}<0)\middle|U_o^{(2)}=u,Y_o=y_i\right). \tag{8}$$

In this paper, we derive probability inequalities for $\psi^{(1)}(u,y_i)$ and $\psi^{(2)}(u,y_i)$. The paper is organized as follows: in section 2, we first give recursive equations for $\psi_n^{(1)}(u,y_i)$ and $\psi_n^{(2)}(u,y_i)$ and an integral equation for $\psi^{(1)}(u,y_i)$ and $\psi^{(2)}(u,y_i)$. We the derive probability inequalities for $\psi^{(1)}(u,y_i)$ and $\psi^{(2)}(u,y_i)$ in section 3 by an inductive approach. Finally, we conclude our paper in Section 4.

2. Integral Equation for Ruin Probabilities

We first give a recursive equation for $\psi_n^{(1)}(u,y_i)$ and an integral equation for $\psi^{(1)}(u,y_i)$.

Theorem 2.1. If model (1) satisfies the assumptions 1.1 to 1.5 then for n = 1, 2, …

$$\psi_{n+1}^{(1)}(u,y_i)=\sum_{j=1}^{M}p_{ij}\left\{\int_0^{+\infty}\int_{h_t}^{+\infty}\psi_n^{(1)}(x-h_t,y_j)dF(x)dG(t)+\int_0^{+\infty}F(h_t)dG(t)\right\}, \tag{9}$$

and

$$\psi^{(1)}(u,y_i)=\sum_{j=1}^{M}p_{ij}\left\{\int_0^{+\infty}\int_{h_t}^{+\infty}\psi^{(1)}(x-h_t,y_j)dF(x)dG(t)+\int_0^{+\infty}F(h_t)dG(t)\right\}, \tag{10}$$

where $h_t=y_j-u(1+t)$.

Proof.

Given $Y_1=y_j\in E$, from (1), we have

$$U_1^{(1)}=U_o^{(1)}(1+I_1)+X_1-Y_1=u(1+I_1)+X_1-y_j.$$

Let

$$B=\left\{U_o^{(1)}=u,Y_o=y_i\right\},A_j=\left\{Y_1=y_j\right\},$$

$$A_1=\left\{X_1<Y_1-u(1+I_1)\right\},\ A_2=\left\{X_1\ge Y_1-u(1+I_1)\right\}.$$

Thus, we have

$$P\left(U_1^{(1)}<0\middle|B\cap A_j\cap A_1\right)=1\Rightarrow P\left(\bigcup_{k=1}^{n+1}(U_k^{(1)}<0)\middle|B\cap A_j\cap A_1\right)=1, \tag{11}$$

and

$$P\left(U_1^{(1)}<0\middle|B\cap A_j\cap A_2\right)=0. \tag{12}$$

Let $\{\tilde{X}_n\}_{n\ge0},\{\tilde{Y}_n\}_{n\ge0},\{\tilde{I}_n\}_{n\ge0}$ be independent copies of $\{X_n\}_{n\ge0}$, $\{Y_n\}_{n\ge0}$, $\{I_n\}_{n\ge0}$ respectively with $\tilde{X}_o=X_1,\tilde{Y}_o=Y_1=y_j,\tilde{I}_o=I_1$.

Thus, (12) and (2) imply that for

$$P\left(\bigcup_{k=1}^{n+1}(U_k^{(1)}<0)\middle|B\cap A_j\cap A_2\right)=P\left(\bigcup_{k=2}^{n+1}(U_k^{(1)}<0)\middle|B\cap A_j\cap A_2\right)$$

$$=P\left(\bigcup_{k=2}^{n+1}\left\{\left(u(1+I_1)+X_1-y_j\right)\prod_{j=2}^{k}(1+I_j)+\sum_{j=2}^{k}(X_j-Y_j)\prod_{p=j+1}^{k}(1+I_p)<0\right\}\middle|B\cap A_j\cap A_2\right)$$

$$=P\left(\bigcup_{k=1}^{n}\left\{\tilde{U}_o^{(1)}\prod_{j=2}^{k}(1+\tilde{I}_j)+\sum_{j=1}^{k}(\tilde{X}_j-\tilde{Y}_j)\prod_{p=j+1}^{k}(1+\tilde{I}_p)<0\right\}\middle|\left\{\tilde{U}_o^{(1)}=u(1+I_1)+X_1-y_j,\tilde{Y}_o=y_j\right\}\cap B\cap A_2\right) \quad (13)$$

That, (5) imply

$$\psi_{n+1}^{(1)}(u,y_i)=P\left\{\bigcup_{k=1}^{n+1}(U_k^{(1)}<0)\middle|U_o^{(1)}=u,Y_o=y_i\right\}$$

Thus, we have

$$\psi_{n+1}^{(1)}(u,y_i)=\sum_{j=1}^{M}p_{ij}P\left\{\bigcup_{k=1}^{n+1}(U_k<0)\middle|B\cap A_j\right\}$$

$$=\sum_{j=1}^{M}p_{ij}\left\{P\left\{\bigcup_{k=1}^{n+1}(U_k<0)\middle|B\cap A_j\cap A_1\right\}.P(A_1|B\cap A_j)+P\left\{\bigcup_{k=1}^{n+1}(U_k<0)\middle|B\cap A_j\cap A_2\right\}.P(A_2|B\cap A_j)\right\}. \quad (14)$$

From (11), we have

$$P\left\{\bigcup_{k=1}^{n+1}\left(U_k^{(1)}<0\right)\middle|B\cap A_j\cap A_1\right\}.P(A_1|B\cap A_j)=\int_0^{+\infty}\int_0^{h_t}dF(x)dG(t),$$

where $h_t=y_j-u(1+t)$

From (13), we have

$$P\left\{\bigcup_{k=1}^{n+1}\left(U_k^{(1)}<0\right)\middle|B\cap A_j\cap A_2\right\}.P(A_2|B\cap A_j)=\int_0^{+\infty}\int_{h_t}^{+\infty}\psi_n^{(1)}\left(x-h_t,y_j\right)dF(x)dG(t).$$

Therefore, (14) is written as

$$\psi_{n+1}^{(1)}(u,y_i)=\sum_{j=1}^{M}p_{ij}\left\{\int_0^{+\infty}\int_{h_t}^{+\infty}\psi_n^{(1)}(x-h_t,y_j)dF(x)dG(t)+\int_0^{+\infty}\int_0^{h_t}dF(x)dG(t)\right\}$$

$$=\sum_{j=1}^{M}p_{ij}\left\{\int_0^{+\infty}\int_{h_t}^{+\infty}\psi_n^{(1)}(x-h_t,y_j)dF(x)dG(t)+\int_0^{+\infty}F(h_t)dG(t)\right\}. \quad (15)$$

Thus, from the dominated convergence theorem, the integaral equation for $\psi^{(1)}(u,y_i)$ in Theorem 2.1 follows immediately by letting $n\to\infty$ in (15).

This completes the proof □.

Similarly, the following recursive equations for $\psi_n^{(2)}(u,y_i)$ and an integral equation for $\psi^{(2)}(u,y_i)$ hold.

Theorem 2.2. If model (3) satisfies the assumptions 1.1 to 1.5 then for n = 1, 2, …

$$\psi_{n+1}^{(2)}(u,y_i)=\sum_{j=1}^{M}p_{ij}\left\{\int_0^{+\infty}\int_{h_t}^{+\infty}\psi_n^{(2)}((u+x)(1+t)-y_j,y_j)dF(x)dG(t)+\int_0^{+\infty}F(h_t)dG(t)\right\}, \quad (16)$$

and

$$\psi^{(2)}(u,y_i)=\sum_{j=1}^{M}p_{ij}\left\{\int_0^{+\infty}\int_{h_t}^{+\infty}\psi^{(2)}((u+x)(1+t)-y_j,y_j)dF(x)dG(t)+\int_0^{+\infty}F(h_t)dG(t)\right\}, \quad (17)$$

where $h_t=\dfrac{y_j-u(1+t)}{1+t}$.

3. Probability Inequality for Ruin Probabilities

To establish probability inequalities for ruin probabilities of model (1), we first proof the following Lemma.

Lemma 3.1. Let model (1) satisfy assumptions 1.1 to 1.5 and $E\left(X_1^k\right)<+\infty(k=1,2)$.

Any $y_i\in E$, if

$$E(Y_1|Y_o=y_i)<E\left(X_1\right) \text{ and } P\left(Y_1-X_1>0|Y_o=y_i\right)>0 \quad (18)$$

then, there exists a unique positive constant R_i satisfying:

$$E\left(e^{R_i(Y_1-X_1)}|Y_o=y_i\right)=1 \quad (19)$$

Proof.

Define

$$f_i(t)=E\left\{e^{t(Y_1-X_1)}|Y_o=y_i\right\}-1;t\in(0,+\infty).$$

We have

$$f_i(t)=E\left\{e^{tY_1}|Y_o=y_i\right\}.E\left(e^{-tX_1}\right)-1=g_i(t).h(t)-1$$

From Y_1 is discrete random variables and it takes values in $E=\left\{y_1,y_2,...,y_M\right\}$ then

$g_i(t)=E\left\{e^{tY_1}|Y_o=y_i\right\}=\sum_{j=1}^{M}p_{ij}e^{ty_j}$ has n-th derivative function on $\left(0,+\infty\right)$ (any $n\in N^*=N\setminus\left\{0\right\}$).

In addition, $h(t)=\int_0^{+\infty}e^{-tx}f(x)dx$ with $f(x)=F'(x)$ satisfying :

$$0\le h(t)=\int_0^{+\infty}e^{-tx}f(x)dx\le\int_0^{+\infty}f(x)dx=1$$

and $0\le\int_0^{+\infty}x^ke^{-tx}f(x)dx\le\int_0^{+\infty}x^kf(x)dx=E\left(X^k\right)<+\infty(k=1,2)$.

This implies that $h(t)$ has n -th derivative function on $\left(0,+\infty\right)$ with $n=1,2$. Thus, $f_i(t)$ has n-th derivative function on $\left(0,+\infty\right)$ with $n=1,2$ and

$$f_i^{'}(t)=E\left\{(Y_1-X_1)e^{t(Y_1-X_1)}|Y_o=y_i\right\}$$

$$f_i^{"}(t)=E\left\{(Y_1-X_1)^2e^{t(Y_1-X_1)}|Y_o=y_i\right\}\ge 0.$$

Which implies that

$$f_i(t) \text{ is a convex function with } f_i(0)=0 \tag{20}$$

and

$$f_i^{'}(0)=E\left\{(Y_1-X_1)\middle|Y_o=y_i\right\}=E\left(Y_1\middle|Y_o=y_i\right)-E(X_1)<0. \tag{21}$$

By $P\left((Y_1-X_1)>0\middle|Y_o=y_i\right)>0$, we can find some constant $\delta_i>0$ such that

$$P\left((Y_1-X_1)>\delta_i>0\middle|Y_o=y_i\right)>0$$

Then, we can get that

$$f_i(t)=E\left\{e^{t(Y_1-X_1)}\middle|Y_o=y_i\right\}-1\ge E\left(\left\{e^{t(Y_1-X_1)}\middle|Y_o=y_i\right\}.1_{\{Y_1-X_1>\delta|Y_o=y_i\}}\right)-1$$

$$\ge e^{t\delta}.P(\left\{(Y_1-X_1)>\delta\middle|Y_o=y_i\right\}-1.$$

Imply $\lim\limits_{t\to+\infty} f_i(t)=+\infty$. (22)

From (20), (21) and (22) suy ra there exists a unique positive constant R_i satisfying (19).
This completes the proof □.

Let: $R_o=\min\left\{R_i>0:E\left(e^{R_i(Y_1-X_1)}\middle|Y_o=y_i\right)=1(y_i\in E)\right\}$

Use Lemma 3.1 and Theorem 2.1, we now obtain a probability inequality for $\psi^{(1)}(u,y_i)$ by an inductive approach.

Theorem 3.1. If model (1) satisfies assumptions 1.1 to 1.5 $E\left(X_1^k\right)<+\infty(k=1,2)$ and (18) then
for any $u>0$ and $y_i\in E$,

$$\psi^{(1)}(u,y_i)\le\beta_1.E\left[e^{-R_ou(1+I_1)}\right], \tag{23}$$

where

$$\beta_1^{-1}=\inf_{t>0}\frac{e^{R_ot}.\int_0^t e^{-R_ox}dF(x)}{F(t)},\beta_1\le 1. \tag{24}$$

Proof.
Firstly, we have

$$\beta_1^{-1}=\inf_{t>0}\frac{e^{R_ot}\int_0^t e^{-R_ox}dF(x)}{F(t)}\ge\inf_{t>0}\frac{e^{R_ot}\int_0^t e^{-R_ot}dF(x)}{F(t)}=\inf_{t\ge0}\frac{\int_0^t dF(x)}{F(t)}=1\Rightarrow\frac{1}{\beta_1}\ge1\Rightarrow\beta_1\le1.$$

For any $t>0$, we have

$$F(t)=\left[\frac{e^{R_ot}.\int_0^t e^{-R_ox}dF(x)}{F(t)}\right]^{-1}.e^{R_ot}.\int_0^t e^{-R_ox}dF(x)$$

$$\le\beta_1.e^{R_ot}.\int_0^t e^{-R_ox}dF(x) \tag{25}$$

$$\le\beta_1.e^{R_ot}.\int_0^t e^{-R_ox}dF(x)\le\beta_1.e^{R_ot}E\left[e^{-R_oX_1}\right]. \tag{26}$$

Then, for $u > 0$ and $y_i \in E$,

$$\psi_1^{(1)}(u, y_i) = P(U_1^{(1)} > 0 \big| U_o^{(1)} = u, Y_o = y_i) = \sum_{j=1}^{M} p_{ij} \int_0^{+\infty} F(h_t) dG(t) \quad (27)$$

Thus, from (26) and (27), we have

$$\psi_1^{(1)}(u, y_i) = \sum_{j=1}^{M} p_{ij} \int_0^{+\infty} F(h_t) dG(t) \le \beta_1 E\left[e^{-R_o X_1}\right] . \sum_{j=1}^{M} p_{ij} . \int_0^{+\infty} e^{R_o\left[y_j - u(1+t)\right]} dG(t)$$

$$= \beta_1 E\left[e^{-R_o X_1}\right] . \sum_{j=1}^{M} p_{ij} . e^{R_o y_j} \int_0^{+\infty} e^{-R_o u(1+t)} dG(t)$$

$$= \beta_1 E\left[e^{-R_o X_1}\right] . E\left[e^{R_o Y_1} \big| Y_o = y_i\right] . E\left[e^{-R_o u(1+I_1)}\right]$$

$$= \beta_1 E\left[e^{R_o (Y_1 - X_1)} \big| Y_o = y_i\right] . E\left[e^{-R_o u(1+I_1)}\right] = \beta_1 E\left[e^{-R_o u(1+I_1)}\right]. \quad (28)$$

Under an inductive hypothesis, we assume for any $u > 0$ and $y_i \in E$,

$$\psi_n^{(1)}(u, y_i) \le \beta_1 . E\left[e^{-R_o u(1+I_1)}\right]. \quad (29)$$

From (28) implies (29) holds with $n = 1$.
For $y_j \in E$, $x > h_t$ and $I_1 \ge 0$, we have

$$\psi_n^{(1)}(x - h_t, y_j) \le \beta_1^* . E\left[e^{-R_o^*\left[(x+u(1+t)-y_j)(1+I_1)\right]}\right] \le \beta_1^* . e^{-R_o^*\left[x+u(1+t)-y_j\right]}.$$

where $\beta_1^{*-1} = \inf_{t>0} \dfrac{e^{R_o^* t} \int_0^t e^{-R_o^* x} dF(x)}{F(t)}$, $E\left(e^{R_o^*(Y_1 - X_1)} \big| Y_o = y_j\right) = 1$ and $R_o^* \ge R_o > 0$.

Any $t > 0$: $\dfrac{e^{R_o t} \int_0^t e^{-R_o x} dF(x)}{F(t)} = \dfrac{\int_0^t e^{R_o(t-x)} dF(x)}{F(t)} \le \dfrac{\int_0^t e^{R_o^*(t-x)} dF(x)}{F(t)} = \dfrac{\int_0^t e^{-R_o^* x} dF(x)}{e^{R_o^* t} . F(t)}$

then

$$\beta_1^{-1} = \inf_{t>0} \frac{e^{R_o t} \int_0^t e^{-R_o x} dF(x)}{F(t)} \le \beta_1^{*-1} = \inf_{t>0} \frac{e^{R_o^* t} \int_0^t e^{-R_o^* x} dF(x)}{F(t)} \Leftrightarrow \frac{1}{\beta_1} \le \frac{1}{\beta_1^*} \Leftrightarrow \beta_1^* \le \beta_1.$$

We get $R_o^*\left[x + u(1+t) - y_j\right] \ge R_o\left[x + u(1+t) - y_j\right] > 0$ then

$$\psi_n^{(1)}(x - h_t, y_j) \le \beta_1 . e^{-R_o\left[x+u(1+t)-y_j\right]} \quad (30)$$

Therefore, by Lemma 3.1, (9), (25) and (30), we get

$$\psi_{n+1}^{(1)}(u, y_i) = \sum_{j=1}^{M} p_{ij} \left\{ \int_0^{+\infty} \int_{h_t}^{+\infty} \psi_n^{(1)}(x - h_t, y_j) dF(x) dG(t) + \int_0^{+\infty} F(h_t) dG(t) \right\}$$

$$\le \sum_{j=1}^{M} p_{ij} \left\{ \int_0^{+\infty} \int_{h_t}^{+\infty} \beta_1 e^{-R_o\left[x+u(1+t)-y_j\right]} dF(y) dG(t) + \int_0^{+\infty} \left(\beta_1 e^{R_o\left[y_j - u(1+t)\right]} \int_0^{h_t} e^{-R_o x} dF(x) \right) dG(t) \right\}$$

$$= \beta_1 . \sum_{j=1}^{M} p_{ij} e^{R_o y_j} \left\{ \int_0^{+\infty} e^{-R_o u(1+t)} dG(t) . \left(\int_{h_t}^{+\infty} e^{-R_o x} dF(x) + \int_0^{h_t} e^{-R_o x} dF(x) \right) \right\}$$

$$= \beta_1 . \sum_{j \in E} p_{ij} e^{R_o y_j} \int_0^{+\infty} e^{-R_o u(1+t)} dG(t) . \int_0^{+\infty} e^{-R_o x} dF(x)$$
$$= \beta_1 E\left[e^{R_o(Y_1 - X_1)} \middle| Y_o = y_i \right] . E\left[e^{-R_o u(1+I_1)} \right] = \beta_1 E\left[e^{-R_o u(1+I_1)} \right].$$

Hence, for any $n = 1, 2, ...$(29) hold.

Therefore, (23) follows by letting $n \to \infty$ in (29).
This completes the proof □.

Remark 3.1. Let $A(u, x_i) = \beta_1 . E\left[e^{-R_o u(1+I_1)} \right]$. From $I_1 \geq 0$ and $\beta_1 \leq 1$, we have

$$A(u, x_i) \leq \beta_1 . E\left[e^{-R_o u} \right] = \beta_1 e^{-R_o u} \leq e^{-R_o u}.$$

Therefore, upper bound for ruin probability in (23) is better than $e^{-R_o u}$.
Similarly, we have Lemma 3.2.
Lemma 3.2. Assume that model (3) satisfies assumptions 1.1 to 1.5 and $E\left(X_1^k \right) < +\infty (k = 1, 2)$.

Any $y_i \in E$, if

$$E\left[Y_1 - X_1(1 + I_1) \middle| Y_o = y_i \right] < 0 \text{ and } P\left(Y_1 - X_1(1 + I_1) > 0 \middle| Y_o = y_i \right) > 0 \quad (31)$$

Then, there exists a unique positive constant R_i satisfying:

$$E\left(e^{R_i[Y_1 - X_1(1+I_1)])} \middle| Y_o = y_i \right) = 1 \quad (32)$$

Let: $\overline{R}_o = \min\left\{ R_i > 0 : E\left(e^{R_i(Y_1 - X_1(1+I_1))} \middle| Y_o = y_i \right) = 1 (y_i \in E) \right\}$

Use Lemma 3.2 and Theorem 2.2, we now obtain a probability inequality for $\psi^{(2)}(u, y_i)$ by an inductive approach.

Theorem 3.2. If model (3) satisfies assumptions 1.1 to 1.5 $E\left(X_1^k \right) < +\infty (k = 1, 2)$ and (31) then

For any $u > 0$ and $y_i \in E$,

$$\psi^{(2)}(u, y_i) \leq \beta_2 . E\left[e^{\overline{R}_o Y_1} \middle| Y_o = y_i \right] E\left[e^{-\overline{R}_o(u + X_1)(1+I_1)} \right]. \quad (33)$$

where

$$\beta_2^{-1} = \inf_{t > 0} \frac{e^{\overline{R}_o t} . \int_0^t e^{-\overline{R}_o x} dF(x)}{F(t)}, \beta_2 \leq 1. \quad (34)$$

Proof.
Similarly, we have $\beta_2 \leq 1$ and any $t > 0$, we have

$$F(t) \leq \beta_2 . e^{\overline{R}_o t} . \int_0^t e^{-\overline{R}_o x} dF(x). \quad (35)$$

Then, for $u > 0$ and $y_i \in E$,

$$\psi_1^{(2)}(u,y_i)=P(U_1^{(2)}>0\big|U_o^{(2)}=u,Y_o=y_i)=\sum_{j=1}^{M}p_{ij}\int_0^{+\infty}F(h_t)dG(t) \tag{36}$$

Thus, from (35) and (36), we have

$$\psi_1^{(2)}(u,y_i)=\sum_{j=1}^{M}p_{ij}\int_0^{+\infty}F(h_t)dG(t)\le\beta_2.\sum_{j=1}^{M}p_{ij}\cdot\int_0^{+\infty}\left\{\int_0^{h_t}e^{\overline{R}_o\frac{y_j-u(1+t)}{1+t}}e^{-\overline{R}_ox}dF(x)\right\}dG(t)$$

$$\le\beta_2.\sum_{j=1}^{M}p_{ij}\cdot\int_0^{+\infty}\left\{\int_0^{h_t}e^{\overline{R}_o\left[-x+\frac{y_j-u(1+t)}{1+t}\right]}dF(x)\right\}dG(t)=\beta_2.\sum_{j=1}^{M}p_{ij}\cdot\int_0^{+\infty}\left\{\int_0^{h_t}e^{\overline{R}_o\frac{y_j-u(1+t)-x(1+t)}{1+t}}dF(x)\right\}dG(t) \tag{37}$$

where $h_t=\dfrac{y_j-u(1+t)}{1+t}$.

That, for $t>0$

$$\int_0^{h_t}e^{\overline{R}_o\frac{y_j-u(1+t)-x(1+t)}{1+t}}dF(x)\le\int_0^{h_t}e^{\overline{R}_o\left[y_j-u(1+t)-x(1+t)\right]}dF(x)\le\int_0^{+\infty}e^{\overline{R}_o\left[y_j-(u+x)(1+t)\right]}dF(x) \tag{38}$$

From (37) and (38), we have

$$\psi_1^{(2)}(u,y_i)\le\beta_2.\sum_{j=1}^{M}p_{ij}\int_0^{+\infty}\int_0^{+\infty}e^{\overline{R}_o\left[y_j-(u+x)(1+t)\right]}dF(x)dG(t)$$

$$\le\beta_2E\left[e^{\overline{R}_oY_1}\Big|Y_o=y_i\right].E\left[e^{-\overline{R}_o(u+X_1)(1+I_1)}\right]. \tag{39}$$

Under an inductive hypothesis, we assume for any $u>0$ and $y_i\in E$,

$$\psi_n^{(2)}(u,y_i)\le\beta_2E\left[e^{\overline{R}_oY_1}\Big|Y_o=y_i\right].E\left[e^{-\overline{R}_o(u+X_1)(1+I_1)}\right] \tag{40}$$

From (39) implies (40) holds with $n=1$.

For $y_j\in E$, $x>h_t$ and $I_1\ge 0$, we have

$$\psi_n^{(1)}((u+x)(1+t)-y_j,y_j)\le\beta_2^*.E\left[e^{\overline{R}_o^*Y_1}\Big|Y_o=y_j\right]E\left[e^{-\overline{R}_o^*\left[(u+x)(1+t)-y_j+X_1)(1+I_1)\right]}\right]$$

$$=\beta_2^*.E\left[e^{\overline{R}_o^*Y_1}\Big|Y_o=y_j\right]E\left[e^{-\overline{R}_o^*\left[(u+x)(1+t)-y_j\right](1+I_1)-\overline{R}_o^*X_1(1+I_1)}\right]$$

$$\le\beta_2^*.E\left[e^{\overline{R}_o^*Y_1}\Big|Y_o=y_j\right]E\left[e^{-\overline{R}_o^*\left[(u+x)(1+t)-y_j\right]-\overline{R}_o^*X_1(1+I_1)}\right]$$

$$=\beta_2^*.e^{-\overline{R}_o^*\left[(u+x)(1+t)-y_j\right]}E\left[e^{\overline{R}_o^*Y_1}\Big|Y_o=y_j\right]E\left[e^{-\overline{R}_o^*X_1(1+I_1)}\right]=\beta_2^*.e^{-\overline{R}_o^*\left[(u+x)(1+t)-y_j\right]}$$

where $\beta_2^{*-1}=\inf\limits_{t>0}\dfrac{e^{\overline{R}_o^*t}\int_0^t e^{-\overline{R}_o^*x}dF(x)}{F(t)}$, $E\left(e^{\overline{R}_o^*(Y_1-X_1(1+I_1))}\Big|Y_o=y_j\right)=1$ and $\overline{R}_o^*\ge\overline{R}_o>0.$

Any $t>0$: $\dfrac{e^{\overline{R}_ot}\int_0^t e^{-\overline{R}_ox}dF(x)}{F(t)}=\dfrac{\int_0^t e^{\overline{R}_o(t-x)}dF(x)}{F(t)}\le\dfrac{\int_0^t e^{\overline{R}_o^*(t-x)}dF(x)}{F(t)}=\dfrac{\int_0^t e^{-\overline{R}_o^*x}dF(x)}{e^{\overline{R}_o^*t}.F(t)}$

then

$$\beta_2^{-1} = \inf_{t>0} \frac{e^{\overline{R}_o t}\int_0^t e^{-\overline{R}_o x} dF(x)}{.F(t)} \le \beta_2^{*-1} = \inf_{t>0} \frac{e^{\overline{R}_o^* t}\int_0^t e^{-\overline{R}_o^* x} dF(x)}{F(t)} \Leftrightarrow \frac{1}{\beta_2} \le \frac{1}{\beta_2^*} \Leftrightarrow \beta_2^* \le \beta_2 .$$

We get $\overline{R}_o^*\left[(u+x)(1+t)-y_j\right] \ge \overline{R}_o\left[(u+x)(1+t)-y_j\right] > 0$ then

$$\psi_n^{(2)}((u+x)(1+t)-y_j, y_j) \le \beta_2 . e^{-\overline{R}_o\left[(u+x)(1+t)-y_j\right]} \tag{41}$$

Therefore, by Lemma 3.2, (16), (35) and (41), we get

$$\psi_{n+1}^{(2)}(u,y_i) = \sum_{j=1}^M p_{ij}\left\{\int_0^{+\infty}\int_{h_t}^{+\infty} \psi_n^{(2)}((u+x)(1+t)-y_j, y_j)dF(x)dG(t) + \int_0^{+\infty} F(h_t)dG(t)\right\}$$

$$\le \sum_{j=1}^M p_{ij}\left\{\int_0^{+\infty}\int_{h_t}^{+\infty} \beta_2 e^{-\overline{R}_o\left[(u+x)(1+t)-y_j\right]} dF(y)dG(t) + \int_0^{+\infty}\left(\beta_2\int_0^{h_t} e^{\overline{R}_o\frac{y_j-u(1+t)}{1+t}} e^{-R_o x} dF(x)\right)dG(t)\right\}$$

$$= \sum_{j=1}^M p_{ij}\left\{\int_0^{+\infty}\int_{h_t}^{+\infty} \beta_2 e^{-\overline{R}_o\left[(u+x)(1+t)-y_j\right]} dF(y)dG(t) + \int_0^{+\infty}\left(\beta_2\int_0^{h_t} e^{\overline{R}_o\frac{y_j-u(1+t)-x(1+t)}{1+t}} dF(x)\right)dG(t)\right\} \tag{42}$$

From (38) and (42), we have

$$\psi_{n+1}^{(2)}(u,y_i) \le \beta_2 . \sum_{j=1}^M p_{ij} \int_0^{+\infty}\int_0^{+\infty} e^{\overline{R}_o\left[y_j-(u+x)(1+t)\right]} dF(x)dG(t)$$

$$= \beta_2 E\left[e^{\overline{R}_o Y_1}\middle| Y_o = y_i\right] . E\left[e^{-\overline{R}_o (u+X_1)(1+I_1)}\right].$$

Hence, for any $n = 1,2,...$(40) hold. Therefore, (33) follows by letting $n \to \infty$ in (40). This completes the proof □.

Remark 3.2. Let $B(u,y_i) = \beta_2 E\left[e^{\overline{R}_o Y_1}\middle| Y_o = y_i\right] E\left[e^{-\overline{R}_o (u+X_1)(1+I_1)}\right]$. From $I_1 \ge 0, X_1 \ge 0$ and $\beta_2 \le 1$, we have

$$\begin{aligned} B(u,y_i) &= \beta_2 E\left[e^{\overline{R}_o Y_1}\middle| Y_o = y_i\right] E\left[e^{-\overline{R}_o u(1+I_1) - \overline{R}_o X_1(1+I_1)}\right] \\ &\le \beta_2 E\left[e^{\overline{R}_o Y_1}\right] E\left[e^{-\overline{R}_o u - \overline{R}_o X_1(1+I_1)}\middle| X_o = x_i\right] \\ &= \beta_2 e^{-\overline{R}_o u} E\left[e^{\overline{R}_o [Y_1 - X_1)(1+I_1)]}\middle| X_o = x_i\right] = \beta_2 e^{-\overline{R}_o u} \le e^{-\overline{R}_o u}. \end{aligned}$$

Therefore, upper bound for ruin probability in (33) is better than $e^{-\overline{R}_o u}$.

4. Conclusion

Our main results in this paper, Theorem 2.1 and Theorem 2.2 give recursive equation for $\psi_n^{(1)}(u,y_i)$ and $\psi_n^{(2)}(u,y_i)$ and integral equation for $\psi^{(1)}(u,y_i)$ and $\psi^{(2)}(u,y_i)$; Theorem 3.1 and Theorem 3.2 give probability inequalities for $\psi^{(1)}(u,y_i)$ and $\psi^{(2)}(u,y_i)$ by an inductive approach.

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