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Optimal Convex Combination Bounds of the weighted geometric and Harmonic Means for the Centroidal Mean

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Abstract

We find the greatest value α and the least value β such that the double inequality

$$\alpha H(a,b) + (1-\alpha)S(a,b) < T(a,b) < \beta H(a,b) + (1-\beta)S(a,b)$$

holds for all $a, b > 0$ with $a \neq b$. Here $S(a,b)$ denotes the weighted geometric mean of a and b with weights $\frac{a}{a+b}$ and $\frac{b}{a+b}$, $T(a,b)$ and $H(a,b)$ denote the Centroidal and harmonic means of two positive numbers a and b, respectively.

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1 Introduction

For $a, b > 0$ with $a \neq b$ the weighted geometric mean $S(a,b)$ with weights $\frac{a}{a+b}$ and $\frac{b}{a+b}$ was introduced as follow:

$$S(a,b) = a^{\frac{a}{a+b}} b^{\frac{b}{a+b}}.$$

This mean is a special case of Gini's mean[3]. For more properties of the mean S see, e.g., [4] and [5]. Recently, the inequalities for means have been the subject of intensive research [1-6] and related references therein.

Let $T(a,b) = 2(a^2+ab+b^2)/3(a+b)$ and $H(a,b) = 2ab/(a+b)$, be the centroidal and harmonicmeans of two positive real numbers a and b with $a \neq b$.

Let $M_r(a,b) = (\frac{a^r+b^r}{2})^{\frac{1}{r}}$ denote the power mean of order $r \neq 0$ of a and b. In [6] E.Neuman and J.Sandor found the sharp bounds for the weighted geometric mean as follow:

$$M_2(a,b) < S(a,b) < \sqrt{2}M_2(a,b).$$

In [2], the authors found the greatest value α and the least value β such that the double inequality

$$\alpha A(a,b) + (1-\alpha)H(a,b) < P(a,b) < \beta A(a,b) + (1-\beta)H(a,b)$$

holds for all $a, b > 0$ with $a \neq b$, where $P(a,b) = \frac{a-b}{4\arctan(\sqrt{a/b})-\pi}$.

The purpose of the present paper is to find the greatest value α and the least value β such that the double inequality

$$\alpha H(a,b) + (1-\alpha)S(a,b) < T(a,b) < \beta H(a,b) + (1-\beta)S(a,b)$$

holds for all $a, b > 0$ with $a \neq b$.

2 Main Results

Theorem 2.1 *The double inequality*

$$\alpha H(a,b) + (1-\alpha)S(a,b) < T(a,b) < \beta H(a,b) + (1-\beta)S(a,b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \geq \frac{1}{3}$ and $\beta \leq \frac{1}{9}$.

Proof. Firstly, we prove that

$$T(a,b) < \frac{1}{9}H(a,b) + \frac{8}{9}S(a,b), \tag{2.1}$$

$$T(a,b) > \frac{1}{3}H(a,b) + \frac{2}{3}S(a,b), \tag{2.2}$$

for all $a, b > 0$ with $a \neq b$.

Without loss of generality, we assume $a > b$. Let $t = a/b > 1$ and $p \in \{\frac{1}{9}, \frac{1}{3}\}$. Then we have

$$\begin{aligned} &\frac{1}{b}[T(a,b) - pH(a,b) - (1-p)S(a,b)] \\ = &\frac{2}{3}\frac{t^2+t+1}{t+1} - \frac{2pt}{1+t} - (1-p)t^{\frac{t}{1+t}}. \end{aligned} \tag{2.3}$$

Let

$$f(t)=\frac{\frac{2}{3}\frac{t^2+t+1}{t+1}-\frac{2pt}{1+t}}{(1-p)t^{\frac{t}{1+t}}}, \tag{2.4}$$

$$g(t)=\ln f(t)=\ln(\frac{2}{3}\frac{t^2+t+1}{t+1}-\frac{2pt}{1+t})-\ln(1-p)-\frac{t}{1+t}\ln t. \tag{2.5}$$

Simple computations lead to

$$\lim_{t\to 1^+} g(t)=0, \tag{2.6}$$

$$\lim_{t\to +\infty} g(t)=\ln\frac{2}{3(1-p)}, \tag{2.7}$$

$$g'(t)=\frac{g_1(t)}{(1+t)^2[\frac{2}{3}(t^2+t+1)-2pt]}, \tag{2.8}$$

where

$$\begin{aligned} g_1(t) &= [\tfrac{2}{3}(2t+1)-2p](1+t)^2-2(1+t)[\tfrac{2}{3}(t^2+t+1)-2pt] \\ &\quad -[\frac{2}{3}(t^2+t+1)-2pt]\ln t. \end{aligned} \tag{2.9}$$

$$\lim_{t\to 1^+} g_1(t)=0, \tag{2.10}$$

$$\lim_{t\to +\infty} g_1(t)=-\infty, \tag{2.11}$$

$$g_1'(t)=(\frac{2}{3}+4p)t-\frac{2}{3}+2p-\frac{2}{3t}-(\frac{4t+2}{3}-2p)\ln t, \tag{2.12}$$

$$\lim_{t\to 1^+} g_1'(t)=6p-\frac{2}{3}, \tag{2.13}$$

$$\lim_{t\to +\infty} g_1'(t)=-\infty. \tag{2.14}$$

$$g_1''(t)=4p-\frac{2}{3}+\frac{2}{3t^2}-\frac{4}{3}\ln t-\frac{2}{3t}+\frac{2p}{t}, \tag{2.15}$$

$$\lim_{t\to 1^+} g_1''(t)=6p-\frac{2}{3}, \tag{2.16}$$

$$\lim_{t\to +\infty} g_1''(t)=-\infty. \tag{2.17}$$

$$g_1^{(3)}(t)=\frac{2}{3t^3}g_2(t), \tag{2.18}$$

where

$$g_2(t)=-2t^2+t-3pt-2, \tag{2.19}$$

$$\lim_{t\to 1^+} g_2(t)=-3-3p, \tag{2.20}$$

$$g_2'(t) = -4t + 1 - 3p, \tag{2.21}$$

$$\lim_{t\to 1^+} g_2'(t) = -3 - 3p, \tag{2.22}$$

$$g_2''(t) = -4. \tag{2.23}$$

From (2.23) and (2.22) we know $g_2'(t) < 0$ for $t > 1$, hence $g_2(t)$ is strictly decreasing in $[1, +\infty)$. It follows from (2.20) and (2.18) we get that $g_1^{(3)}(t) < 0$ for $t > 1$, hence $g_1''(t)$ is strictly decreasing in $[1, +\infty)$.

Now we divide the proof into two cases:

Case 1 If $p = \frac{1}{9}$.

(2.16) leads to $g_1''(t) < 0$ for $t > 1$, hence $g_1'(t)$ is strictly decreasing in $[1, +\infty)$. From (2.6), (2.8), (2.10), (2.12), (2.13) and $\frac{2}{3}(t^2 + t + 1) - 2pt > 0$ for $p = \frac{1}{9}$, we can get $g(t) < 0$ for $t > 1$. Then inequality (2.1) follows from (2.3)-(2.5).

Case 2 If $p = \frac{1}{3}$.

Then from (2.16) and (2.17) together with the monotonicity of $g_1''(t)$ we clearly see that there exists $\lambda_1 > 1$ such that $g_1''(t) > 0$ for $t \in [1, \lambda_1)$ and $g_1''(t) < 0$ for $t \in (\lambda_1, +\infty)$, hence $g_1'(t)$ is strictly increasing in $[1, \lambda_1]$ and strictly decreasing in $[\lambda_1, +\infty)$.

From (2.13) and (2.14) together with the monotonicity of $g_1'(t)$ in $[1, \lambda_1]$ and $[\lambda_1, +\infty)$ we know that there exists $\lambda_2 > \lambda_1$ such that $g_1'(t) > 0$ for $t \in [1, \lambda_2)$ and $g_1'(t) < 0$ for $t \in (\lambda_2, +\infty)$, hence $g_1(t)$ is strictly increasing in $[1, \lambda_2]$ and strictly decreasing in $[\lambda_2, +\infty)$.

From (2.10) and (2.11) together with the monotonicity of $g_1(t)$ we clearly see that there exists $\lambda_3 > \lambda_2$ such that $g_1(t) > 0$ for $t \in [1, \lambda_3)$ and $g_1(t) < 0$ for $t \in (\lambda_3, +\infty)$, hence $g(t)$ is strictly increasing in $[1, \lambda_3]$ and strictly decreasing in $[\lambda_3, +\infty)$.

From (2.6) and (2.7) together with the monotonicity of $g(t)$ in $[1, \lambda_3]$ and $[\lambda_3, +\infty)$, we can get that $g(t) > 0$ for $t \in [1, +\infty)$, from (2.3) and (2.4) we get (2.2).

Secondly, we prove that $\frac{1}{9}H(a, b)+\frac{8}{9}S(a, b)$ is the best possible upper convex combination bound of the weighted geometric and harmonic means for the centroidal mean T(a,b).

For any $t > 1$ and $\beta \in R$, we have

$$T(t, 1) - \beta H(t, 1) - (1 - \beta)S(t, 1) = \frac{h(t)}{3(1 + t)}, \tag{2.26}$$

where

$$h(t) = 2(t^2 + t + 1) - 6\beta t - 3(1 - \beta)(1 + t)t^{\frac{t}{1+t}}. \tag{2.27}$$

It follows from (2.27) that

$$h(1) = h'(1) = 0, \tag{2.28}$$

$$h''(1) = \frac{1}{2}(9\beta - 1). \tag{2.29}$$

If $\beta > \frac{1}{9}$, then (2.29) leads to

$$h''(1) > 0. \tag{2.30}$$

From (2.30) and the continuity of $h''(t)$ we see that there exists $\delta = \delta(\beta) > 0$ such that

$$h''(t) > 0, \quad \forall t \in [1, 1+\delta). \tag{2.31}$$

Then (2.28) and (2.31) imply that

$$h(t) > 0, \quad \forall t \in [1, 1+\delta). \tag{2.32}$$

Therefore, $\beta H(t,1) + (1-\beta)S(t,1) < T(t,1)$ for $t \in [1, 1+\delta)$ follows from (2.26) and (2.32).

Finally, we prove that $\frac{1}{3}H(a,b) + \frac{2}{3}S(a,b)$ is the best possible lower convex combination bound of the weighted geometric and harmonic means for the centroidal mean T(a,b).

In fact, for $\alpha < \frac{1}{3}$, we have

$$\lim_{x \to +\infty} \frac{\alpha H(1,x) + (1-\alpha)S(1,x)}{T(1,x)} = \frac{3}{2}(1-\alpha) > 1. \tag{2.33}$$

Inequality (2.33) implies that for any $\alpha < \frac{1}{3}$ there exists $X = X(\alpha) > 1$ such that $\alpha H(1,x) + (1-\alpha)S(1,x) > T(1,x)$ for $x \in (X, +\infty)$.

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