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EXISTENCE AND REGULARITY OF SOLUTIONS TO NONLINEAR ELLIPTIC EQUATIONS WITHOUT SIGN CONDITION

Jaouad IGBIDA
Département de Mathématiques,
Centre Régional des Métiers
de l'Education et de la Formation
El Jadida, Maroc.

Ahmed JAMEA
Département de Mathématiques,
Centre Régional des Métiers
de l'Education et de la Formation
El Jadida, Maroc.

Abderrahmane EL HACHIMI
Département de Mathématiques,
Faculté des Sciences, Université Mohammed V- Agdal
Rabat, Maroc.

Abstract

In this paper we study the existence of bounded weak solutions for some nonlinear Dirichlet problems in a bounded domains. Without any sign condition on the term which growth quadratically on the gradient and for a given function f in $L^m(\Omega)$ with $m > N/2$, we prove the existence of bounded weak solutions via L^∞-estimates. Our methods relay on Schauder fixed point theorem, a priori estimates and Stampacchia's L^∞-regularity.

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1 Introduction

This article is devoted to study the Dirichlet problem for some non linear elliptic equations whose simplest model is

$$a(x,u) - \Delta u + g(x,u)|\nabla u|^2 = f(x) \text{ in } \Omega,$$

$$u|_{\partial\Omega} = 0.$$

For a bounded domain Ω in $\mathbb{R}^N$, $N > 2$, Δ denotes the Laplace operator.

Our research are considered without imposing any limitation on the growth of $g(x,s)$ as tends to infinity and without assuming any sign condition on the Carathéodory function g. Let us note that $g(x,s)$ is a Carathéodory function on $\Omega \times (0,+\infty)$ which may have a singularity at $s = 0$ and my change of sign. The Carathéodory function $a(.,.)$ grows in u like $|u|^r$ and satisfy some supplementary conditions which are described below. Under suitable conditions on the data, we shall study existence and regularity of solutions for problem (P).

On the on hand, in some papers, it is proved existence of solutions when the source therm f is small in a suitable norm. On the other hand, condition on the function g have been considered in order to get a solution for f in a given Lebesgue space. Let us not that in [11] the function g is considerer such that $g(x,s) = b(s)$ and b is an increasing function with $b(0) = 0$.

In this paper, to prove existence of bounded weak solutions, we assume that $f \in L^m$, $m > \frac{N}{2}$, $g(x,s)$ is a Carathéodory function on $\Omega \times (0,+\infty)$ may have a singularity at $s = 0$ and my change of sign and $a(.,.)$ is a Carathéodory function satisfying some conditions as bellow. We shall obtain solution by approximating process. Using a priori estimates, Schauder fixed point theorem and Stampacchia's L^∞-regularity results we shall show that the approximated solutions converges to a solution of problem (P).

After the classical references of Ladyzenskaja and Lions (see [19] and [20]), many works have been devoted to elliptic problems with lower order terms having quadratic growth with respect to the gradients (see e.g. [8], [9], [14], [16], [17], [18], [23] and the references therein).

This kind of problems though being physically natural, does not seem to have been studied in the literature. For the simpler case where $a = 0$, g is a constant (we can assume $g = 1$ without loss of generality) and $f \in L^{\frac{N}{2}}$; that is when (P) of the form

$$\begin{aligned} -\Delta u &= |\nabla u|^2 + f(x) \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{1}$$

the problem has been studied in [18], where the change of variable $v = e^u - 1$ leads to the following problem

$$\begin{aligned} -\Delta v &= f(x)(v+1) \text{ in } \Omega, \\ v &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{2}$$

Then, provided that $f \in L^{\frac{N}{2}}$, it is proved there that (1) admits a unique solution in $W_0^{1,2}(\Omega)$.

In the case where g is constant and $f = 0$, $a = 0$, which is special situations, this equation may be considered as the stationary part of equation

$$u_t - \Delta u = \epsilon |\nabla u|^2 .$$

This equation is well known as the Kardar-Parisi-Zhang equation (see [17]). It presents also the viscosity approximation as $\epsilon \to +\infty$ of Hamilton-Jacobi equations from stochastic control theory (see [22]).

Finally, let us recall that the case where $f \in L^q$ with $q > \frac{N}{2}$, $a = 0$, and g is a Carathéodory function satisfying sign conditions, have been considered in [1, 4, 8]. Existence and regularity results have been obtain in [6, 8, 9] and the uniqueness results have been shown in [2].

2 Assumptions and main result

We are interested in establishing an existence result for the following elliptic problem in Ω

$$-\Delta u + a(x,u) + h(x,u,\nabla u) = f(x) \text{ in } \Omega,$$

$$u|_{\partial\Omega} = 0.$$

Ω is a bounded domain of $\mathbb{R}^N$ with $N > 2$, with boundary $\Gamma = \partial\Omega$. Let us set

$$h(x,u,\nabla u) = g(x,u)|\nabla u|^2.$$

We assume that there exist an increasing function

$$b : (0, +\infty) \to (0, +\infty)$$

and $\beta \in (0,1)$ such that the Carathéodory function g satisfies the following hypothesis

$$-\beta \leq s\, g(x,s) \leq b(s), \quad \forall s > 0, \text{ and } a.e.\ x \in \Omega \tag{3}$$

We remarque that no condition on the upper growth of $g(x,s)$ as s goes to infinity is imposed. In particular, the nonlinearity can have a singularity at zero.

The function $a(.,.)$ satisfy the following hypothesis:

(A_1) $a(x,s) : \Omega \times R \to R$ is measurable in $x \in R^N$ for any fixed $s \in R$ and continuous in s for a.e. x.

(A_2) There exists a constant $c > 0$ and a positive function $\gamma(x) \in L^\infty(\Omega)$ such that for all s and almost every x

$$a(x,s) \geq \gamma(x)s + c.$$

(A_3) For any $\alpha > 0$ the function

$$a_\alpha(x) = \sup_{|s|\leq\alpha} \{a(x,s)|s|^{r-1}\}$$

is integrable over Ω.

The source terms f satisfies the following hypotheses

$$f \in L^m(\Omega),\ m > \frac{N}{2}.$$

We look for a solution u in $H_0^1(\Omega)$ such that both function $a(x,u)$ and $h(x,u,\nabla u)$ are integrable, and the following equality holds

$$\int_\Omega a(x,u)\phi + \int_\Omega \nabla u \nabla \phi + \int_\Omega h(x,u,\nabla u)\phi = \int_\Omega f\,\phi, \tag{4}$$

for any test function ϕ in $C_0^\infty(\Omega)$.

For all $k \in \mathbb{R}^+$, we recall the definition of a truncated function $T_k(s)$ defined by

$$T_k(z) = \begin{cases} z & \text{if } |z| \leq k \\ k & \text{if } z > k \\ -k & \text{if } z < -k \end{cases}$$

We also consider

$$G_k(z) = z - T_k(z) = (|z| - k)^+ \,sign\,(z).$$

Theorem 2.1. *Let $r > 1$ and $m > \frac{N}{2}$. If we assume that (3) and $(A_1 - A_3)$ are satisfied, then the problem (P) has at least one solution which belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$.*

□

The proof follows by considering a method of penalization and approximation to approach the solutions of the problem. Next, a priori estimates are obtained on the solutions and we give some consequences of these estimates. Finally, we passe to the limit.

3 Penalization and approximation

This sections is devoted to prove some basic estimates for a sequence of approximated problems. We denote by c a positive constant which may only depend on the parameters of our problem, its value my vary from line to line.

We define also the following Carathéodory function on $\Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^+$ as follows:

$$h_n(x,s,\zeta) = g_n(x,s)\frac{|\zeta|^2}{1+\frac{1}{n}|\zeta|^2},$$

where

$$g_n(x,s) = \begin{cases} 0 & \text{if} \quad s \leq 0 \\ n^2 s^2 T_n g(x,s) & \text{if} \quad 0 < s < \frac{1}{n} \\ T_n g(x,s) & \text{if} \quad \frac{1}{n} \leq s \end{cases}.$$

For any $n \in \mathbb{N}$ we consider

$$A_n(x,s) = a_n(x,s)\frac{|s|^{r-1}}{1+\frac{1}{n}|s|^{r-1}},$$

where

$$a_n(x,s) = \gamma(x) T_n(s).$$

We define now the function Λ_n by

$$\Lambda_n(x,s,\zeta) = h_n(x,s,\zeta) + A_n(x,s).$$

This function is bounded. Indeed, $g_n(x,s)$, $\gamma(x)$ and $\frac{|\zeta|^2}{1+\frac{1}{n}|\zeta|^2}$ are bounded.

Then we have the following approximating problems

$$-\Delta u + \Lambda_n(x,u,\nabla u) = f(x) \text{ in } \Omega. \tag{5}$$

We recall that $f \in L^m(\Omega)$ with $m > N/2$. We consider the following operator

$$\mathbb{L}_n : H_0^1(\Omega) \to L^m(\Omega),$$

defined by

$$\mathbb{L}_n(u) = -A_n(x,u) - h_n(x,u,\nabla u) + f(x).$$

Denoting by u_k a sequence of function in $H_0^1(\Omega)$ such that $u_k \to u$ in $H_0^1(\Omega)$ and denoting

$$G_{n,k} = |[h_n(x,u_k,\nabla u_k) - h_n(x,u,\nabla u)]|^m,$$

and

$$A_{n,k} = |[A_n(x,u_k) - A_n(x,u)]|^m.$$

As an application of the dominated convergence theorem, we infer, from the convergence of u_k to u and ∇u_k to ∇u a.e. $x \in \Omega$ and from the boundedness of the operators $g_n(x,s)$ and $a_n(x,s)$ that

$$\lim_{k\to\infty}\int_\Omega G_{n,k} = 0,$$

and

$$\lim_{k\to\infty}\int_\Omega A_{n,k} = 0.$$

Then,

$$\mathbb{L}_n\, u_k \to \mathbb{L}_n\, u \ \text{ in } \ L^m(\Omega), \ \text{as } k \text{ tends to infinity.}$$

Therefor, $\mathbb{L}_n$ is continuous.

We consider now the inverse of the Laplacian operator

$$(-\Delta)^{-1} : L^m(\Omega) \to H_0^1(\Omega).$$

Then, the solutions of (5) are the fixed points of the composition operator

$$\Gamma_n \equiv (-\Delta)^{-1} \circ \mathbb{L}_n.$$

Since $m > N/2$, we deduce that the operator $(-\Delta)^{-1}$ is compact and hence the composition of it with the continuous operator $\mathbb{L}_n$ i.e. that Γ_n is also compact.

Let us now observe that,

$$\begin{aligned}||\mathbb{L}_n\, u||_m &\leq ||h_n(x,u,\nabla u)||_m + ||A_n(x,u)||_m + ||f||_m \\ &\leq ||f||_m + 2n^2|\Omega|^{1/m}.\end{aligned}$$

By the continuity of $(-\Delta)^{-1}$, this implies that there exists a positive constante R which depond on n such that

$$||\Gamma_n\, u||_{H_0^1(\Omega)} \leq R(n).$$

So that, The operator Γ_n maps the ball in $H_0^1(\Omega)$ centered at zero and with radius $R(n)$ into it self. Finally from the Schauder fixed point theorem there exists a fixed point $u_n \in H_0^1(\Omega)$ of Γ_n. That means that a solution of the approximating problems.

Let us consider the folowing notation

$$\gamma = 2^*\left[\frac{1}{(2^*)'} - \frac{1}{m}\right],$$

where $2^* = \frac{2N}{N-2}$.

$$\eta = \lambda^{-2^*} 2^{2^* \frac{\gamma}{\gamma-1}} \lambda^{-2^*} (1-\beta)^{-2^*} ||f||_m^{2^*} |\Omega|^{\gamma-1}, \tag{6}$$

where λ is the Sobolev constants defined by

$$\lambda = \inf_{w \in H_0^1(\Omega)-\{0\}} \frac{||w||^2}{||w||_{2^*}^2}.$$

Let us note that, there exist $\kappa > 1$ such that

$$s\, g_n(x,s) \leq \kappa,\ \forall s \in]0,\eta],\ \text{a.e}\, x \in \Omega, \tag{7}$$

and from 3 that

$$s\, g_n(x,s) + \beta \geq 0,\ \text{a.e}\, x \in \Omega,\quad \text{for all } s \in \mathbb{R}, \tag{8}$$

for every n.

We prove now that u_n is bounded in $H_0^1(\Omega) \cap L^r(\Omega)$. From the approximated problem on has

$$\int_\Omega |\nabla u_n|^2 + \int_\Omega \Lambda_n(x,u_n,\nabla u_n)u_n \leq \int_\Omega f\, u_n.$$

It follows

$$\int_\Omega |\nabla u_n|^2 + \int_\Omega h_n(x,u_n,\nabla u_n)u_n + \int_\Omega A_n(x,u_n)u_n \leq \int_\Omega f\, u_n.$$

Then

$$\int_\Omega |\nabla u_n|^2 + \int_\Omega g_n(x,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2}u_n + \int_\Omega a_n(x,u_n)\frac{|u_n|^{r-1}}{1+\frac{1}{n}|u_n|^{r-1}}u_n \leq \int_\Omega f\, u_n.$$

That is equivalent to

$$(1-\beta)\int_\Omega |\nabla u_n|^2 + \int_\Omega [g_n(x,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2}u_n + \beta|\nabla u_n|^2] + \int_\Omega a_n(x,u_n)\frac{|u_n|^{r-1}}{1+\frac{1}{n}|u_n|^{r-1}}u_n \leq \int_\Omega f\, u_n.$$

From (8) on has

$$u_n g_n(x,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2} + \beta\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2} \geq 0,\ a.e\, x \in \Omega.$$

By consequence

$$u_n g_n(x,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2}+\beta|\nabla u_n|^2\geq 0.$$

Therefor

$$\int_\Omega |u_n|^r + ||\nabla u_n||^2 \leq c\,||f||_m\,||u_n||_{m'},$$

where $m\prime$ is the conjugate exponent of m ($m\prime=\frac{m}{m-1}$). Since $m\prime\leq 2^*$ indeed $m>N/2$. Then, by Sobolev embedding theorem the sequence u_n is bounded in $H_0^1(\Omega)\cap L^r(\Omega)$.

Let us now consider the truncating function $G_k(s)=s-T_k(s)$ and from the approximating problem we obtain

$$\int_\Omega |\nabla G_k(u_n)|^2+\int_\Omega \Lambda_n(x,u_n,\nabla u_n)u_n G_k(u_n)\leq\int_\Omega fG_k(u_n)\ .$$

Then, on has

$$\int_\Omega |\nabla G_k(u_n)|^2+\int_\Omega h_n(x,u_n,\nabla u_n)G_k(u_n)+\int_\Omega A_n(x,u_n)G_k(u_n)\leq\int_\Omega f\,G_k(u_n).$$

It follows

$$(1-\beta)\int_\Omega |\nabla G_k(u_n)|^2+\int_\Omega [h_n(x,u_n,\nabla u_n)G_k(u_n)+\beta|\nabla G_k(u_n)|^2]\leq\int_\Omega f\,G_k(u_n)\,.$$

which implies that

$$(1-\beta)\int_\Omega |\nabla G_k(u_n)|^2+\int_\Omega [g_n(x,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2}G_k(u_n)+\beta|\nabla G_k(u_n)|^2]\leq\int_\Omega f\,G_k(u_n)\,.$$

Then, on obtain

$$\int_\Omega |\nabla G_k(u_n)|^2\leq c\int_\Omega f\,G_k(u_n)\,.$$

From this inequality, it follows by Stampacchia's L^∞-regularity that

$$||u_n||_\infty\leq\eta,$$

where $\eta = 2^{2^*\frac{\gamma}{\gamma-1}}\lambda^{-2^*}(1-\beta)^{-2^*}||f||_m^{2^*}|\Omega|^{\gamma-1}$.

Since the solution u_n is bounded independently on n in $H_0^1(\Omega)$. Then, up to a subsequence, that we denote again by u_n, there exist $u \in H_0^1(\Omega)$, such that u_n converge to u weakly in $H_0^1(\Omega)$.

From Rellich-Kondrachov's theorem we have the almost every where convergence in Ω. That is

$$u_n \to u \text{ weakly in } H_0^1(\Omega),$$

$$u_n \to u \text{ almost every where in } \Omega\ , \tag{9}$$

and

$$a_n(x,u_n) \to a(x,u) \text{ almost every where in } \Omega.$$

Taking into account the equi-integrability of u_n in $L^r(\Omega)$, it follows that of $a_n(x,u_n)|u_n|^{r-1}$ in $L^1(\Omega)$.

Hence, we have

$$a_n(x,u_n)|u_n|^{r-1} \to a(x,u)|u|^{r-1} \text{ in } L^1(\Omega)\,. \tag{10}$$

Since on has up to a subsequence u_n, that

$$\nabla u_n \to \nabla u \text{ almost every where in } \Omega, \tag{11}$$

and ∇u_n is equi-integrable in $L^2(\Omega)$, then on has

$$\nabla u_n \to \nabla u \text{ in } L^2(\Omega)\,.$$

We conclude that

$$\Delta u_n \to \Delta u \text{ in } L^1(\Omega)\,.$$

From the precedent section there exists a solution u_n of the following problem

$$\begin{aligned} -\Delta u_n + \Lambda_n(x,u_n,\nabla u_n) &= f \text{ in } \Omega, \\ u_n &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

in the sense that u_n belongs to $H_0^1(\Omega)$, $\Lambda_n(x,u_n,\nabla u_n)$ belongs to $L^1(\Omega)$ and

$$\int_\Omega \nabla u_n.\nabla\varphi + \int_\Omega \Lambda_n(x,u_n,\nabla u_n)\varphi = \int_\Omega f\,\varphi,$$

holds for every φ in $H_0^1(\Omega)\cap L^\infty(\Omega)$.

Then, it follows that

$$\int_\Omega \nabla u_n.\nabla(\phi(u_n-u)\psi) + \int_\Omega \Lambda_n(x,u_n,\nabla u_n)\phi(u_n-u)\psi = \int_\Omega f\,\phi(u_n-u)\psi,$$

where ψ is a positive function in $C_0^\infty(\Omega)$ and

$$\phi(r) = r\,e^{\alpha r^2},\ \ \alpha \text{ a positive constant.}$$

Which implies that

$$\int_\Omega \nabla u_n\,\nabla(u_n-u)\phi'(u_n-u)\psi + \int_\Omega \nabla u_n \nabla\psi\,\phi(u_n-u) + \int_\Omega h_n(x,u_n,\nabla u_n)\varphi \le \int_\Omega f\,\varphi.$$

It follows

$$\int_\Omega \nabla u_n\,\nabla(u_n-u)\phi'(u_n-u)\psi + \int_\Omega \nabla u_n \nabla\psi\,\phi(u_n-u) + I_n \le \int_\Omega f\,\phi(u_n-u)\psi,$$

where

$$I_n = \int_\Omega g_n(x,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2}\,\phi(u_n-u)\psi.$$

By consequence

$$\int_\Omega |\nabla(u_n-u)|^2\phi'(u_n-u)\psi + \int_\Omega \nabla u\,\nabla(u_n-u)\phi'(u_n-u)\psi + \int_\Omega \nabla u_n\nabla\psi\phi(u_n-u) + \le \int_\Omega f\,\phi(u_n-u)\psi.$$

Since we have

$$I_n \ge -c|\nabla u_n|^2|\phi(u_n-u)|\psi,\ \ \text{a.e. } x\in\Omega,$$

then

$$\int_\Omega |\nabla(u_n-u)|^2\phi'(u_n-u)\psi - c|\nabla u_n|^2|\,\phi(u_n-u)\psi$$
$$\le -\int_\Omega \nabla u\,\nabla(u_n-u)\,\phi'(u_n-u)\psi - \int_\Omega \nabla u_n\nabla\psi\ \phi(u_n-u) + \int_\Omega f\ \phi(u_n-u)\psi.$$

On the other hand

$$\int_\Omega |\nabla(u_n-u)|^2\phi(u_n-u)\psi = \int_\Omega |\nabla u_n|^2|\,\phi(u_n-u)|\psi + \int_\Omega |\nabla u|^2|\,\phi(u_n-u)|\psi - 2\int_\Omega \nabla u_n\nabla u|\ \phi(u_n-u)|\psi.$$

Let us note that

$$\phi'(r) - c|\phi(r)| = e^{\alpha r^2}(1 + 2\alpha r^2 - c|r|),$$

and taking a count to the fact that for large value of α we have

$$e^{\alpha r^2}(1 + 2\alpha r^2 - c|r|) \geq \frac{1}{2}, \ \forall s \in \mathbb{R}. \tag{12}$$

On deduce that

$$\begin{aligned}\frac{1}{2}\int_\Omega |\nabla(u_n - u)|^2\psi \leq -c\int_\Omega |\nabla u|^2|\phi(u_n - u)|\psi + 2c\int_\Omega \nabla u_n \nabla u|\,\phi(u_n - u)|\psi \\ - \int_\Omega \nabla u\,\nabla(u_n - u)\phi'(u_n - u)\psi - \int_\Omega \nabla u_n \nabla\psi\,\phi(u_n - u) \\ + \int_\Omega f\,\phi(u_n - u)\psi.\end{aligned}$$

From the dominated convergence theorem, we obtain

$$\int_\Omega |\nabla(u_n - u)|^2\psi \to 0, \quad \text{for } n \text{ tending to} + \infty, \quad \text{for all } \psi \in C_0^\infty(\Omega)$$

We cane deduce now, up to a subsequence, that there exist $\vartheta \in L^2(\Omega)$ such that

$$|\nabla u_n(x)| \leq \vartheta(x), \ \text{ a.e. } x \in \Omega, \tag{13}$$

and

$$\nabla u_n(x) \to \nabla u(x), \ \text{ a.e. } x \in \Omega.$$

Since (7) and (8) are satisfied, then fore some $c > 0$ we have

$$|g_n(x, u_n)| \leq c \ \text{ a.e. } x \in \Omega.$$

By the ue of (13) on has

$$|g_n(x, u_n)|\frac{|\nabla u_n|^2}{1 + \frac{1}{n}|\nabla u_n|^2} \leq c\vartheta^2 \text{ a.e. } x \in \Omega.$$

Passing now to the limit, we obtain

$$g_n(x, u_n)\frac{|\nabla u_n|^2}{1 + \frac{1}{n}|\nabla u_n|^2} \to g(x, u)|\nabla u|^2 \quad \text{a.e. } x \in \Omega.$$

Using the dominated convergence theorem, it yields that

$$\int_\Omega g_n(x, u_n)\frac{|\nabla u_n|^2}{1 + \frac{1}{n}|\nabla u_n|^2}\phi \to \int_\Omega g(x, u)|\nabla u|^2\phi.$$

References

[1] B. Abdellaoui , A. Dallaglio, I. Peral, Some remarks on elliptic problems with critical growth in the gradient, J. Differential equations 222 (2006) 21-62.

[2] A. Ambrosetti, H. Brezis, G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, J. Funct. Anal. 122 (2) (1994) 519-543.

[3] PH. Bénilan, L. Boccardo, TH. Gallouët, R. Gariepy, M. Pierre, J.L. Vázquez, An L^1 theory of existence and uniqueness of solutions of nonlinear elliptic equations, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 22 (1995) 241-273.

[4] A. Bensoussan , L. Boccardo, F. Murat, On a nonlinear partial differential equation having natural growth terms and unbounded solutions, Ann. Inst. H. Poincaré Anal. Non Linéaire 5 (1988) 347-364.

[5] A. Ben-Artzi, P. Souplet, F.B. Weissler, The local theory for the viscous Hamilton-Jacobi equations in Lebesgue spaces, J. Math. Pure. Appl. 9 (2002), 343-378.

[6] L. Boccardo, A. Dallaglio , L. Orsina, Existence and regularity results for some elliptic equations with degenerate coercivity, Atti Sem. Mat. Fis. Univ. Modena 46 (1998) 51-81.

[7] L. Boccardo, T. Gallouet, Strongly nonlinear elliptic equations having natural growth terms and L^1 data, Nonlinear Anal. 19 (1992) 573-579.

[8] L. Boccardo, F. Murat, J.-P. Puel, L^∞ estimates for some nonlinear elliptic partial differential equations and application to an existence result, SIAM J. Math. Anal. 2 (1992) 326-333.

[9] K. Cho, H.J. Choe, Non-linear degenerate elliptic partial differential equations with critical growth conditions on the gradient, Proc. Am. Math. Soc. 123 (12) (1995) 3789-3796.

[10] A. El Hachimi, J.-P. Gossez, A note on nonresonance condition for a quasilinear elliptic problem. Nonlinear Analysis, Theory Methods and Applications, Vol 22, No 2 (1994), pp. 229-236.

[11] A. El Hachimi, Jaouad Igbida, Bounded weak solutions to nonlinear elliptic equations , Elec. J. Qual. Theo. Diff. Eqns., 2009, No. 10, 1-16.

[12] A. El Hachimi, Jaouad Igbida, Nonlinear parabolic equations with critical growth and superlinear reaction terms, IJMS, Vol 2, No S08 (2008) 62-72.

[13] A. El Hachimi, M. R. Sidi Ammi, Thermistor problem: a nonlocal parabolic problem, ejde, 11 (2004), pp. 117-128.

[14] V. Ferone, F. Murat, Nonlinear problems having natural growth in the gradient: an existence result when the source terms are small, Nonlinear Anal. Theory Methods Appl. 42 (7) (2000) 1309-1326.

[15] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, 2nd edition, Springer-Verlag (1983).

[16] N. Grenon, C. Trombetti, Existence results for a class of nonlinear elliptic problems with p-growth in the gradient, Nonlinear Anal. 52 (3) (2003) 931-942.

[17] M. Kardar, G. Parisi, Y.C. Zhang, Dynamic scaling of growing interfaces, Phys. Rev. Lett. 56 (1986) 889-892.

[18] J.L. Kazdan, R.J. Kramer, Invariant criteria for existence of solutions to second-order quasi-linear elliptic equations, Comm. Pure Appl. Math. 31 (5) (1978) 619-645.

[19] O.A. Ladyzhenskaja, N.N. Ural'ceva, Linear and quasi-linear elliptic equations, Academic Press, New York - London, 1968.

[20] J. Leray, J. L. Lions, Quelques résultats de Viš̃ik sur les problèmes elliptiques semi-linéaires par les méthodes de Minty et Browder, Bull. Soc. Math. France, 93 (1965), 97-107.

[21] J. L.Lions, Quelques méthodes de résolution des problèmes aux limites non linéaire, Dunod et Gautier-Villars, (1969).

[22] P.L. Lions, Generalized solutions of Hamilton-Jacobi Equations, Pitman Research Notes in Mathematics, vol. 62, 1982.

[23] C. Maderna, C.D. Pagani, S. Salsa, Quasilinear elliptic equations with quadratic growth in the gradient, J. Differential Equations 97 (1) (1992) 54-70.

[24] G. Stampacchia, Equations elliptiques du second ordre à coefficients discontinus, Séminaire de Mathématiques Supérieures, vol. 16, Les Presses de l'Université de Montréal, Montréal, 1966.

[25] G. Stampacchia, Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus, Ann. Inst. Fourier, Grenoble 15 (1965) 189-258.

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