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Generalizations of Parallelogram law in Hilbert C*-modules

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Abstract

In this paper we present a new equality in the framework of Hilbert C*-modules. As a consequence, we get generalizations of parallelogram law in the Hilbert C*-module case.

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1 Introduction

Suppose that $B(\mathcal{H})$ denotes the algebra of all bounded linear operators on a separable complex Hilbert space $\mathcal{H}$. The classical parallelogram law state that

$$|a+b|^2 + |a-b|^2 = 2|a|^2 + 2|b|^2,$$

for $a, b \in \mathbb{C}$. There are several extensions of parallelogram law among them we could refer the interested reader to [1, 2, 4, 9].

Also several authors have presented generalizations of parallelogram law for operators on a Hilbert space.

M. Fujii and H. Zuo [3] showed that if A, B belong to the algebra $B(H)$ and $\lambda \neq 0$ then

$$|A-B|^2 + \frac{1}{\lambda}|\lambda A + B|^2 = (1+\lambda)|A|^2 + (1+\frac{1}{\lambda})|B|^2. \tag{1}$$

where $|C| = (C^*C)^{\frac{1}{2}}$ denotes the absolute value of $C \in B(H)$. Note that $A \geq 0$ means that $\langle Ax, x\rangle \geq 0$ for all $x \in H$, and $A \leq 0$ represents that $-A \geq 0$, and $A \geq B$ if A and B are self-adjoint operators and $A - B \geq 0$, for any $A, B \in B(H)$.

We generalized (1) in the Hilbert C*-module case.

2 Preliminaries

Let us recall some definitions and basic properties of C*-algebras and Hilbert C*-modules that we need in the rest of the parer. A Banach $*$-algebra $\mathcal{A}$ is called a C*-algebra if it satises $\|a^*a\|^2 = \|a\|$ for any $a \in \mathcal{A}$. An element a of a C*-algebra $\mathcal{A}$ is positive if there exists $b \in \mathcal{A}$ such that $a = b^*b$. We write $a \geq 0$ to mean that a is positive. The relation " $\leq$ " given by

$$a \leq b \text{ if and only if } b - a \text{ is positive}$$

defines a partial ordering on $\mathcal{A}$. Let $\mathcal{A}$ be a C*-algebra then the absolute value of a is defined by $|a| = (a^*a)^{\frac{1}{2}}$. For undefined notions and more details on C*-algebra theory, we refer to [6].

Let $\mathcal{A}$ be a C*-algebra and let $\mathcal{H}$ be a right $\mathcal{A}$-module. $\mathcal{H}$ is a pre-Hilbert $\mathcal{A}$-module if $\mathcal{H}$ is equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$ that possesses the following properties,
(i) $\langle u, u \rangle \geq 0$, for all $u \in \mathcal{H}$ and $\langle u, u \rangle = 0$ if and only if $u = 0$;
(ii) $\langle u, \alpha v + \beta w \rangle = \alpha \langle u, v \rangle + \beta \langle u, w \rangle$, for all $\alpha, \beta \in \mathbb{C}$ and $u, v, w \in \mathcal{H}$;
(iii) $\langle u, v \rangle = \langle v, u \rangle^*$, for all $u, v \in \mathcal{H}$;
(iv) $\langle u, va \rangle = \langle u, v \rangle a$, for all $a \in \mathcal{A}$ and $u, v \in \mathcal{H}$;
The action of $\mathcal{A}$ on $\mathcal{H}$ is $\mathbb{C}$- and $\mathcal{A}$-linear, i.e., $\mu(ua) = u(\mu a) = (\mu u)a$ for every $\mu \in \mathbb{C}$, $a \in \mathcal{A}$ and $u \in \mathcal{H}$. For $u \in \mathcal{H}$, we define $\|u\| = \|\langle u, u \rangle\|^{\frac{1}{2}}$. If $\mathcal{H}$ is complete with $\|.\|$, it is called a Hilbert $\mathcal{A}$-module or a Hilbert C*-module over $\mathcal{A}$.
The C*-algebra $\mathcal{A}$ itself can be recognized as a Hilbert $\mathcal{A}$-module with the inner product $\langle a, b \rangle = a^*b$ for any $a, b \in \mathcal{A}$. For a C*-algebra $\mathcal{A}$ the standard Hilbert $\mathcal{A}$-module $\ell^2(\mathcal{A})$ is defined by

$$\ell^2(\mathcal{A}) = \{\{a_j\}_{j\in\mathbb{N}} : \sum_{j\in\mathbb{N}} a_j^* a_j \ \ \textit{converges in } \mathcal{A}\}$$

with $\mathcal{A}$-inner product $\langle \{a_j\}_{j\in\mathbb{N}}, \{b_j\}_{j\in\mathbb{N}} \rangle = \sum_{j\in\mathbb{N}} a_j^* b_j$. Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert modules over C*-algebra $\mathcal{A}$. A map $T : \mathcal{H} \to \mathcal{K}$ is said to be adjointable if there exists a mapping $T^* : \mathcal{K} \to \mathcal{H}$ satisfying $\langle Tu, v \rangle = \langle u, T^*v \rangle$ where $u \in \mathcal{H}$ and $v \in \mathcal{K}$. The mapping T^* is called the adjoint of T.

For every u in Hilbert C*-module $\mathcal{H}$ we define the absolute value of u as the unique positive square root of $\langle u, u \rangle$, that is, $|u| = \langle u, u \rangle^{\frac{1}{2}}$. We refer the reader to [5, 7, 8] for more information on Hilbert C*-modules.

In the sequel we denote $\mathcal{H}$ and $\mathcal{K}$ as Hilbert modules over a unital C*-algebra $\mathcal{A}$ with unit e.

3 Some Identities

In this section we give some equalities to obtain some results.

Theorem 3.1. *Let $w, z, u, v \in \mathcal{H}$ and $a, b \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{R}$. Then we have*

$$\begin{aligned}\alpha\beta a|w+z|^2 + b|\beta u - \alpha v|^2 &= \beta(\alpha a|w|^2 + \beta b|u|^2) + \alpha(\beta a|z|^2 + \alpha b|v|^2)\\ &\quad + \alpha\beta\left[a(\langle w, z\rangle + \langle z, w\rangle) - b(\langle u, v\rangle + \langle v, u\rangle)\right]. \quad (2)\end{aligned}$$

Proof. For $w, z, u, v \in \mathcal{H}$ we have

$$\begin{aligned}|w+z|^2 &= \langle w+z, w+z\rangle\\ &= \langle w, w\rangle + \langle w, z\rangle + \langle z, w\rangle + \langle z, z\rangle\\ &= |w|^2 + \langle w, z\rangle + \langle z, w\rangle + |z|^2, \quad (3)\end{aligned}$$

Also,

$$\begin{aligned}|\beta u - \alpha v|^2 &= \langle \beta u - \alpha v, \beta u - \alpha v\rangle\\ &= \beta^2\langle u, u\rangle - \alpha\beta\langle u, v\rangle - \alpha\beta\langle v, u\rangle + \alpha^2\langle v, v\rangle\\ &= \beta^2|u|^2 - \alpha\beta\langle u, v\rangle - \alpha\beta\langle v, u\rangle + \alpha^2|v|^2, \quad (4)\end{aligned}$$

By using (3) and (4) we have

$$\begin{aligned}\alpha\beta a|w+z|^2 + b|\beta u - \alpha v|^2 &= \alpha\beta a|w|^2 + \alpha\beta a\langle w, z\rangle + \alpha\beta a\langle z, w\rangle + \alpha\beta a|z|^2\\ &\quad + \beta^2 b|u|^2 - \alpha\beta b\langle u, v\rangle - \alpha\beta b\langle v, u\rangle + \alpha^2 b|v|^2\\ &= \beta(\alpha a|w|^2 + \beta b|u|^2) + \alpha(\beta a|z|^2 + \alpha b|v|^2)\\ &\quad + \alpha\beta[a(\langle w, z\rangle + \langle z, w\rangle) - b(\langle u, v\rangle + \langle v, u\rangle)].\end{aligned}$$

Which complete the proof. □

Theorem 3.2. *Suppose that $w, z, u, v \in \mathcal{H}$ with $\langle w, z\rangle = \mu\langle u, v\rangle$ for nonzero $\mu \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$. Then*

$$\frac{\alpha\beta}{\mu}|w+z|^2 + |\beta u - \alpha v|^2 = \beta(\frac{\alpha}{\mu}|w|^2 + \beta|u|^2) + \alpha(\frac{\beta}{\mu}|z|^2 + \alpha|v|^2). \quad (5)$$

In particular when $\langle w, z\rangle = \langle u, v\rangle$ so we have

$$\alpha\beta|w+z|^2 + |\beta u - \alpha v|^2 = \beta(\alpha|w|^2 + \beta|u|^2) + \alpha(\beta|z|^2 + \alpha|v|^2). \quad (6)$$

Proof. Put $\mu a = b = e$ in (2) then by assumption we conclude (5). □

Applying formula (6) of Theorem 3.2 on the elements of the Hilbert C*-module $\ell^2(\mathcal{A})$ we obtain the following result.

Corollary 3.3. *Let $\{a_j\}_{j\in J}, \{b_j\}_{j\in J}, \{c_j\}_{j\in J}$ and $\{d_j\}_{j\in J} \in \ell^2(\mathcal{A})$ with $\sum_{j\in J} a_j^* b_j = \sum_{j\in J} c_j^* d_j$ and $\alpha, \beta \in \mathbb{R}$. Then*

$$\begin{aligned}&\alpha\beta\sum_{j\in J}|a_j + b_j|^2 + \sum_{j\in J}|\beta c_j - \alpha d_j|^2\\ &\qquad = \beta(\alpha\sum_{j\in J}|a_j|^2 + \beta\sum_{j\in J}|c_j|^2) + \alpha(\beta\sum_{j\in J}|b_j|^2 + \alpha\sum_{j\in J}|d_j|^2).\end{aligned}$$

Let H and K be Hilbert spaces and let $B(H,K)$ be the set of all bounded linear operators from H into K then $B(H,K)$ is a Hilbert $B(H)$-module with a $B(H)$-valued inner product $\langle T,S\rangle = T^*S$ for all $T,S \in B(H,K)$, and with a linear operation of $B(H)$ on $B(H,K)$ by the composition of operators. Then the space $\underbrace{B(H,K)\oplus\cdots\oplus B(H,K)}_{n} = \{(T_1,\ldots,T_n) : T_i \in B(H,K), i = 1,\ldots,n\}$ is a Hilbert $B(H)$-module via $\langle (T_i)_i,(S_i)_i\rangle = \sum_{i=1}^n T^*S$. Applying formula (6) of Theorem 3.2 for elements of this Hilbert module we get the following result.

Corollary 3.4. *Suppose that $T_1,...,T_n, S_1,...,S_n$ and $Q_1,...,Q_n, L_1,...,L_n$ are elements of $B(\mathcal{H},\mathcal{K})$, i.e., the set of all bounded linear operators from H into K, with $\sum_{j=1}^n T_j^*S_j = \sum_{j\in J}^n Q_j^*L_j$ and $\alpha,\beta \in \mathbb{R}$. Then*

$$\alpha\beta\sum_{j=1}^n |T_j+S_j|^2 + \sum_{j=1}^n |\beta Q_j - \alpha L_j|^2$$
$$= \beta(\alpha\sum_{j=1}^n |T_j|^2 + \beta\sum_{j=1}^n |Q_j|^2) + \alpha(\beta\sum_{j=1}^n |S_j|^2 + \alpha\sum_{j=1}^n |L_j|^2).$$

4 Generalized parallelogram law

In the following Theorem we get a generalization of parallelogram law in the framework of Hilbert C*-modules, whose this is a Theorem 4.1 of [3] in operator equality the moreover part is the Theorem 3.2 of [3] in operator inequality where the Hilbert module is the $B(\mathcal{H})$ over itself.

Theorem 4.1. *Let $u,v \in \mathcal{H}$ and λ be nonzero element of $\mathbb{R}$, then the following statements hold.*

$$(i) \quad |u-v|^2 + \frac{1}{\lambda}|\lambda u + v|^2 = (1+\lambda)|u|^2 + (1+\frac{1}{\lambda})|v|^2.$$

$$(ii) \quad |u+v|^2 + \frac{1}{\lambda}|\lambda u - v|^2 = (1+\lambda)|u|^2 + (1+\frac{1}{\lambda})|v|^2.$$

Moreover,
(I) $|u\mp v|^2 + |\lambda u \pm v|^2 \leq (1+\lambda)|u|^2 + (1+\frac{1}{\lambda})|v|^2 \Leftrightarrow 0 < \lambda \leq 1$ *or* $\lambda u = \mp v$,
(II) $|u\mp v|^2 + |\lambda u \pm v|^2 \geq (1+\lambda)|u|^2 + (1+\frac{1}{\lambda})|v|^2 \Leftrightarrow \lambda < 0$ *or* $\lambda \geq 1$ *or* $\lambda u = \mp v$.

Proof. (i) Put $w = u, z = -v, \alpha = -1, \beta = \lambda$ in (5) of Theorem 3.2. Note that in this case $\mu = -1$. For part (ii) let us put $w = u, z = v, \alpha = 1, \beta = \lambda$ in (6). □

The following Example shows that the equality in Theorem 4.1 holds for every nonzero $\lambda \in \mathbb{R}$, and the first part of the moreover part is true for $\lambda \in (0,1]$ and another part is holds for $\lambda \in (-\infty,0) \cup [1,+\infty)$.

Example 4.2. *Let $\mathcal{A}$ be the C^*-algebra of the set of all diagonal matrices in $M_{2\times 2}(\mathbb{C})$ and suppose $\mathcal{A}$ is the Hilbert $\mathcal{A}$-module over itself. (Here, diagonal matrix means a 2×2 matrix (a_{ij}) such that $a_{11} = a, a_{22} = b$ and $a_{12} = a_{21} = 0$, for $a, b \in \mathbb{C}$.) Consider,*

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

Then we have

$$|A \mp B|^2 + \frac{1}{\lambda}|\lambda A \mp B|^2 - (1+\lambda)|A|^2 - (1+\frac{1}{\lambda})|B|^2 = 0$$

for every nonzero real number λ. Also we have

$$\begin{aligned} C &:= |A - B|^2 + |\lambda A + B|^2 - (1+\lambda)|A|^2 - (1+\frac{1}{\lambda})|B|^2 \\ &= \begin{pmatrix} \frac{\lambda^3-5\lambda^2+8\lambda-4}{\lambda} & 0 \\ 0 & \frac{16\lambda^3+8\lambda^2-15\lambda-9}{\lambda} \end{pmatrix} \end{aligned}$$

The matrix $C \le 0$ if $\lambda \in (0,1]$ and $C \ge 0$ if $\lambda \in (-\infty,0) \cup [1,+\infty)$.

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