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The Jordan canonical form of homogeneous linear mappings

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Abstract

This paper researches the Jordan canonical form of homogeneous linear mappings on low dimensional complex $\mathbb{Z}_2$-graded vector spaces.

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1 Preliminary Notes

It is well-known that the Jordan canonical form of linear mappings on low dimensional vector spaces over a complex field [1]. The aim of this paper is to research the Jordan canonical form of homogeneous linear mappings on low dimensional $\mathbb{Z}_2$-graded vector spaces over a complex field. Throughout this paper, we assume that all vector spaces are $\mathbb{Z}_2$-graded over a complex number field and all linear mappings are homogeneous.

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space over a complex number field. If a linear mapping $\mathcal{A}$ satisfies $\mathcal{A}(V_{\bar{i}}) \subseteq V_{\bar{i}}, (i = 0, 1)$, then $\mathcal{A}$ is called an even mapping, i.e., $\mathcal{A} \in (EndV)_{\bar{0}}$. If a linear mapping $\mathcal{A}$ satisfies $\mathcal{A}(V_{\bar{0}}) \subseteq V_{\bar{1}}$ and $\mathcal{A}(V_{\bar{1}}) \subseteq V_{\bar{0}}$, then $\mathcal{A}$ is called an odd mapping, i.e., $\mathcal{A} \in (EndV)_{\bar{1}}$.

We let $(\text{card}\mathcal{A}\alpha_1, \text{card}\mathcal{A}\alpha_2, \cdots, \text{card}\mathcal{A}\alpha_n)$ be the matrix of $\mathcal{A}$ with respect to the basis $\alpha_1, \alpha_2, \cdots, \alpha_n$ of V. This matrix will be denoted by $M(\mathcal{A};\alpha_1, \alpha_2,\cdots,\alpha_n)$ or simply by $M(\mathcal{A})$.

Let V be a 2 or 3-dimensional $\mathbb{Z}_2$-graded vector space over the complex number field $\mathbb{C}$ and the characteristic polynomial of $\mathcal{A}$ be $f(\lambda)$.

2 Main Results

Theorem 2.1 *Let V be a 2-dimensional $\mathbb{Z}_2$-graded vector space over a complex number field $\mathbb{C}$. If $\dim V_{\bar{0}} = 2$ and $\mathcal{A} \in EndV$, then following statements hold:*

1. *When $\mathcal{A} \in (EndV)_{\bar{0}}$, the Jordan canonical form of $\mathcal{A}$ is well-known.*
2. *When $\mathcal{A} \in (EndV)_{\bar{1}}$, the Jordan canonical form of $\mathcal{A}$ with respect to any basis is zero matrix.*

Proof. 1. (1) If $f(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2), \lambda_1 \neq \lambda_2$,then the matrix of $\mathcal{A}$ with respect to some basis of V is $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

(2) If $f(\lambda) = (\lambda - \lambda_0)^2$ and $\mathcal{A} = \lambda_0\text{id}$,then the matrix of $\mathcal{A}$ with respect to some basis of V is $\lambda_0 I_2$.If $\mathcal{A} \neq \lambda_0\text{id}$,we let $\varepsilon_1, \varepsilon_2$ be a basis of $V_{\bar{0}}$ such that $\varepsilon_2 = (\mathcal{A} - \lambda_0\text{id})\varepsilon_1 \neq 0$ and $(\mathcal{A} - \lambda_0\text{id})\varepsilon_2 = 0$, then $\mathcal{A}\varepsilon_1 = \lambda_0\varepsilon_1 + (\mathcal{A} - \lambda_0\text{id})\varepsilon_2 = \lambda_0\varepsilon_1 + \varepsilon_2$, $\mathcal{A}\varepsilon_2 = \lambda_0\varepsilon_2$. Hence, the matrix of $\mathcal{A}$ with respect to this basis is $\begin{pmatrix} \lambda_0 & 0 \\ 1 & \lambda_0 \end{pmatrix}$.

2. When $\mathcal{A} \in (EndV)_{\bar{1}}$, we let $\varepsilon_1, \varepsilon_2$ be a basis of $V_{\bar{0}}$. According to the definition $\mathcal{A}(V_{\bar{0}}) \subseteq V_{\bar{1}}$,then the matrix of $\mathcal{A}$ with respect to any basis is zero matrix. □

Theorem 2.2 *Let V be a 2-dimensional $\mathbb{Z}_2$-graded vector space over a complex number field $\mathbb{C}$. If $\dim V_{\bar{0}} = \dim V_{\bar{1}} = 1$ and $\mathcal{A} \in EndV$, then the following statements hold:*

1. *When $\mathcal{A} \in (EndV)_{\bar{0}}$, if $f(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2), \lambda_1 \neq \lambda_2$, then the matrix of $\mathcal{A}$ with respect to some basis of V is $diag(\lambda_1, \lambda_2)$; if $f(\lambda) = (\lambda - \lambda_0)^2$, then $\mathcal{A} - \lambda_0\text{id} = 0$, the the matrix of $\mathcal{A}$ with respect to some basis of V is $diag(\lambda_0, \lambda_0)$.*
2. *When $\mathcal{A} \in (EndV)_{\bar{1}}$, the matrix of $\mathcal{A}$ with respect to some basis of V is $diag(\lambda_1, -\lambda_1)$.*

Proof. 1. If $f(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2), \lambda_1 \neq \lambda_2$, then the matrix of $\mathcal{A}$ with respect to some basis of V is $diag(\lambda_1, \lambda_2)$.

If $f(\lambda)=(\lambda-\lambda_1)(\lambda-\lambda_2), \lambda_1=\lambda_2$, then the matrix of $\mathcal{A}$ with respect to some basis of V is $diag(\lambda_1,\lambda_1)$.

2. When $\mathcal{A}\in(EndV)_{\bar{1}}$, we let $\varepsilon_0,\varepsilon_1$ be a basis of V such that $\varepsilon_0\in V_{\bar{0}}$ and $\varepsilon_1\in V_{\bar{1}}$. It is easy to see that the matrix of $\mathcal{A}$ with respect to this basis is $\begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix}$, so $f(\lambda)=\lambda^2-a_{12}\cdot a_{21}=(\lambda-\lambda_1)(\lambda+\lambda_1)$. Hence, there exists a basis of V such that the matrix of $\mathcal{A}$ with respect to the basis is $diag(\lambda_1,-\lambda_1)$. □

Theorem 2.3 *Let V be a 2-dimensional $\mathbb{Z}_2$-graded vector space over a complex number field $\mathbb{C}$. If $\dim V_{\bar{1}}=2$ and $\mathcal{A}\in EndV$, then the following statements hold:*

1. *When $\mathcal{A}\in(EndV)_{\bar{0}}$, the matrix of $\mathcal{A}$ with respect to any basis of V is zero matrix.*

2. *When $\mathcal{A}\in(EndV)_{\bar{1}}$, if $f(\lambda)=(\lambda-\lambda_1)(\lambda-\lambda_2), \lambda_1\neq\lambda_2$, then the matrix of $\mathcal{A}$ with respect to some basis of V is $diag(\lambda_1,\lambda_2)$; if $f(\lambda)=(\lambda-\lambda_0)^2$, then the matrix of $\mathcal{A}$ with respect to some basis of V is $\begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{pmatrix}$ or $\begin{pmatrix} \lambda_0 & 0 \\ 1 & \lambda_0 \end{pmatrix}$.*

Proof. 1. When $\mathcal{A}\in(EndV)_{\bar{0}}$, we let $\varepsilon_1,\varepsilon_2$ be a basis of $V_{\bar{1}}$, according to the definition $\mathcal{A}(V_{\bar{1}})\subseteq V_{\bar{0}}$, then the matrix of $\mathcal{A}$ with respect to any basis is zero matrix.

2. When $\mathcal{A}\in(EndV)_{\bar{1}}$, we let ε_i be an eigenvector of $\mathcal{A}$ belonging to an eigenvalue λ_i, then $\varepsilon_1,\varepsilon_2$ is a basis of V and $M(\mathcal{A};\varepsilon_1,\varepsilon_2)=diag(\lambda_1,\lambda_2)$.

If $f(\lambda)=(\lambda-\lambda_0)^2$ and $\mathcal{A}-\lambda_0\mathrm{id}=0$, then the matrix of $\mathcal{A}$ with respect to some basis of V is $\begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{pmatrix}$.

If $f(\lambda)=(\lambda-\lambda_0)^2$ and $\mathcal{A}-\lambda_0\mathrm{id}\neq0$, then the matrix of $\mathcal{A}$ with respect to some basis of V is $M(\mathcal{A};\varepsilon_1,\varepsilon_2)=\begin{pmatrix} \lambda_0 & 0 \\ 1 & \lambda_0 \end{pmatrix}$. □

Theorem 2.4 *Let V be a 3-dimensional $\mathbb{Z}_2$-graded vector space over a complex number field $\mathbb{C}$. If $\dim V_{\bar{0}}=3$ and $\mathcal{A}\in EndV$, then the following statements hold:*

1. *When $\mathcal{A}\in(EndV)_{\bar{0}}$.*

(1) *If $f(\lambda)=(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)$, where $\lambda_1,\lambda_2,\lambda_3$ is not equal to each other, then the matrix of $\mathcal{A}$ with respect to this basis is $diag(\lambda_1,\lambda_2,\lambda_3)$.*

(2) *If $f(\lambda)=(\lambda-\lambda_1)^2(\lambda-\lambda_2)$ and $\lambda_1\neq\lambda_2$, then the matrix of $\mathcal{A}$ with respect to this basis is*

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \ or \ \begin{pmatrix} \lambda_1 & 0 & 0 \\ 1 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}.$$

(3) *If* $f(\lambda) = (\lambda - \lambda_0)^3$, *then the matrix of* $\mathcal{A}$ *with respect to this basis is*

$$\begin{pmatrix} \lambda_0 & 0 & 0 \\ 0 & \lambda_0 & 0 \\ 0 & 0 & \lambda_0 \end{pmatrix} \ or \ \begin{pmatrix} \lambda_0 & 0 & 0 \\ 1 & \lambda_0 & 0 \\ 0 & 0 & \lambda_0 \end{pmatrix} \ or \ \begin{pmatrix} \lambda_0 & 0 & 0 \\ 1 & \lambda_0 & 0 \\ 0 & 1 & \lambda_0 \end{pmatrix}.$$

2. *When* $\mathcal{A} \in (EndV)_{\bar{1}}$, *, the matrix of* $\mathcal{A}$ *with respect to any basis of* V *is zero matrix.*

Proof. 1. Take ε_i be an eigenvector of $\mathcal{A}$ belonging to an eigenvalue λ_i, then $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ is a basis of $V_{\bar{0}}$ and $M(\mathcal{A}; \varepsilon_1, \varepsilon_2, \varepsilon_3) = diag(\lambda_1, \lambda_2, \lambda_3)$.

(2) If $f(\lambda) = (\lambda - \lambda_1)^2(\lambda - \lambda_2)$, then the root space decomposition of $V_{\bar{0}}$ is $V_{\bar{0}} = R_{\lambda_1}(\mathcal{A}) \oplus R_{\lambda_2}(\mathcal{A})$, where $\dim R_{\lambda_1}(\mathcal{A}) = 2$, $\dim R_{\lambda_2}(\mathcal{A}) = 1$.

If $(\mathcal{A} - \lambda_1 \mathrm{id})^2 \mid_{R_{\lambda_1}(\mathcal{A})} = 0$, let $\{\varepsilon_1, \varepsilon_2\}$ be a basis of $R_{\lambda_1}(\mathcal{A})$. Take ε_3 be an eigenvector of $R_{\lambda_2}(\mathcal{A})$ belonging to an eigenvalue λ_2 , then $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ is a basis of $V_{\bar{0}}$, and $M(\mathcal{A}; \varepsilon_1, \varepsilon_2, \varepsilon_3) = diag(\lambda_1, \lambda_1, \lambda_2)$.

If $(\mathcal{A} - \lambda_0 \mathrm{id})^2 \mid_{R_{\lambda_1}(\mathcal{A})} \neq 0$, then take $\varepsilon_1 \in R_{\lambda_1}(\mathcal{A})$ such that $\varepsilon_2 = (\mathcal{A} - \lambda_1 \mathrm{id})\varepsilon_1 \neq 0$, take ε_3 be an eigenvector of $\mathcal{A}$ belonging to an eigenvalue λ_2, then $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ is a basis of $V_{\bar{0}}$ and

$$M(\mathcal{A}; \varepsilon_1, \varepsilon_2, \varepsilon_3) = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 1 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}.$$

(3) If $\mathcal{A} - \lambda_0 \mathrm{id} = 0$, then the matrix of $\mathcal{A}$ with respect to this basis is $\lambda_0 I_3$.

If $(\mathcal{A} - \lambda_0 \mathrm{id}) \mid_{V_{\bar{0}}} \neq 0$ and $(\mathcal{A} - \lambda_0 \mathrm{id})^2 = 0$, we need to prove $\dim E_{\lambda_0}(\mathcal{A}) = 2$. $\dim E_{\lambda_0}(\mathcal{A}) \leq 2$ is straight-forward. Suppose $\dim E_{\lambda_0}(\mathcal{A}) = 1$ and $\{\beta_1, \beta_2, \beta_3\}$ be a basis of $E_{\lambda_0}(\mathcal{A})$, then take $\beta_3 \in E_{\lambda_0}(\mathcal{A})$, so $(\mathcal{A} - \lambda_0 \mathrm{id})\beta_1 = k\beta_3 \neq 0, (\mathcal{A} - \lambda_0 \mathrm{id})\beta_2 = l\beta_3 \neq 0$. But $(\mathcal{A} - \lambda_0 \mathrm{id})(l\beta_1 - k\beta_2) = 0$, i.e., $(l\beta_1 - k\beta_2) \in E_{\lambda_0}(\mathcal{A}) = L(\beta_3)$, this is a contradiction.

Let $\varepsilon_1 \in V_{\bar{0}}$ such that $\varepsilon_2 = (\mathcal{A} - \lambda_1 \mathrm{id})\varepsilon_1 \neq 0$, then we have $\varepsilon_2 \in E_{\lambda_0}(\mathcal{A})$. Take $\varepsilon_3 \in E_{\lambda_0}(\mathcal{A})$ such that $\{\varepsilon_2, \varepsilon_3\}$ is the basis of $E_{\lambda_0}(\mathcal{A})$. If $k_1\varepsilon_1 + k_2\varepsilon_2 + k_3\varepsilon_3 = 0$ and $\mathcal{A} - \lambda_1 \mathrm{id} = 0$, then $k_1\varepsilon_2 = 0$, so $k_1 = 0$. Let $\{\varepsilon_2, \varepsilon_3\}$ be the basis of $E_{\lambda_0}(\mathcal{A})$, then $k_2 = k_3 = 0$, so $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ is the basis of $V_{\bar{0}}$, and

$$M(\mathcal{A}; \varepsilon_1, \varepsilon_2, \varepsilon_3) = \begin{pmatrix} \lambda_0 & 0 & 0 \\ 1 & \lambda_0 & 0 \\ 0 & 0 & \lambda_0 \end{pmatrix}.$$

If $(\mathcal{A} - \lambda_0 \mathrm{id})^2 \neq 0$, we can take $\varepsilon_1 (\in V_{\bar{0}})$ such that $\varepsilon_3 = (\mathcal{A} - \lambda_1 \mathrm{id})^2 \varepsilon_1 \neq 0$, then $\varepsilon_2 = (\mathcal{A} - \lambda_1 \mathrm{id})\varepsilon_1 \neq 0$.

If $k_1\varepsilon_1 + k_2\varepsilon_2 + k_3\varepsilon_3 = 0$, then $(\mathcal{A} - \lambda_1 \mathrm{id})^2 k_1\varepsilon_1 = 0$, i.e., $k_1\varepsilon_3 = 0$, so $k_1 = 0$. Then we have $(\mathcal{A} - \lambda_1 \mathrm{id})(k_2\varepsilon_2 + k_3\varepsilon_3) = k_2\varepsilon_3 = 0$, so $k_3\varepsilon_3 = 0$. Hence, $k_3 = 0$ and $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ is a basis of $V_{\bar{0}}$. Therefore

$$M(\mathcal{A}; \varepsilon_1, \varepsilon_2, \varepsilon_3) = \begin{pmatrix} \lambda_0 & 0 & 0 \\ 1 & \lambda_0 & 0 \\ 0 & 1 & \lambda_0 \end{pmatrix}.$$

2.When $\mathcal{A} \in (EndV)_{\bar{1}}$, let $\varepsilon_1, \varepsilon_2, \varepsilon_3$ be a basis of $V_{\bar{0}}$ according to the definition $\mathcal{A}(V_{\bar{0}}) \subseteq V_{\bar{1}}$, then the matrix of $\mathcal{A}$ with respect to any basis is zero matrix. □

Theorem 2.5 *Let V be a 3-dimensional $\mathbb{Z}_2$-graded vector space over a complex number field $\mathbb{C}$. If $\dim V_{\bar{0}} = 2, \dim V_{\bar{1}} = 1$ and $\mathcal{A} \in EndV$, then the following statements hold:*

1. *When $\mathcal{A} \in (EndV)_{\bar{0}}$, if $f(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)$, where $\lambda_1, \lambda_2, \lambda_3$ is not equal to each other, then the matrix of $\mathcal{A}$ with respect to this basis is $diag(\lambda_1, \lambda_2, \lambda_3)$.*

If $f(\lambda) = (\lambda-\lambda_1)^2(\lambda-\lambda_2)$ and $\lambda_1 \neq \lambda_2$, then the matrix of $\mathcal{A}$ with respect to this basis is

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \text{ or } \begin{pmatrix} \lambda_1 & 0 & 0 \\ 1 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}.$$

If $f(\lambda) = (\lambda-\lambda_0)^3$, then the matrix of $\mathcal{A}$ with respect to this basis is

$$\begin{pmatrix} \lambda_0 & 0 & 0 \\ 0 & \lambda_0 & 0 \\ 0 & 0 & \lambda_0 \end{pmatrix} \text{ or } \begin{pmatrix} \lambda_0 & 0 & 0 \\ 1 & \lambda_0 & 0 \\ 0 & 0 & \lambda_0 \end{pmatrix}.$$

2. *When $\mathcal{A} \in (EndV)_{\bar{1}}$, the matrix of $\mathcal{A}$ with respect to this basis is $diag(\lambda_1, -\lambda_1, 0)$ or zero matrix.*

Proof. 1. (1) Take ε_i be an eigenvector of $\mathcal{A}$ belonging to an eigenvalue λ_i, then $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ is a basis of $V = V_{\bar{0}} \oplus V_{\bar{1}}$ and $M(\mathcal{A}; \varepsilon_1, \varepsilon_2, \varepsilon_3) = diag(\lambda_1, \lambda_2, \lambda_3)$.

(2) If $(\mathcal{A} - \lambda_2 \mathrm{id}) \mid_{V_{\bar{1}}} \neq 0$, then we can take $\{\varepsilon_1, \varepsilon_2\}$ be a basis of $V_{\bar{0}}$ and $\varepsilon_3 (\in V_{\bar{1}})$ be an eigenvector of $\mathcal{A}$ belonging to an eigenvalue λ_1. So $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ is the basis of V is and $M(\mathcal{A}; \varepsilon_1, \varepsilon_2, \varepsilon_3) = diag(\lambda_1, \lambda_1, \lambda_2)$.

If $(\mathcal{A}-\lambda_2 \mathrm{id}) \mid_{V_{\bar{1}}} \neq 0$, then we can take $\varepsilon_1 \in V_{\bar{1}}$ such that $\varepsilon_2 = (\mathcal{A}-\lambda_2 \mathrm{id})\varepsilon_1 \neq 0$. Take ε_3 be an eigenvector of $\mathcal{A}$ belonging to an eigenvalue λ_1, so $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ is the basis of V and

$$M(\mathcal{A}; \varepsilon_1, \varepsilon_2, \varepsilon_3) = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 1 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}.$$

(3) If $(\mathcal{A} - \lambda_0 \mathrm{id}) = 0$, then the matrix of $\mathcal{A}$ with respect to any basis is $\lambda_0 I_3$.

If $(\mathcal{A} - \lambda_0 \mathrm{id}) \mid_{V_{\bar{0}}} \neq 0$ and $(\mathcal{A} - \lambda_0 \mathrm{id})^2 = 0$, then $\dim E_{\lambda_0}(\mathcal{A}) = 2$. Take $\varepsilon_1 (\in V_{\bar{0}})$ be an eigenvector of $\mathcal{A}$ belonging to an eigenvalue λ_0 such that $\varepsilon_2 = (\mathcal{A}-\lambda_2 \mathrm{id})\varepsilon_1 \neq 0$, take $\varepsilon_3 \in V_{\bar{1}}$ be an eigenvector of $\mathcal{A}$ belonging to an eigenvalue λ_0 and we have

$$M(\mathcal{A}; \varepsilon_1, \varepsilon_2, \varepsilon_3) = \begin{pmatrix} \lambda_0 & 0 & 0 \\ 1 & \lambda_0 & 0 \\ 0 & 0 & \lambda_0 \end{pmatrix}.$$

2. (1) Let $\mathcal{A} \in (EndV)_{\bar{1}}$ and ε_i be an eigenvector of $\mathcal{A}$ belonging to an eigenvalue λ_i, then $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ is a basis of $V = V_{\bar{0}} \oplus V_{\bar{1}}$ and $M(A_{\bar{1}}; \varepsilon_1, \varepsilon_2, \varepsilon_3) = diag(\lambda_1, -\lambda_1, 0)$.

(2) Let $\varepsilon_1, \varepsilon_2, \varepsilon_3$ be a basis of $V_{\bar{0}}$. According to the definition $\mathcal{A}(V_{\bar{0}}) \subseteq V_{\bar{1}}$ we have the matrix of $\mathcal{A}$ with respect to any basis is zero matrix. □

Theorem 2.6 *Let* $\dim V_{\bar{0}} = 1, \dim V_{\bar{1}} = 2$.

1. *When* $\mathcal{A} \in (EndV)_{\bar{0}}$, *if* $f(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$, *where* $\lambda_1, \lambda_2, \lambda_3$ *is not equal to each other, then the matrix of* $\mathcal{A}$ *with respect to this basis is* $diag(\lambda_1, \lambda_2, \lambda_3)$. *If* $f(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)^2$, *then the matrix of* $\mathcal{A}$ *with respect to this basis is*

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \; or \; \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 1 & \lambda_2 \end{pmatrix}.$$

If $f(\lambda) = (\lambda - \lambda_0)^3$, *then the matrix of* $\mathcal{A}$ *with respect to this basis is*

$$\begin{pmatrix} \lambda_0 & 0 & 0 \\ 0 & \lambda_0 & 0 \\ 0 & 0 & \lambda_0 \end{pmatrix} \; or \; \begin{pmatrix} \lambda_0 & 0 & 0 \\ 0 & \lambda_0 & 0 \\ 0 & 1 & \lambda_0 \end{pmatrix}.$$

2. *When* $\mathcal{A} \in (EndV)_{\bar{1}}$, *then the matrix of* $\mathcal{A}$ *with respect to this basis is* $diag(\lambda_1, -\lambda_1, 0)$ *or zero matrix.*

Proof. 1. (1) Let ε_i be an eigenvector of $\mathcal{A}$ belonging to an eigenvalue λ_i, then $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ is a basis of $V = V_{\bar{0}} \oplus V_{\bar{1}}$ and $M(\mathcal{A}; \varepsilon_1, \varepsilon_2, \varepsilon_3) = diag(\lambda_1, \lambda_2, \lambda_3)$.

(2) If $(\mathcal{A} - \lambda_2 \mathrm{id}) |_{V_{\bar{1}}} \neq 0$ and $\{\varepsilon_1, \varepsilon_2\}$ be a basis of $V_{\bar{1}}$, then take $\varepsilon_3 \in V_{\bar{0}}$ be an eigenvector of $\mathcal{A}$ belonging to an eigenvalue λ_1. So $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ is the basis of V and $M(\mathcal{A}; \varepsilon_1, \varepsilon_2, \varepsilon_3) = diag(\lambda_1, \lambda_2, \lambda_2)$.

If $(\mathcal{A} - \lambda_2 \mathrm{id}) |_{V_{\bar{1}}} \neq 0$, then take $\varepsilon_2 (\in V_{\bar{1}})$ such that $\varepsilon_3 = (\mathcal{A} - \lambda_2 \mathrm{id})\varepsilon_2 \neq 0$. Take ε_1 be an eigenvector of $\mathcal{A}$ belonging to an eigenvalue λ_1 , so $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ is the basis of V and

$$M(\mathcal{A}; \varepsilon_1, \varepsilon_2, \varepsilon_3) = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 1 & \lambda_2 & 0 \\ 0 & 1 & \lambda_2 \end{pmatrix}.$$

(3) If $\mathcal{A} - \lambda_0 \mathrm{id} = 0$, then the matrix of $\mathcal{A}$ with respect to any basis is $\lambda_0 I_3$.

If $(\mathcal{A} - \lambda_0 \mathrm{id}) |_{V_{\bar{1}}} \neq 0$ and $(\mathcal{A} - \lambda_0 \mathrm{id})^2 = 0$, then $\dim E_{\lambda_0}(\mathcal{A}) = 2$. Let $\varepsilon_1 \in V_{\bar{0}}$ be an eigenvector of $\mathcal{A}$ belonging to an eigenvalue λ_0 and $\varepsilon_2 \in V_{\bar{1}}$ be an eigenvector of $\mathcal{A}$ belonging to an eigenvalue λ_0, then $\varepsilon_3 = (\mathcal{A} - \lambda_2 \mathrm{id})\varepsilon_2 \neq 0$ and

$$M(\mathcal{A}; \varepsilon_1, \varepsilon_2, \varepsilon_3) = \begin{pmatrix} \lambda_0 & 0 & 0 \\ 0 & \lambda_0 & 0 \\ 0 & 1 & \lambda_0 \end{pmatrix}.$$

2. (1) Let ε_i be an eigenvector of $\mathcal{A}$ belonging to an eigenvalue λ_i, then $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ is a basis of $V = V_{\bar{0}} \oplus V_{\bar{1}}$ and $M(A_{\bar{1}}; \varepsilon_1, \varepsilon_2, \varepsilon_3) = diag(\lambda_1, -\lambda_1, 0)$.

(2) When $\mathcal{A} \in (EndV)_{\bar{1}}$ and $\varepsilon_1, \varepsilon_2, \varepsilon_3$ be a basis of $V_{\bar{0}}$. According to the definition $\mathcal{A}(V_{\bar{0}}) \subseteq V_{\bar{1}}$, we have the matrix of $\mathcal{A}$ with respect to any basis is zero matrix. □

Theorem 2.7 *Let* $\dim V_{\bar{1}} = 3$*, we have*

Case 1 *If* $\mathcal{A} \in \mathrm{End}(V_{\bar{0}})$*, then the matrix of* $\mathcal{A}$ *with respect to any basis is zero matrix.*

Case 2 *If* $\mathcal{A} \in \mathrm{End}(V_{\bar{1}})$*, then we have*

(1) $f(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$*, where* $\lambda_1, \lambda_2, \lambda_3$ *is not equal to each other, then the matrix of* $\mathcal{A}$ *with respect to this basis is* $diag(\lambda_1, \lambda_2, \lambda_3)$*.*

(2) $f(\lambda) = (\lambda - \lambda_1)^2(\lambda - \lambda_2)$*, then the matrix of* $\mathcal{A}$ *with respect to this basis is*

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \begin{pmatrix} \lambda_1 & 0 & 0 \\ 1 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$$

(3) $f(\lambda) = (\lambda - \lambda_0)^3$*, if* $A_{\bar{1}} - \lambda_0 \mathrm{id} = 0$*, then the matrix of* $\mathcal{A}$ *with respect to this basis is*

$$\begin{pmatrix} \lambda_0 & 0 & 0 \\ 0 & \lambda_0 & 0 \\ 0 & 0 & \lambda_0 \end{pmatrix},$$

if $A_{\bar{1}} - \lambda_0 \mathrm{id} \neq 0$ *and* $(A_{\bar{1}} - \lambda_0 \mathrm{id})^2 = 0$*, then the matrix of* $\mathcal{A}$ *with respect to this basis is*

$$\begin{pmatrix} \lambda_0 & 0 & 0 \\ 1 & \lambda_0 & 0 \\ 0 & 0 & \lambda_0 \end{pmatrix},$$

if $(A_{\bar{1}} - \lambda_0 \mathrm{id})^2 \neq 0$ *and* $(A_{\bar{1}} - \lambda_0 \mathrm{id})^3 = 0$*, then the matrix of* $\mathcal{A}$ *with respect to this basis is*

$$\begin{pmatrix} \lambda_0 & 0 & 0 \\ 1 & \lambda_0 & 0 \\ 0 & 1 & \lambda_0 \end{pmatrix}.$$

Proof. 1. When $\mathcal{A} \in (EndV)_{\bar{0}}$ and $\varepsilon_1, \varepsilon_2, \varepsilon_3$ be a basis of $V_{\bar{1}}$, then according to the definition we have $\mathcal{A}(V_{\bar{1}}) \subseteq V_{\bar{0}}$. Hence, the matrix of $\mathcal{A}$ with respect to any basis is zero matrix.

2. (1) Let $\mathcal{A} \in (EndV)_{\bar{1}}$ and ε_i be an eigenvector of $\mathcal{A}$ belonging to an eigenvalue λ_i, then $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ is a basis of $V_{\bar{1}}$ and $M(A_{\bar{1}}; \varepsilon_1, \varepsilon_2, \varepsilon_3) = diag(\lambda_1, \lambda_2, \lambda_3)$.

(2) If $f(\lambda) = (\lambda - \lambda_1)^2(\lambda - \lambda_2)$, then the root space decomposition of $V_{\bar{1}}$ is $V_{\bar{1}} = R_{\lambda_1}(A_{\bar{1}}) \oplus R_{\lambda_2}(A_{\bar{1}})$, where $\dim R_{\lambda_1}(A_{\bar{1}}) = 2$ and $\dim R_{\lambda_2}(A_{\bar{1}}) = 1$.

If $(\mathcal{A} - \lambda_1 \mathrm{id})^2 |_{R_{\lambda_1}(\mathcal{A})} = 0$ and $\{\varepsilon_1, \varepsilon_2\}$ is a basis of $R_{\lambda_1}(\mathcal{A})$, then take ε_3 be an eigenvector of $R_{\lambda_2}(\mathcal{A})$ belonging to an eigenvalue λ_2. So $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ is a basis of $V_{\bar{1}}$ and $M(\mathcal{A}; \varepsilon_1, \varepsilon_2, \varepsilon_3) = diag(\lambda_1, \lambda_1, \lambda_2)$.

If $(\mathcal{A} - \lambda_0 \mathrm{id})^2 |_{R_{\lambda_1}(\mathcal{A})} \neq 0$, then take $\varepsilon_1 \in R_{\lambda_1}(\mathcal{A})$ such that $\varepsilon_2 = (\mathcal{A} - \lambda_1 \mathrm{id})\varepsilon_1 \neq 0$. Let ε_3 be an eigenvector of $\mathcal{A}$ belonging to an eigenvalue λ_2, then

$\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ is a basis of $V_{\bar{1}}$ and

$$M(\mathcal{A}; \varepsilon_1, \varepsilon_2, \varepsilon_3) = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 1 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}.$$

(3) If $\mathcal{A} - \lambda_0 \mathrm{id} = 0$, then the matrix of $\mathcal{A}$ with respect to this basis is $\lambda_0 I_3$.

If $(\mathcal{A}-\lambda_0\mathrm{id})\mid_{V_{\bar{1}}} \neq 0$ and $(\mathcal{A}-\lambda_0\mathrm{id})^2 = 0$, then we need to prove $\mathrm{dim}E_{\lambda_0}(\mathcal{A}) = 2$. $\mathrm{dim}E_{\lambda_0}(\mathcal{A}) \leq 2$ is straight-forward. If $\mathrm{dim}E_{\lambda_0}(\mathcal{A}) = 1$, let $\{\beta_1, \beta_2, \beta_3\}$ be a basis of $E_{\lambda_0}(\mathcal{A})$, then take $\beta_3 \in E_{\lambda_0}(\mathcal{A})$. So we have $(\mathcal{A} - \lambda_0\mathrm{id})\beta_1 = k\beta_3 \neq 0, (\mathcal{A} - \lambda_0\mathrm{id})\beta_2 = l\beta_3 \neq 0$. But $(\mathcal{A} - \lambda_0\mathrm{id})(l\beta_1 - k\beta_2) = 0$, i.e., $(l\beta_1 - k\beta_2) \in E_{\lambda_0}(\mathcal{A}) = L(\beta_3)$, this is a contradiction.

Let $\varepsilon_1 \in V_{\bar{1}}$ such that $\varepsilon_2 = (\mathcal{A} - \lambda_1\mathrm{id})\varepsilon_1 \neq 0$, then $\varepsilon_2 \in E_{\lambda_0}(\mathcal{A})$. Take $\varepsilon_3 \in E_{\lambda_0}(\mathcal{A})$ such that $\{\varepsilon_2, \varepsilon_3\}$ is a basis of $E_{\lambda_0}(\mathcal{A})$. If $k_1\varepsilon_1 + k_2\varepsilon_2 + k_3\varepsilon_3 = 0$ and $\mathcal{A} - \lambda_1\mathrm{id} = 0$, then $k_1\varepsilon_2 = 0$, so $k_1 = 0$. Let $\{\varepsilon_2, \varepsilon_3\}$ be a basis of $E_{\lambda_0}(\mathcal{A})$, then $k_2 = k_3 = 0$. Hence, $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ is a basis of $V_{\bar{1}}$ and

$$M(\mathcal{A}; \varepsilon_1, \varepsilon_2, \varepsilon_3) = \begin{pmatrix} \lambda_0 & 0 & 0 \\ 1 & \lambda_0 & 0 \\ 0 & 0 & \lambda_0 \end{pmatrix}.$$

If $(\mathcal{A}-\lambda_0\mathrm{id})^2 \neq 0$, then we can take $\varepsilon_1(\in V_{\bar{1}})$ such that $\varepsilon_3 = (\mathcal{A}-\lambda_1\mathrm{id})^2\varepsilon_1 \neq 0$. So $\varepsilon_2 = (\mathcal{A} - \lambda_1\mathrm{id})\varepsilon_1 \neq 0$.

If $k_1\varepsilon_1 + k_2\varepsilon_2 + k_3\varepsilon_3 = 0$, then $(\mathcal{A}-\lambda_1\mathrm{id})^2 k_1\varepsilon_1 = 0$, i.e., $k_1\varepsilon_3 = 0$, so $k_1 = 0$. Then $(\mathcal{A} - \lambda_1\mathrm{id})(k_2\varepsilon_2 + k_3\varepsilon_3) = k_2\varepsilon_3 = 0$ and $k_3\varepsilon_3 = 0$, so $k_3 = 0$. Hence, $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ is a basis of $V_{\bar{1}}$ and

$$M(\mathcal{A}; \varepsilon_1, \varepsilon_2, \varepsilon_3) = \begin{pmatrix} \lambda_0 & 0 & 0 \\ 1 & \lambda_0 & 0 \\ 0 & 1 & \lambda_0 \end{pmatrix}.$$

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