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A Riemannian Mean Inequality

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Abstract

In this paper, we shall show a Riemannian mean inequality from grand Furuta inequality.

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1 Introduction

A capital letter (such as T) stands for a $m \times m$ matrix on $\mathbb{C}$.

In 2005, R. Bhatia et al defined the weighted Riemannian mean for positive definite matrice as follows,

Definition 1.1([6, 9]). For $A_1, A_2, \cdots, A_n > 0$. If $\omega = (w_1, w_2, \cdots, w_n)$ is a probability vector, then the weighted Reimannian Mean $\mathscr{R}_\delta(\omega; A_1, A_2, \cdots, A_n)$ is defined as

$$\mathscr{R}_\delta(\omega; A_1, A_2, \cdots, A_n) = \underset{X>0}{arg\ min} \sum_{i=1}^{n} w_i \delta_2^2(A_i, X),$$

where $arg\ min f(X)$ stands for the point X_0 which $f(X_0) \le f(X)$, and $\delta_2(S,T) = \| \log S^{-\frac{1}{2}} T S^{-\frac{1}{2}} \|_2$.

Recently, M. Ito[8] obtained a Riemannian mean inequalitiy from grand Furuta inequality. As a continuation, we shall show another Riemannian mean inequality which is different from Ito's result.

Thoerem 1.1 (Grand Furuta Inequality, [7]) If $A \ge B \ge 0$ with $A > 0$, then $A^{1-t+r} \ge [A^{\frac{r}{2}}(A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^s A^{\frac{r}{2}}]^{\frac{1-t+r}{(p-t)s+r}}$ holds for p, $s \ge 1$, $t \in [0,1]$ and $r \ge t$.

Theorem 1.2 ([10]) For $A_1, A_2, \cdots, A_n > 0$ and a probability vector $\omega = (w_1, w_2, \cdots, w_n)$, if $w_1 \log A_1 + w_2 \log A_2 + \cdots + w_n \log A_n \le 0$, then $\mathscr{R}_\delta(\omega; A_1, A_2, \cdots, A_n) \le I$.

2 Main Result

Theorem 2.1. For $A_1, A_2, \cdots, A_n > 0$, if $A_i \geq A_n$ holds for each $i = 1, 2, \cdots, n-1$, $t_i \in [0,1], p_i, s_i \geq 1$ and $r_i \geq t_i$ hold for each $i = 1, 2, \cdots, n$, then

$$\begin{aligned}\mathscr{R}_\delta\Big(\omega; &A_n^{-\frac{r_1-t_1}{2}}\left(A_n^{-t_1}\natural_{s_1}A_1^{-p_1}\right)A_n^{-\frac{r_1-t_1}{2}}, A_n^{-\frac{r_2-t_2}{2}}\left(A_n^{-t_2}\natural_{s_2}A_2^{-p_2}\right)A_n^{-\frac{r_2-t_2}{2}},\\ &\cdots\ \cdots, A_n^{-\frac{r_i-t_i}{2}}\left(A_n^{-t_i}\natural_{s_i}A_i^{-p_i}\right)A_n^{-\frac{r_i-t_i}{2}}, \cdots\ \cdots,\\ &A_n^{-\frac{r_{n-1}-t_{n-1}}{2}}\left(A_n^{-t_{n-1}}\natural_{s_{n-1}}A_{n-1}^{-p_{n-1}}\right)A_n^{-\frac{r_{n-1}-t_{n-1}}{2}}, A_n\Big) \leq I,\end{aligned}$$

where $\omega = \frac{\bar{\omega}}{\|\bar{\omega}\|_1}$ and $\bar{\omega} = (\frac{1}{(p_1-t_1)s_1+r_1}, \frac{1}{(p_2-t_2)s_2+r_2}, \cdots, \frac{1}{(p_{n-1}-t_{n-1})s_{n-1}+r_{n-1}}, n-1)$.

Proof. Applying $A_n^{-1} \geq A_1^{-1} > 0$ to grand Furuta inequality, and taking reverse, we have

$$\left\{A_n^{\frac{r_1}{2}}\left(A_n^{-\frac{t_1}{2}}A_1^{p_1}A_n^{-\frac{t_1}{2}}\right)^{s_1}A_n^{\frac{r_1}{2}}\right\}^{\frac{1-t_1+r_1}{(p_1-t_1)s_1+r_1}} \geq A_n^{1-t_1+r_1}. \tag{2.1}$$

Taking logarithm to both side of (2.1), and deleting $1 - t_1 + r_1$, we can obtain

$$\frac{1}{(p_1-t_1)s_1+r_1}\log\left\{A_n^{\frac{r_1}{2}}\left(A_n^{-\frac{t_1}{2}}A_1^{p_1}A_n^{-\frac{t_1}{2}}\right)^{s_1}A_n^{\frac{r_1}{2}}\right\} \geq \log A_n. \tag{2.2}$$

It follows that

$$\frac{1}{(p_1-t_1)s_1+r_1}\log\left\{A_n^{-\frac{r_1}{2}}\left(A_n^{\frac{t_1}{2}}A_1^{-p_1}A_n^{\frac{t_1}{2}}\right)^{s_1}A_n^{-\frac{r_1}{2}}\right\} + \log A_n \leq 0. \tag{2.3}$$

By the same way, for each $i = 2, 3, \cdots, n-1$, the follow inequality holds.

$$\frac{1}{(p_i-t_i)s_i+r_i}\log\left\{A_n^{-\frac{r_i}{2}}\left(A_n^{\frac{t_i}{2}}A_i^{-p_i}A_n^{\frac{t_i}{2}}\right)^{s_i}A_n^{-\frac{r_i}{2}}\right\} + \log A_n \leq 0. \tag{2.4}$$

Summing up (2.3) and (2.4) for $i = 2, 3, \cdots, n-1$, we have

$$\sum_{i=1}^{n-1}\frac{1}{(p_i-t_i)s_i+r_i}\log\left\{A_n^{-\frac{r_i}{2}}\left(A_n^{\frac{t_i}{2}}A_i^{-p_i}A_n^{\frac{t_i}{2}}\right)^{s_i}A_n^{-\frac{r_i}{2}}\right\} + (n-1)\log A_n \leq 0. \tag{2.5}$$

For $i = 1, 2, \cdots, n-1$, let $w_i = \frac{\frac{1}{(p_i-t_i)s_i+r_i}}{\sum_{i=1}^{n-1}\frac{1}{(p_i-t_i)s_i+r_i}+(n-1)}$, and let $w_n = \frac{n-1}{\sum_{i=1}^{n-1}\frac{1}{(p_i-t_i)s_i+r_i}+(n-1)}$. By (2.5), the follows inequality holds.

$$\begin{aligned}w_1\log A_n^{-\frac{r_1-t_1}{2}}&\left(A_n^{-t_1}\natural_{s_1}A_1^{-p_1}\right)A_n^{-\frac{r_1-t_1}{2}} + w_2\log A_n^{-\frac{r_2-t_2}{2}}\left(A_n^{-t_2}\natural_{s_2}A_2^{-p_2}\right)A_n^{-\frac{r_2-t_2}{2}}\\ +\cdots\ \cdots + w_i\log A_n^{-\frac{r_i-t_i}{2}}&\left(A_n^{-t_i}\natural_{s_i}A_i^{-p_i}\right)A_n^{-\frac{r_i-t_i}{2}} + \cdots\ \cdots\\ +w_{n-1}\log A_n^{-\frac{r_{n-1}-t_{n-1}}{2}}&\left(A_n^{-t_{n-1}}\natural_{s_{n-1}}A_{n-1}^{-p_{n-1}}\right)A_n^{-\frac{r_{n-1}-t_{n-1}}{2}} + w_n\log A_n \leq 0.\end{aligned}$$

Applying Theorem 1.2 to the inequality above, we complete the proof. □

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