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A Different Proof of the Beta Function

Charng-Yih Yu
Department of Applied Statistics and Information Science
Ming Chuan University, Taipei, Taiwan
cyyu@mail.mcu.edu.tw

ABSTRACT: This article will show a particular proof for beta function by using Laplace transform and convolution formula rather than probability theory. It is possible that this idea and method could be applied to deal with other problems.

KEYWORDS: Beta function; Laplace transform; Convolution formula; Gamma function

1. INTRODUCTION

Because the beta function $\int_0^1 (1-x)^{\alpha-1} x^{\beta-1} dx = \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ is related to beta distribution or gamma distribution, thus the conventional proof of beta function is dependent on the probability theory as referred to [1], [2] and [3]. It is more convenient to use Laplace transform as opposed to the probability theory in derivation process. The proof can be sketched by following steps: (1) convert the product $\Gamma(\alpha)\Gamma(\beta)$ to the product of Laplace transforms, (2) apply the convolution formula as referred to [4] and take inverse Laplace transform , (3) use the change of variable for convolution, (4) apply Laplace transform to obtain beta function. The details of the proof will be shown in the next section. There are some necessary definitions for the proof.

Definition 1. If the integral $\int_0^\infty f(x)e^{-sx}dx$ exists, then it is called the Laplace of $f(x)$ and denoted by $L[f(x)]$.

Definition 2. If the integral $\int_0^\infty x^{\alpha-1}e^{-x}dx$ exists, then it is called gamma function and denoted by $\Gamma(\alpha)$.

Definition 3. The formula $L\left[\int_0^x f(x-y)g(y)dy\right]=L[f(x)]L[g(x)]$ is called the convolution of $f(x)$ and $g(x)$. Please refer to the reference [4] for the proof.

2. MAIN RESULTS

First to show the relationship between the gamma function and the Laplace transform.

Theorem 1. If $s>0$, then $L[x^{\alpha-1}]=s^{-\alpha}\Gamma(\alpha)$. (1)

Proof. To establish (1), we use the change of variable to show $L[x^{\alpha-1}]$ in terms of $\Gamma(\alpha)$ as follows:

$$L[x^{\alpha-1}]=\int_0^\infty x^{\alpha-1}e^{-sx}dx$$

$$=s^{-\alpha}\int_0^\infty y^{\alpha-1}e^{-y}dy$$

$$=s^{-\alpha}\Gamma(\alpha).$$

Q.E.D.

Second, using convolution formula to show the different proof of the beta function.

Theorem 2. If $\alpha>0$, $\beta>0$, then $\int_0^1 (1-x)^{\alpha-1}x^{\beta-1}dx=\dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

Proof. By (1), we obtain that

$$\Gamma(\alpha)=s^{\alpha}L[x^{\alpha-1}] \text{ and } \Gamma(\beta)=s^{\beta}L[x^{\beta-1}].$$

Hence $\Gamma(\alpha)\Gamma(\beta) = s^{\alpha+\beta} L\left[x^{\alpha-1}\right] L\left[x^{\beta-1}\right],$

or $$\frac{\Gamma(\alpha)\Gamma(\beta)}{s^{\alpha+\beta}} = L\left[x^{\alpha-1}\right] L\left[x^{\beta-1}\right]. \qquad (2)$$

To apply (2) in the computation of inverse Laplace transforms, we rewrite it as

$$L^{-1}\left[\frac{\Gamma(\alpha)\Gamma(\beta)}{s^{\alpha+\beta}}\right] = L^{-1}\left[L\left[x^{\alpha-1}\right] L\left[x^{\beta-1}\right]\right].$$

By convolution formula and making the change of variable, $u = \frac{y}{x}$, we obtain

$$L^{-1}\left[\frac{\Gamma(\alpha)\Gamma(\beta)}{s^{\alpha+\beta}}\right] = \int_0^x (x-y)^{\alpha-1} y^{\beta-1} dy$$

$$= x^{\alpha+\beta-1} \int_0^1 (1-u)^{\alpha-1} u^{\beta-1} du .$$

We now apply the Laplace transform and obtain

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{s^{\alpha+\beta}} = \frac{\Gamma(\alpha+\beta)}{s^{\alpha+\beta}} \int_0^1 (1-u)^{\alpha-1} u^{\beta-1} du .$$

Hence $\displaystyle\int_0^1 (1-x)^{\alpha-1} x^{\beta-1} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$,

and the proof is complete. Q.E.D.

It is quite different from the conventional proof being dependent on the probability theory. This idea of the proof might be an elegant technique for other problems.

References

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