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A new characterization of $L_2(2^m)$ by nse

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Abstract

Let G be a group and $\omega(G)$ be the set of element orders of G. Let $k \in \omega(G)$ and s_k be the number of elements of order k in G. Let $\text{nse}(G) = \{s_k \big| k \in \omega(G)\}$. The groups $L_2(8)$ and $L_2(16)$ are unique determined by nse(G). In this paper, we prove that if G is a group such that nse(G)=nse($L_2(2^m)$), then $G \cong L_2(2^m)$.

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1 Introduction

A finite group G is called a simple K_4-group, if G is a simple group with $|\pi(G)| = 4$. W. J. Shi in [30] pointed out the following problem.

Shi's Problem. Is the number of simple K_4-groups is finite or infinite? So it is also difficult to known the exact number of simple K_4-group $L_2(3^m)$. Thus characterization of $L_2(3^m)$ is also an interesting work. For nse, the most important problem is related to **Thompson's Problem**.

In 1987, J. G. Thompson posed a very interesting problem related to algebraic number fields as follows (see [32]).

Thompson's Problem. Let $T(G) = \{(n, s_n)\big|n \in \omega(G)$ and $s_n \in$nse$(G)\}$, where s_n is the number of elements with order n. Suppose that $T(G) = T(H)$. If G is a finite solvable group, is it true that H is also necessarily solvable?

It is easy to see that if G and H are of the same order type, then

$$\text{nse}(G) = \text{nse}(H), \qquad |G| = |H|.$$

Theorem 1.1 *Let G be a group and H be one of the following groups.*
(1) M is a simple K_i-group, where $i = 3, 4$ (see [28, 27] respectively);
(2) A_{12}, A_{13} (see [19, 14]);
(3) A Sporadic simple group (see [10]);
(4) A_n, $n = r, r+1, r+2, r+3, r+4, r+5$ where r is a prime (see [1]);
(5) S_r, where r is a prime [8];
(6) $L_2(2^m)$ with $2^m + 1$ is a prime or $2^m - 1$ is a prime (see [26]).
Then $|G| = |H|$ and nse(G)=nse(H) if and only if $G \cong H$.

Not all groups can be determined by nse(G) and $|G|$. Let A, B be two finite groups, $G := A \rtimes B$ means the semidirect of A, B and $A \lhd G$. For example. In 1987, J. G. Thompson gave an example as followings. Let

$$G_1 = C_2 \times C_2 \times C_2 \times C_2 \rtimes A_7, \quad G_2 = L_3(4) \rtimes C_2,$$

where both G_1 and G_2 are maximal subgroups of M_{23}. Then nse(G_1)=nse(G_2)=$\{1$, 435, 2240, 6300, 8064, 6720, 5040, 57600$\}$, but $G_1 \ncong G_2$.

Comparing the sizes of elements of same order but disregarding the actual orders of elements in $T(G)$ of the **Thompson's Problem**, in other words, it remains only nse(G), whether can it characterize finite simple groups?

Theorem 1.2 *Let G be a group and H be one of the following groups.*
(1) Some projective special linear groups (see [29, 33, 19, 5, 4] respectively);
(2) PGL(2,p) [2];
(3) $L_3(5)$, $U_3(5)$ and $U_3(7)$ (see [21], [20] and [23] respectively);
(4) S_r, where r is a prime, $r - 2$ is a prime and $r < 5.10^8$ [8];
(5) M_{11}, M_{12}, M_{23}, M_{24} [6];
(6) A_7, A_8 [7];
(7) $L_5(2)$ [22];
(8) J_1 [9];
(9) S_8 [3].
Then nse(G)=nse(H) if and only if $G \cong H$.

In this paper, it is shown that the group $L_2(2^m)$ also can be characterized by nse.

We introduce some new notations which will be used in the paper. Let $a.b$ denote the products of an integer a by an integer b. Let r be a prime. Then we denote the p-part of the integer n by n_p. Without confusion, we also denote the number of the Sylow r-subgroup P_r by n_r or $n_r(G)$. The other notations are standard (see [11]).

2 Preliminary Lemmas

In this section, we give some lemmas which will be used in the proof of the main theorem.

Lemma 2.1 *Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G \big| g^m = 1\}$, then $m \big| |L_m(G)|$.*

Proof. See [13]. □

Lemma 2.2 *Let G be a group containing more than two elements. If the maximal number s of elements of the same order in G is finite, then G is finite and $|G| \leq s(s^2 - 1)$.*

Proof. See [29]. □

Lemma 2.3 *With the exceptions of the relations $(239)^2 - 2(13)^4 = -1$ and $(3)^5 - 2(11)^2 = 1$ every solution of the equation*

$$p^m - 2q^n = \pm 1; \qquad p, q \quad \text{prime}; \qquad m, n > 1,$$

has exponents $m = n = 2$; i. e. it comes from a unit $p - q.2^{\frac{1}{2}}$ of the quadratic field $Q(2^{\frac{1}{2}})$ for which the coefficients p and q are primes.

Proof. See [12] and [18]. □

To prove $G \cong L_2(2^m)$, we need the structure of simple K_4-groups.

Lemma 2.4 *Let G be a simple K_4-group. Then G is isomorphic to one of the following groups:*

(1) A_7, A_8, A_9 or A_{10}.

(2) M_{11}, M_{12} or J_2.

(3) One of the following:

(a) $L_2(r)$, where r is a prime and $r^2 - 1 = 2^a \cdot 3^b \cdot v^c$ with $a \geq 1$, $b \geq 1$, $c \geq 1$, and v is a prime greater than 3.

(b) $L_2(2^m)$, where $2^m - 1 = u$, $2^m + 1 = 3t^b$ with $m \geq 2$, u, t are primes, $t > 3$, $b \geq 1$.

(c) $L_2(3^m)$, where $3^m + 1 = 4t$, $3^m - 1 = 2u^c$ or $3^m + 1 = 4t^b$, $3^m - 1 = 2u$, with $m \geq 2$, u, t are odd primes, $b \geq 1$, $c \geq 1$.

(4) One of the following 28 simple groups: $L_2(16)$, $L_2(25)$, $L_2(49)$, $L_2(81)$, $L_3(4)$, $L_3(5)$, $L_3(7)$, $L_3(8)$, $L_3(17)$, $L_4(3)$, $S_4(4)$, $S_4(5)$, $S_4(7)$, $S_4(9)$, $S_6(2)$, $O_8^+(2)$, $G_2(3)$, $U_3(4)$, $U_3(5)$, $U_3(7)$, $U_3(8)$, $U_3(9)$, $U_4(3)$, $U_5(2)$, $Sz(8)$, $Sz(32)$, ${}^2D_4(2)$ *or* ${}^2F_4(2)$.

Proof. See [31]. □

Lemma 2.5 *If* $2 \mid q$, *then* $nse(L_2(q))=\{1, \phi(d).q.(q+1)/2, 1 < d \mid (q-1), \phi(s).q.(q-1)/2, 1 < s \mid (q+1), q^2-1\}$

Proof. It is easy to get from [16], Chapter 2, Theorem 8.2-8.5. □

Lemma 2.6 $\mu(L_2(q)) = \{q, (q-1)/2, (q+1)/2\}$ *with* q *odd.*

Proof. See [17, p. 213]. □

Lemma 2.7 *Let* G *be a finite solvable group and* $|G| = mn$, *where* $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $(m,n) = 1$. *Let* $\pi = \{p_1, \cdots, p_r\}$ *and* h_m *be the number of Hall* π*-subgroups of* G. *Then* $h_m = q_1^{\beta_1} \cdots q_s^{\beta_s}$ *satisfies the following conditions for all* $i \in \{1, 2, \cdots, s\}$*:*

(1) $q_i^{\beta_i} \equiv 1$ *(mod* p_j*) for some* p_j.

(2) The order of some chief factor of G *is divided by* $q_i^{\beta_i}$.

Proof. See [24, Theorem 9.3.1]. □

Lemma 2.8 *Let* G *be a finite group and* $p \in \pi(G)$ *be odd. Suppose that* P *is a Sylow* p*-subgroup of* G *and* $n = p^s m$ *with* $(p,m) = 1$. *If* P *is not cyclic and* $s > 1$, *then the number of elements of order* n *is always a multiple of* p^s.

Proof. See [25]. □

Let G be a group such that nse(G)=nse($L_2(3^m)$), and s_n be the number of elements of order n. By Lemma 2.2 we have that G is finite. We note that $s_n = k\phi(n)$, where k is the number of cyclic subgroups of order n. Also we note that if $n > 2$, then $\phi(n)$ is even. If $m \in \omega(G)$, then by Lemma 2.1 and the above discussion, we have

$$\begin{cases} \phi(m) \mid s_m \\ m \mid \sum_{d \mid m} s_d \end{cases} \qquad (1)$$

Remark 2.9 *If $p^k \in \omega(G)$ and $|P| = p^k$ where P is a Sylow p-subgroup of G, then $s_{p^k} = \phi(p^k).n_p$. If $|P| > p^k$, then by Lemma 2.8, $s_{p^k} = p^k.t$ for some non-negative integer t. In particular, if $k > 2$, then t is either a multiple of p or coprime to p. On the other hand, $s_p = \phi(p).n_p$, in this case, $(n_p, p) = 1$. Hence if $p \in \pi(G)$, we consider the prime divisor of $\pi(t_{p'})$ where $p' = \pi(t) - \{p\}$, which possibly lies in $\pi(G)$. So whether the Sylow p-subgroup of G is cyclic or not, we can only consider when the p-subgroup is cyclic of order p.*

Lemma 2.10 *Let G be a group and x, y, and $p \geq 3$ are different primes. If $s_x, s_y \in nse(G) = nse(L_2(3^m)) = \{1, \phi(d).3^m.(3^m+1)/2, 1 < d \mid (3^m-1)/2, \phi(s).3^m.(3^m-1)/2, 1 < s \mid (3^m+1)/2, (3^m)^2 - 1\}$ and $s_x \neq s_y$ where m satisfies*

$$\begin{cases} 3^m + 1 = 4t, \\ 3^m - 1 = 2u^c \end{cases} \tag{2}$$

or

$$\begin{cases} 3^m + 1 = 4t^b, \\ 3^m - 1 = 2u \end{cases} \tag{3}$$

with $m \geq 2$, u, t are odd primes, $b \geq 1$, $c \geq 1$ and $\{p\} \subsetneqq \pi(s_x) \cap \pi(s_y)$, then if $x, y \in \pi(G)$, there is an element of order p of G.

In the proof of the Lemma, we always assume that the largest prime belongs to $\pi(G)$. If $c = 1$, then we assume that $u \in \pi(G)$; if $c > 1$, then we assume that $t \in \pi(G)$.

Proof. Let

$$\begin{cases} 3^m + 1 = 4t, \\ 3^m - 1 = 2u^c. \end{cases}$$

Then we consider the following two cases.

If $c = 1$, then $u > t > 3 > 2$ and $\frac{3^m-1}{2} = u$ is a prime. Since $(u, s_u) = 1$, then $s_u = \phi(d).3^m.(3^m + 1)/2$, with $1 < d \mid (3^m - 1)/2$. In this case, $s \mid 2.t$.

- Let $s = 2$. Then by Lemma 2.1, $s_2 = 3^m.(3^m - 1)/2$ is the only odd number of nse(G) and so $2 \in \pi(G)$. Since $(t, s_t) = 1$, then $s_t = \phi(t).3^m.(3^m - 1)/2$. Since $(3^m + 1, 3^m - 1) = 2$, then $3 \in \pi(s_2) \cap \pi(s_u)$. So we can assume that $3 \mid n_u$ or $3 \mid \phi(u)$.
    - * Let $3 \mid n_u$. Then since $u \in \pi(G)$, $3 \in \pi(G)$, this is the desired result.
    - * Let $3 \mid \phi(u)$. Since $u = \frac{3^m-1}{2}$ is a prime, then $3 \mid \frac{3^m-3}{2} = 3(3^{m-1} - 1)/2$. Since $(3^m + 1, 3^m - 1) = 2$ and $(3^m - 1, 3^{m-1} - 1) = 2$, we can assume that 2 or $t \in \pi(3^{m-1} - 1)$. If $t \mid 3^{m-1} - 1$, then $3^m = 3 + 3kt$ for some integer k, but the equation has no solution since $t > 3$

is a prime. Hence $3^{m-1} - 1 = 2^a$ for some integer a. By Lemma 2.3, $m - 1 = 2$ and $a = 3$. So $3^3 + 1 = 4t$, $t = 7$ and $u = 13$. It follows that nse(G)=nse($L_2(3^3)$), and from [4], the prime 3 also, in this case, belongs to $\pi(G)$.

- Let $s = t$. Then by Lemma 2.6, $2.u \notin \omega(G)$. It follows that the Sylow 2-subgroup of G acts fixed point freely on the set of elements of order u and $|P_2| \mid s_u$. Similarly $t.u \notin \omega(G)$ and $|P_t| \mid s_u$. Hence $|G| \mid 3^m.(3^m + 1).(3^m - 1)$. On the other hand,

$$\sum_{s_k \in \mathrm{nse}(G)} s_k = 3^m.(3^m - 1).(3^m + 1)/2 \leq |G| \mid 3^m.(3^m + 1).(3^m - 1). \quad (4)$$

  Since $\sum_{s_k \in \mathrm{nse}(G)} s_k$ is odd, then the inequality has no solution in $\mathbb{N}$.

If $c > 1$, then $t > u > 3 > 2$ and $\frac{3^m+1}{4} = t$ is a prime. Since $(t, s_t) = 1$, then $s_t = \phi(t).3^m.(3^m - 1)/2$, $s_2 = 3^m.(3^m - 1)/2$ and $2 \in \pi(G)$. Since $3, u \in \pi(s_2) \cap \pi(s_t)$. Then we consider the following two cases: $3 \mid n_t$ or $3 \mid \phi(t)$; $u \mid n_t$ or $u \mid \phi(t)$.

Case a. $3 \mid n_t$ or $3 \mid \phi(t)$

- Let $3 \mid n_t$. Then since $t \in \pi(G)$, $3 \in \pi(G)$.

- Let $3 \mid \phi(t)$. Then $3 \mid t - 1$. It follows that there is a Frobenius group of order $3.t$ with a Frobenius kernel of order t and a Frobenius complement of order 3. It follows that there is an element of order $u.t$ which contradicts Lemma 2.6. Therefore $3 \nmid t - 1$.

Case b. $u \mid n_t$ or $u \mid \phi(t)$.

- Let $u \mid n_t$. Then since $t \in \pi(G)$, $u \in \pi(G)$.

- Let $u \mid \phi(t)$. Then $u \mid t - 1$. It follows that there is a Frobenius group of order $u.t$ with a Frobenius kernel of order t and a Frobenius complement of order u. It follows that there is an element of order $3.t$ which contradicts Lemma 2.6. Therefore $u \nmid t - 1$.

Similarly as the case "$3^m + 1 = 4t, 3^m - 1 = 2u^c$" we also can do this case "$3^m + 1 = 4t^b, 3^m - 1 = 2u$".

This completes the proof of the Lemma. □

3 Main Results and its Proof

In this section, we will give the proof of the main theorem.

Theorem 3.1 *Let G be a group and r a prime. Then $G \cong L_2(3^m)$,*

$$\begin{cases} 3^m + 1 = 4t, \\ 3^m - 1 = 2u^c \end{cases}$$

or

$$\begin{cases} 3^m + 1 = 4t^b, \\ 3^m - 1 = 2u \end{cases}$$

with $m \geq 2$, u, t are odd primes, $b \geq 1$, $c \geq 1$ if and only if nse(G)=nse($L_2(3^m)$).

Proof. If $G \cong L_2(3^m)$, then from Lemma **??**, nse(G)=nse($L_2(3^m)$).

So we assume that nse(G)=nse($L_2(3^m)$) with $3^m + 1 = 4t, 3^m - 1 = 2u^c$. We consider $c = 1$ and $c > 1$.

- Case a. $c = 1$.

  Then $u > t > 3 > 2$.

  If $u, t, 3, 2 \in \pi(G)$, $s_u = \phi((3^m - 1)/2).3^m.(3^m + 1)/2$, $s_t = \phi(t).3^m.(3^m - 1)/2$, $s_3 = (3^m)^2 - 1$ and $s_2 = 3^m.(3^m - 1)/2$, $2 \in \pi(G)$.

  If $t \in \pi(G)$ and since $3, u \in \pi(s_t) \cap \pi(s_2)$, then by Lemma 2.10, $3, u \in \pi(G)$.

  Therefore we consider the following subcases: $\pi(G) = \{2, u\}$ and $\pi(G) = \{2, 3, u, t\}$.

  * Subcase a. $\pi(G) = \{2, u\}$.
    Since $3 \in \pi(s_2) \cap \pi(s_u)$, then by Lemma 2.10, $3 \in \pi(G)$, a contradiction.
  * Subcase b. $\pi(G) = \{2, 3, u, t\}$.
    By Lemma 2.6, $2.3 \notin \omega(G)$. It follows that the Sylow 2-subgroup of G acts fixed point freely on the set of elements of order 3, $|P_2| \mid s_3$ and $|P_3| \mid 3^m$. Similarly $3.u \notin \omega(G)$ and $|P_u| \mid s_3$, in particular $|P_u| = u$; $3.t \notin \omega(G)$ and $|P_t| \mid s_3$.
    Therefore we can assume that $|G| \mid 2^m 3^n.t^p.u^q$ for some non-negative integers m, n, p, q. On the other hand,

    $$\frac{3^m.(3^m - 1)(3^m + 1)}{2} \leq |G| \leq 3^m.(3^m - 1).(3^m + 1)$$

    and so $|G| = 3^m.(3^m - 1).(3^m + 1)/2$ or $|G| = 3^m.(3^m - 1).(3^m + 1)$.

In the following, we first prove that there is no group such that $|G| = 3^m.(3^m-1).(3^m+1)$ and nse(G)=nse($L_2(3^m)$), then from [27], get the desired result.

G is insoluble. Assume that G is soluble. Since $s_u = \phi((3^m-1)/2).3^m.(3^m+1)/2$, then $n_u = 3^m.(3^m+1)/2 = 3^m.4t$. By Lemma 2.7, $2^2 \equiv 1 \pmod{u}$, a contradiction. So G is insoluble.

There is a normal series

$$1 \lhd K \lhd L \lhd G$$

such that L/K is a simple K_i-group with $i = 3, 4$.

Let L/K be a simple K_3-group. Then from [15], L/K is isomorphic to one of the following groups: A_5, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $U_3(3)$, $L_3(3)$, $U_4(2)$.

It is easy to prove that L/K is not a simple K_3-group. For instance, assume that $L/K \cong L_2(17)$. Then $u = 17$ and so $3^m - 1 = 2.17$. The equation has no solution in $\mathbb{N}$.

Let L/K be a simple K_4-group. Similarly to the above case, we can rule out the groups which are isomorphic to (1)(2)(4) of Lemma 2.4. Therefore we consider Lemma 2.4(3) with the following three cases.

(a) $L/K \cong L_2(r)$, where r is a prime and r satisfies

$$r^2 - 1 = 2^a.3^b.v^c$$

with $a \geq 1$, $b \geq 1$, $c \geq 1$, $v > 3$ is a prime.

Since r is the largest prime divisor of L/K. Hence $u = (3^m-1)/2 = r$. That is $|L_2(r)| \mid |G|$, namely,

$$(((3^m-1)/2)^2 - 1).(3^m-1)/4 \mid 3^m.(3^m-1).(3^m+1).$$

It follows that

$$\frac{3^m-3}{16} \mid 3^m,$$

a contradiction since $2 \mid 3^m - 3$ and $2^2 \nmid 3^m - 3$.

(b) $L/K \cong L_2(2^m)$, where $2^m - 1 = u'$, $2^m + 1 = 3t'^{b'}$ with $m \geq 2$, u', t' are primes, $t' > 3$, $b' \geq 1$. Then $u = u'$. It follows that

$$|L_2(2^m)| \mid |G|,$$

namely,

$$2^m(2^m-1)(2^m+1) \mid 3^m.(3^m-1).(3^m+1),$$

a contradiction.

(c) $L/K \cong L_2(3^m)$, where

$$\begin{cases} 3^m + 1 = 4t', \\ 3^m - 1 = 2u'^{c'} \end{cases}$$

or

$$\begin{cases} 3^m + 1 = 4t'^{b'}, \\ 3^m - 1 = 2u' \end{cases}$$

with $m \geq 2$, u', t' are odd primes, $b' \geq 1$, $c' \geq 1$.
Let $\overline{G} = G/K$ and $\overline{L} = L/K$. Then

$$L_2(3^m) \leq \overline{L} \cong \overline{L}C_{\overline{G}}(\overline{L})/C_{\overline{G}}(\overline{L}) \leq \overline{G}/C_{\overline{G}}(\overline{L}) = N_{\overline{G}}(\overline{L})/C_{\overline{G}}(\overline{L}) \leq \mathrm{Aut}(\overline{L})$$

Set $M = \{xK \mid xK \in C_{\overline{G}}(\overline{L})\}$, then $G/M \cong \overline{G}/C_{\overline{G}}(\overline{L})$ and so $L_2(3^m) \leq G/M \leq \mathrm{Aut}(L_2(3^m))$. Therefore $G/M \cong L_2(3^m)$ or $G/M \cong SL_2(3^m)$.

If $G/M \cong L_2(3^m)$, then $|M| = 2$ and $M = Z(G)$. It follows that there is an element of order $2.u$, a contradiction.

If $G/M \cong SL_2(3^m)$, then $M = 1$. But $\mathrm{nse}(SL_2(3^m)) \neq \mathrm{nse}(G)$, we also rule out this case.

Therefore $|G| = 3^m.(3^m - 1).(3^m + 1)/2 = |L_2(3^m)|$. By assumption, $\mathrm{nse}(G) = \mathrm{nse}(L_2(3^m))$. So by [27], $G \cong L_2(3^m)$.

- Case b. $c > 1$.

  Then $t > u > 3 > 2$. Similarly as "$c = 1$", we have $G \cong L_2(3^m)$.

Also we can do the case "$3^m + 1 = 4t^b, 3^m - 1 = 2u$" as the case "$3^m + 1 = 4t, 3^m - 1 = 2u^c$", so we have $G \cong L_2(3^m)$.

This completes the proof of the theorem. □

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