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On (α, β, γ)-derivations of Lie color algebras

Bing Sun

School of Mathematics and Statistics
Northeast Normal University

Abstract

This paper is primarily concerned with (α, β, γ)-derivations of finite dimensional Lie color algebras over the field of complex numbers. Some properties of (α, β, γ)-derivations of the Lie color algebras are obtained. In particular, an example for (α, β, γ)-derivations of low dimensional non-simple Lie color algebras are presented.

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1 Introduction

As a natural generalization of Lie algebras and Lie superalgebras [3], Lie color algebras play an important role in theoretical physics [4]. Ree introduced generalized Lie algebras, which are called Lie color algebras today [6]. In recent years, Lie color algebras have become an interesting subject of mathematics [2, 6]. The search for a new concept of invariant characteristics of Lie algebras led to the definition of (α, β, γ)-derivations in [5]. The aim of this paper is to partially generalize some beautiful results about (α, β, γ)-derivations in [5, 7].

This paper is organized as follows. we introduce (α, β, γ)-derivations and show their pertinent properties. In particular, an example for (α, β, γ)-derivations of low dimensional non-simple Lie color algebras are presented.

In [1] the readers could find all notations and notions of Lie color algebras which are not precisely defined in this paper.

2 Main Results

Definition 2.1 *A linear transformation $A \in \mathrm{Pl}_\theta(L)$ is called an (α, β, γ)-*derivation *of degree θ if there exist $\alpha, \beta, \gamma \in \mathbb{C}$ such that*

$$\alpha A([x, y]) = \beta[A(x), y] + \varepsilon(\theta, x)\gamma[x, A(y)],$$

for all $x, y \in \mathrm{hg}(L)$. For given $\alpha, \beta, \gamma \in \mathbb{C}$, the set of all (α, β, γ)-derivations of degree θ is denoted by $\mathfrak{D}(\alpha, \beta, \gamma)_\theta$, i.e. $\mathfrak{D}(\alpha, \beta, \gamma)_\theta$ is equal to the set

$$\{A \in \mathrm{Pl}_\theta(L)\,|\alpha A([x, y]) = \beta[A(x), y] + \varepsilon(\theta, x)\gamma[x, A(y)], \forall x, y \in \mathrm{hg}(L)\}.$$

Denote by $\mathfrak{D}(\alpha, \beta, \gamma) = \bigoplus_{\theta \in G} \mathfrak{D}(\alpha, \beta, \gamma)_\theta$ the set of all (α, β, γ)-derivations of L.

In particular, $\mathfrak{D}(\alpha, \beta, \gamma)$ coincides with the centroid if $\alpha = \beta$ and $\gamma = 0$ (or $\alpha = \gamma$ and $\beta = 0$). Therefore, (α, β, γ)-derivations are the natural generalization of centroids.

Proposition 2.2 *For any $\alpha, \beta, \gamma \in \mathbb{C}$ and $k \in \mathbb{C}\backslash\{0\}$,*

$$\mathfrak{D}(\alpha, \beta, \gamma) = \mathfrak{D}(\alpha k, \beta k, \gamma k) = \mathfrak{D}(\alpha, \gamma, \beta). \tag{1}$$

Proof. In fact, it is sufficient to check the homogeneous elements of $\mathfrak{D}(\alpha, \beta, \gamma)$. For any $x, y \in \mathrm{hg}(L)$, we have

$$\begin{aligned} & A \in \mathfrak{D}(\alpha, \beta, \gamma)_\theta \Leftrightarrow \alpha A([x, y]) = \beta[A(x), y] + \varepsilon(\theta, x)\gamma[x, A(y)] \\ \Leftrightarrow \quad & \alpha k A([x, y]) = \beta k[A(x), y] + \varepsilon(\theta, x)\gamma k[x, A(y)] \Leftrightarrow A \in \mathfrak{D}(\alpha k, \beta k, \gamma k)_\theta \end{aligned}$$

and

$$\begin{aligned} & A \in \mathfrak{D}(\alpha, \beta, \gamma)_\theta \Leftrightarrow \alpha A([x, y]) = \beta[A(x), y] + \varepsilon(\theta, x)\gamma[x, A(y)] \\ \Leftrightarrow \quad & -\varepsilon(x, y)\alpha A([y, x]) = -\varepsilon(\theta + x, y)\beta[y, A(x)] - \varepsilon(\theta, x)\varepsilon(x, \theta + y)\gamma[A(y), x] \\ \Leftrightarrow \quad & \alpha A([y, x]) = \gamma[A(y), x] + \varepsilon(\theta, y)\beta[y, A(x)] \Leftrightarrow A \in \mathfrak{D}(\alpha, \gamma, \beta)_\theta. \end{aligned}$$

Thus (1) holds. □

Lemma 2.3 *For any $\alpha, \beta, \gamma \in \mathbb{C}$,*

$$\mathfrak{D}(\alpha, \beta, \gamma) = \mathfrak{D}(0, \beta - \gamma, \gamma - \beta) \cap \mathfrak{D}(2\alpha, \gamma + \beta, \gamma + \beta).$$

Proof. Suppose that $A \in \mathfrak{D}(\alpha, \beta, \gamma)_\theta$, for given $\alpha, \beta, \gamma \in \mathbb{C}$ and arbitrary $x, y \in \mathrm{hg}(L)$, then we have

$$\alpha A([x, y]) = \beta[A(x), y] + \varepsilon(\theta, x)\gamma[x, A(y)], \tag{2}$$
$$\alpha A([y, x]) = \beta[A(y), x] + \varepsilon(\theta, y)\gamma[y, A(x)]. \tag{3}$$

By subtracting Eq. (3) from Eq. (2), we obtain

$$0 = (\beta - \gamma)\left([A(x), y] - \varepsilon(\theta, x)[x, A(y)]\right). \tag{4}$$

By suming Eqs. (2) and (3), we have

$$2\alpha A([y, x]) = (\beta + \gamma)\left([A(y), x] + \varepsilon(\theta, y)[y, A(x)]\right). \tag{5}$$

Thus $\mathfrak{D}(\alpha, \beta, \gamma) \subseteq \mathfrak{D}(0, \beta - \gamma, \gamma - \beta) \cap \mathfrak{D}(2\alpha, \beta + \gamma, \beta + \gamma)$.

Conversely, Eq. (2) can also be obtained by Eqs. (4) and (5). Therefore, the proof is completed. □

Theorem 2.4 *For any $\alpha,\beta,\gamma\in\mathbb{C}$, there exists $\delta\in\mathbb{C}$ such that the space $\mathfrak{D}(\alpha,\beta,\gamma)$ is equal to one of the four following spaces:*
1. $\mathfrak{D}(\delta,0,0)$, 2. $\mathfrak{D}(\delta,1,-1)$, 3. $\mathfrak{D}(\delta,1,0)$, 4. $\mathfrak{D}(\delta,1,1)$.

Proof.

1. Suppose that $\beta+\gamma=0$. Then either $\beta=\gamma=0$ or $\beta=-\gamma\neq 0$.

   (i) For $\beta=\gamma=0$, we can easily obtain $\mathfrak{D}(\alpha,\beta,\gamma)=\mathfrak{D}(\alpha,0,0)$.

   (ii) For $\beta=-\gamma\neq 0$, it follows from Eq. (1) and Lemma 2.3 that

$$\begin{aligned}\mathfrak{D}(\alpha,\beta,\gamma) &= \mathfrak{D}(0,\beta-\gamma,\gamma-\beta)\cap\mathfrak{D}(2\alpha,0,0)\\ &= \mathfrak{D}(0,1,-1)\cap\mathfrak{D}(\alpha,0,0).\end{aligned}$$

   On the other hand, it shows that

$$\mathfrak{D}(\alpha,1,-1)=\mathfrak{D}(0,2,-2)\cap\mathfrak{D}(2\alpha,0,0)=\mathfrak{D}(0,1,-1)\cap\mathfrak{D}(\alpha,0,0).$$

   Therefore, we have $\mathfrak{D}(\alpha,\beta,\gamma)=\mathfrak{D}(\alpha,1,-1)$.

2. Suppose that $\beta+\gamma\neq 0$. Then either $\beta-\gamma\neq 0$ or $\beta=\gamma\neq 0$.

   (i) For $\beta-\gamma\neq 0$, it also follows from Eq. (1) and Lemma 2.3 that

$$\begin{aligned}\mathfrak{D}(\alpha,\beta,\gamma) &= \mathfrak{D}(0,\beta-\gamma,\gamma-\beta)\cap\mathfrak{D}(2\alpha,\beta+\gamma,\beta+\gamma)\\ &= \mathfrak{D}(0,1,-1)\cap\mathfrak{D}(\frac{2\alpha}{\beta+\gamma},1,1).\end{aligned}$$

   According to Lemma 2.3, we have

$$\mathfrak{D}(\frac{\alpha}{\beta+\gamma},1,0)=\mathfrak{D}(0,1,-1)\cap\mathfrak{D}(\frac{2\alpha}{\beta+\gamma},1,1).$$

   Then we have $\mathfrak{D}(\alpha,\beta,\gamma)=\mathfrak{D}(\frac{\alpha}{\beta+\gamma},1,0)$.

   (ii) For $\beta=\gamma\neq 0$, it easily shows $\mathfrak{D}(\alpha,\beta,\gamma)=\mathfrak{D}(\frac{\alpha}{\beta},1,1)$.

In conclusion, the proof is completed. □

Next we will discuss in detail the possible (α,β,γ)-derivations of L which only depends on the value of the parameter $\delta\in\mathbb{C}$.

1. $\mathfrak{D}(\delta,0,0)$:

   (i) For $\delta=0$, it is clear to show that $\mathfrak{D}(0,0,0)=\mathrm{Pl}(L)$.

   (ii) For $\delta\neq 0$, the space $\mathfrak{D}(\delta,0,0)$ sends derived algebras to the zero vector:

$$\mathfrak{D}(\delta,0,0)=\left\{A\in\mathrm{Pl}(L)\,\middle|\,A(L^2)=0\right\}.$$

   Clearly, its dimension is as follows:

$$\dim(\mathfrak{D}(\delta,0,0))=\dim(L/L^2)\dim(L).$$

2. $\mathfrak{D}(\delta,1,-1)$:

(i) For $\delta=0$, The space of $\mathfrak{D}(0,1,-1)$ is given by

$$\begin{aligned}\mathfrak{D}(0,1,-1) &= \bigoplus_{\theta\in G}\{A\in \mathrm{Pl}_\theta(L)\,|[A(x),y]=\varepsilon(\theta,x)[x,A(y)],\forall x,y\in \mathrm{hg}(L)\}\\ &= \mathrm{QC}(L).\end{aligned}$$

(ii) $\delta\neq 0$, we obtain a algebra $\mathfrak{D}(1,1,-1)$ as an intersection of two algebras:

$$\begin{aligned}\mathfrak{D}(\delta,1,-1) &= \mathfrak{D}(0,2,-2)\cap\mathfrak{D}(2\delta,0,0)\\ &= \mathfrak{D}(0,2,-2)\cap\mathfrak{D}(2,0,0)\\ &= \mathfrak{D}(1,1,-1).\end{aligned}$$

3. $\mathfrak{D}(\delta,1,0)$:

(i) For $\delta=0$, we sends the whole L into its center $\mathcal{Z}(L)$:

$$\mathfrak{D}(0,1,0)=\{A\in \mathrm{Pl}(L)\,|A(L)\subseteq\mathcal{Z}(L)\}$$

and therefore its dimension is as follows:

$$\dim(\mathfrak{D}(0,1,0))=\dim(\mathcal{Z}(L))\dim(L).$$

(ii) For $\delta=1$, the space $\mathfrak{D}(1,1,0)$ is given by

$$\mathfrak{D}(1,1,0)=\bigoplus_{\theta\in G}\{A\in \mathrm{Pl}_\theta(L)\,|A[x,y]=[A(x),y],\forall x,y\in \mathrm{hg}(L)\}.$$

Note that $A\in \mathrm{Pl}_\theta(L)$ also satisfies

$$A[x,y]=-\varepsilon(x,y)A[y,x]=-\varepsilon(x,y)[A(y),x]=\varepsilon(\theta,x)[x,A(y)].$$

Hence

$$\begin{aligned}\mathfrak{D}(1,1,0) &= \bigoplus_{\theta\in G}\{A\in \mathrm{Pl}_\theta(L)\,|A\circ \mathrm{ad}(x)=\varepsilon(\theta,x)\mathrm{ad}(x)\circ A,\forall x\in \mathrm{hg}(L)\}\\ &= \mathcal{C}(L).\end{aligned}$$

(iii) For the remaining values of δ and the general case of Lie color algebra L, the space $\mathfrak{D}(\delta,1,0)$ is only the vector subspace of $\mathrm{Pl}(L)$. Thus, we have the one-parametric set of vector spaces:

$$\mathfrak{D}(\delta,1,0)=\mathfrak{D}(0,1,-1)\cap\mathfrak{D}(2\delta,1,1).$$

4. $\mathfrak{D}(\delta,1,1)$:

   (i) For $\delta=0$, we have a Lie color algebra $\mathfrak{D}(0,1,1)$ which equals to the set

$$\bigoplus_{\theta\in G}\{A\in \mathrm{Pl}_\theta(L)\,|[A(x),y]=-\varepsilon(\theta,x)[x,A(y)],\forall x,y\in \mathrm{hg}(L)\}.$$

   (ii) For $\delta=1$, the space $\mathfrak{D}(1,1,1)$ is just the derivation algebra of L in the ordinary sense, i.e. $\mathfrak{D}(1,1,1)=\mathrm{Der}(L)$.

   (iii) For the remaining values of δ, the space $\mathfrak{D}(\delta,1,1)$ is only the vector subspace of $\mathrm{Pl}(L)$ in the general case of Lie color algebra L.

According to the discussions above, we immediately obtain the following proposition.

Proposition 2.5 *Let L be a complex Lie color algebra and $\alpha,\beta,\gamma\in\mathbb{C}$. Then $\mathfrak{D}(\alpha,\beta,\gamma)$ equals one of the following subspaces of* $\mathrm{Pl}(L)$*:*
(1) $\mathfrak{D}(0,0,0)=\mathrm{Pl}(L)$,
(2) $\mathfrak{D}(1,0,0)=\{A\in\mathrm{End}(L)\,|A(L^2)=0\}$,
(3) $\mathfrak{D}(0,1,-1)=\mathrm{QC}(L)$,
(4) $\mathfrak{D}(1,1,-1)=\mathfrak{D}(0,1,-1)\cap\mathfrak{D}(1,0,0)$,
(5) $\mathfrak{D}(\delta,1,1)$, $\delta\in\mathbb{C}$,
(6) $\mathfrak{D}(\delta,1,0)=\mathfrak{D}(0,1,-1)\cap\mathfrak{D}(2\delta,1,1)$, $\delta\in\mathbb{C}$.

Example 2.6 *Consider a non-abelian two-dimensional Lie color algebra L_2 with a basis $\{e_1,e_2\}$ and its only non-zero commutation relation is $[e_1,e_2]=e_2$, where $\varepsilon(\phi,e_1)=\varepsilon(e_1,e_1)=\varepsilon(e_2,e_2)=1$. Then all (α,β,γ)-derivations of L_2 are given as follows:*

- $\mathfrak{D}(1,1,1)=\mathrm{span}_{\mathbb{C}}\left\{\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}\cong L_2.$
- $\mathfrak{D}(0,1,1)=\mathrm{span}_{\mathbb{C}}\left\{\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right\}.$
- $\mathfrak{D}(1,1,0)=\mathrm{span}_{\mathbb{C}}\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}\right\}.$
- $\mathfrak{D}(0,1,-1)=\mathrm{span}_{\mathbb{C}}\left\{\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&1\end{pmatrix}\right\}.$
- $\mathfrak{D}(1,0,0)\cap\mathfrak{D}(0,1,1)=\mathrm{span}_{\mathbb{C}}\left\{\begin{pmatrix}0&0\\1&0\end{pmatrix}\right\}.$
- $\mathfrak{D}(\delta,1,0)=\{0\}$ *for* $\delta\neq1$.

- $\mathfrak{D}(\delta,1,1)=\mathrm{span}_{\mathbb{C}}\left\{\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}\delta-1&0\\0&1\end{pmatrix}\right\}\ for\ \delta\neq 0.$
- $\mathfrak{D}(0,1,0)=\mathfrak{D}(1,0,0)\cap\mathfrak{D}(0,1,0)=\{0\}.$
- $\mathfrak{D}(1,0,0)=\mathrm{span}_{\mathbb{C}}\left\{\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix}\right\}\cong L_2.$
- $\mathfrak{D}(1,1,-1)=\mathrm{span}_{\mathbb{C}}\left\{\begin{pmatrix}0&0\\1&0\end{pmatrix}\right\}.$

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