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On the Neutrix Composition of Distributions of the Delta Function and the Function $[\cosh_+^{-1}(x+1)]^r$

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Abstract

Let F be a distribution in $\mathcal{D}'$ and let f be a locally summable function. The composition $F(f(x)))$ of F and f is said to exist and be equal the distribution $h(x)$ if the neutrix limit of the sequence $\{F_n(f(x))\}$ is equal to $h(x)$, where $F_n(x) = F(x) * \delta_n(x)$ for $n = 1, 2, \ldots,$ and $\{\delta_n(x)\}$ is a certain sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. The function $\cosh_+^{-1}(x+1)$ is defined by

$$\cosh_+^{-1}(x+1) = H(x)\cosh^{-1}(|x|+1),$$

where $H(x)$ denotes Heaviside's function. It is proved that the neutrix composition $\delta^{(s)}[\cosh_+^{-1}(x+1)]^r$ exists and

$$\delta^{(s)}[\cosh_+^{-1}(x+1)]^r = \sum_{k=0}^{rs+r-2}\sum_{j=0}^{k}\sum_{i=0}^{j}\frac{(-1)^{s+k-j}s!}{r2^{j+2}}\binom{k}{j}\binom{j}{i} \times\frac{[(j-2i+1)^{rs+r-1}-(j-2i-1)^{rs+r-1}]}{(rs+r-1)!}\delta^{(k)}(x),$$

for $r, s = 1, 2, \ldots$.

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1 Introduction

In the following, let $\mathcal{D}$ be the space of infinitely differentiable functions φ with compact support and let $\mathcal{D}[a, b]$ be the space of infinitely differentiable functions with support contained in the interval $[a,b]$. let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$ and let $\mathcal{D}'[a,b]$ be the space of distributions defined on $\mathcal{D}[a,b]$.

Now let $\rho(x)$ be a function in $\mathcal{D}$ satisfying the following properties:
(i) $\rho(x) = 0$ for $|x| \geq 1$,
(ii) $\rho(x) \geq 0$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\displaystyle\int_{-1}^{1} \rho(x)\,dx = 1$.

By putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \dots$, we have $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. Further, if F is a distribution in $\mathcal{D}'$ and $F_n(x) = F(x) * \delta_n(x) = \langle F(x-t), \varphi(x)\rangle$, then $\{F_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to $F(x)$.

Now let $f(x)$ be an infinitely differentiable function having a single simple root at the point $x = x_0$. Gel'fand and Shilov defined the distribution $\delta^{(r)}(f(x))$ by the equation

$$\delta^{(r)}(f(x)) = \frac{1}{|f'(x_0)|}\left[\frac{1}{|f'(x)|}\frac{d}{dx}\right]^r \delta(x-x_0),$$

for $r = 0, 1, 2, \dots$, see [9].

The following definition [2] is a generalization of Gel'fand and Shilov's definition of the composition involving the delta function [9].

Definition 1.1 *Let F be a distribution in $\mathcal{D}'$ and let f be a locally summable function. We say that the neutrix composition $F(f(x))$ exists and is equal to h on the open interval (a,b) if*

$$\mathop{\mathrm{N-lim}}_{n\to\infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle$$

for all φ in $\mathcal{D}[a,b]$, where $F_n(x) = F(x) * \delta_n(x)$ for $n = 1, 2, \dots$ and N is the neutrix, see [1], having domain N' the positive integers and range N'' the real numbers, with negligible functions which are finite linear sums of the functions

$$n^\lambda \ln^{r-1} n,\ \ln^r n : \qquad \lambda > 0,\ r = 1, 2, \dots$$

and all functions which converge to zero in the normal sense as n tends to infinity.

In particular, we say that the composition $F(f(x))$ exists and is equal to h on the open interval (a,b) if

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle$$

for all φ in $\mathcal{D}[a,b]$.

Note that taking the neutrix limit of a function $f(n)$, is equivalent to taking the usual limit of Hadamard's finite part of $f(n)$. If f, g are two distributions then in the ordinary sense the composition $f(g)$ does not necessary exist. Thus the deifintion of the neutrix composition of distributions was originally given in [2] but was then simply called the composition of distributions.

The following theorems were proved in [3], [4], [7], and [8] respectively.

Theorem 1.2 *The neutrix composition $\delta^{(s)}(\operatorname{sgn} x|x|^\lambda)$ exists and*

$$\delta^{(s)}(\operatorname{sgn} x|x|^\lambda) = 0$$

for $s = 0, 1, 2, \ldots$ and $(s+1)\lambda = 1, 3, \ldots$ and

$$\delta^{(s)}(\operatorname{sgn} x|x|^\lambda) = \frac{(-1)^{(s+1)(\lambda+1)}s!}{\lambda[(s+1)\lambda-1]!}\delta^{((s+1)\lambda-1)}(x)$$

for $s = 0, 1, 2, \ldots$ and $(s+1)\lambda = 2, 4, \ldots$.

Theorem 1.3 *The compositions $\delta^{(2s-1)}(\operatorname{sgn} x|x|^{1/s})$ and $\delta^{(s-1)}(|x|^{1/s})$ exist and*

$$\begin{aligned} \delta^{(2s-1)}(\operatorname{sgn} x|x|^{1/s}) &= \frac{(2s)!}{2}\delta'(x), \\ \delta^{(s-1)}(|x|^{1/s}) &= (-1^s\delta(x) \end{aligned}$$

for $s = 1, 2, \ldots$.

Theorem 1.4 *The neutrix composition $\delta^{(s)}[(\sinh^{-1} x_+)^{1/r}]$ exists and*

$$\delta^{(s)}[(\sinh^{-1} x_+)^{1/r}] = \sum_{k=0}^{M-1}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{i+k}rc_{s,k,i}}{2^{k+1}k!}\delta^{(k)}(x)$$

for $s = 0, 1, 2, \ldots$ and $r = 1, 2, \ldots$, where M is the smallest positive integer greater than $(s - r^2 + 1)/r$ and

$$c_{s,k,i} = \begin{cases} \dfrac{[(k-2i+1)^p + (k-2i-1)^p](-1)^s s!}{2p!}, & p = \dfrac{s-r+1}{r} \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 1.5 *The neutrix composition* $\delta^{(s)}[(\sinh^{-1} x_+)^r]$ *exists and*

$$\delta^{(s)}[(\sinh^{-1} x_+)^r] = \sum_{k=0}^{rs-r-1} \sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^k r c_{s,k,i}}{2^{k+1} k!} \delta^{(k)}(x)$$

for $s = 0, 1, 2, \ldots$ *and* $r = 1, 2, \ldots,$ *where*

$$c_{s,k,i} = \frac{(-1)^s s! [(k-2i+1)^{rs-r+1} + (k-2i-1)^{rs+r-1}]}{2(rs+r-1)!}$$

The following two theorems were proved in [6]

Theorem 1.6 *The neutrix composition* $\delta^{(rs+r-1)}[\cosh_+^{-1}(x+1)]^{1/r}$ *exists and*

$$\begin{aligned}\delta^{(rs+r-1)}[\cosh_+^{-1}(x+1)]^{1/r} &= \sum_{k=0}^{s-1} \sum_{j=0}^{k} \sum_{i=0}^{j} \frac{(-1)^{rs+r-j-1} r}{2^{j+2}} \binom{k}{j} \binom{j}{i} \\ &\quad \times \frac{[(j-2i+1)^s - (j-2i-1)^s](rs+r-1)!}{k! s!} \delta^{(k)}(x),\end{aligned}$$

for $r, s = 1, 2, \ldots.$

Theorem 1.7 *The neutrix composition* $\delta^{(rs+r-1)}[\cosh_+^{-1}(x+1)^{1/r}]$ *exists and*

$$\begin{aligned}\delta^{(rs+r-1)}[\cosh_+^{-1}(x+1)^{1/r}] &= \\ = \sum_{k=0}^{s-1} \sum_{j=0}^{rk+r-1} \sum_{i=0}^{j} \frac{(-1)^{rs+rk+k-j} r}{2^{j+2}} &\binom{rk+r-1}{j} \binom{j}{i} \\ &\times \frac{[(j-2i+1)^{rs+r-1} - (j-2i-1)^{rs+r-1}]}{k!} \delta^{(k)}(x),\end{aligned}$$

for $r, s = 1, 2, \ldots.$

2 Main Results

In the following we define the functions $\cosh_+^{-1}(x+1)$ and $\cosh_-^{-1}(|x|+1)$ by

$$\cosh_+^{-1}(x+1) = H(x)\cosh^{-1}(|x|+1), \quad \cosh_-^{-1}(x+1) = H(-x)\cosh^{-1}(|x|+1).$$

It follows that

$$\cosh^{-1}(|x|+1) = \cosh_+^{-1}(x+1) + \cosh_-^{-1}(|x|+1).$$

Now we need the following lemma, which can be proved by induction:

Lemma 2.1

$$\int_{-1}^{1} t^i \rho^{(s)}(t)\,dt = \begin{cases} 0, & 0 \le i < s, \\ (-1)^s s!, & i = s \end{cases}$$

and

$$\int_{0}^{1} t^s \rho^{(s)}(t)\,dt = \tfrac{1}{2}(-1)^s s!$$

for $s = 0, 1, 2, \ldots$.

Now we prove the following theorem:

Theorem 2.2 *The neutrix composition* $\delta^{(s)}[\cosh_+^{-1}(x+1)]^r$ *exists and*

$$\begin{aligned} \delta^{(s)}[\cosh_+^{-1}(x+1)]^r &= \sum_{k=0}^{rs+r-2} \sum_{j=0}^{k} \sum_{i=0}^{j} \frac{(-1)^{s+j} s!}{r 2^{j+2}} \binom{k}{j} \binom{j}{i} \\ &\times \frac{[(j-2i+1)^{rs+r-1} - (j-2i-1)^{rs+r-1}]}{(rs+r-1)!} \delta^{(k)}(x), \end{aligned} \tag{1}$$

for $r, s = 1, 2, \ldots$.

Proof. It is clear that $\delta^{(s)}[\cosh_+^{-1}(x+1)]^r = 0$ on any interval not containing the origin and so we only need prove equation (1) on the interval $[-1, 1]$. To do this, we first of all have to evaluate

$$\begin{aligned} \int_{-1}^{1} x^k \delta_n^{(s)}[\cosh_+^{-1}(x+1)]^r\,dx &= \int_{0}^{1} x^k \delta_n^{(s)}[\cosh^{-1}(x+1)]^r\,dx \\ &\quad + \int_{-1}^{0} x^k \delta_n^{(s)}(0)\,dx \\ &= n^{s+1} \int_{0}^{1} x^k \rho^{(s)}[n(\cosh^{-1}(x+1)]^r\,dx \\ &\quad + n^{s+1} \int_{-1}^{0} x^k \rho^{(s)}(0)\,dx \\ &= I_1 + I_2. \end{aligned} \tag{2}$$

It is obvious that

$$\operatorname*{N-lim}_{n\to\infty} I_2 = 0 \tag{3}$$

Using the substitution $t = n[\cosh^{-1}(x+1)]^r$ or

$$x = \cosh(t/n)^{1/r} - 1,$$

we have

$$\begin{aligned}
I_1 &= \frac{n^{s+1-1/r}}{r}\int_0^1 t^{\frac{1-r}{r}}[\cosh(t/n)^{1/r}-1]^k\sinh(t/n)^{1/r}\rho^{(s)}(t)\,dt \\
&= \frac{n^{s+1-1/r}}{r}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\int_0^1 t^{\frac{1-r}{r}}\cosh^j(t/n)^{1/r}\sinh(t/n)^{1/r}\rho^{(s)}(t)\;dt \\
&= \frac{n^{s+1-1/r}}{r}\sum_{j=0}^{k}\frac{(-1)^{k-j}}{2^{j+1}}\binom{k}{j}\int_0^1 t^{\frac{1-r}{r}}(e^{(t/n)^{1/r}}+e^{-(t/n)^{1/r}})^j \\
&\qquad\times(e^{(t/n)^{1/r}}-e^{-(t/n)^{1/r}})\rho^{(s)}(t)\,dt \\
&= \frac{n^{s+1-1/r}}{r}\sum_{j=0}^{k}\sum_{i=0}^{j}\frac{(-1)^{k-j}}{2^{j+1}}\binom{k}{j}\binom{j}{i}\int_0^1 t^{\frac{1-r}{r}} \\
&\qquad\times(e^{(j-2i+1)(t/n)^{1/r}}-e^{(j-2i-1)(t/n)^{1/r}})\rho^{(s)}(t)\,dt
\end{aligned}$$

and it follows that

$$\begin{aligned}
\mathrm{N}\!-\!\lim_{n\to\infty} I_1 &= \sum_{j=0}^{k}\sum_{i=0}^{j}\frac{(-1)^{k-j}}{r2^{j+1}}\binom{k}{j}\binom{j}{i}\frac{(j-2i+1)^{rs+r-1}-(j-2i-1)^{rs+r-1}}{(rs+r-1)!} \\
&\qquad\times\int_0^1 t^s\rho^{(s)}(t)\,dt \\
&= \sum_{j=0}^{k}\sum_{i=0}^{j}\frac{(-1)^{s+k-j}s!}{r2^{j+2}}\binom{k}{j}\binom{j}{i} \\
&\qquad\times\frac{[(j-2i+1)^{rs+r-1}-(j-2i-1)^{rs+r-1}]}{(rs+r-1)!}. \qquad (4)
\end{aligned}$$

and it now follows from equations (2), (3) and (4) that

$$\begin{aligned}
\mathrm{N}\!-\!\lim_{n\to\infty}\int_{-1}^{1} x^k\delta_n^{(s)}[\cosh_+^{-1}(x+1)]^{1/r}\,dx &= \sum_{j=0}^{k}\sum_{i=0}^{j}\frac{(-1)^{s+k-j}s!}{r2^{j+2}}\binom{k}{j}\binom{j}{i} \\
&\qquad\times\frac{[(j-2i+1)^{rs+r-1}-(j-2i-1)^{rs+r-1}]}{(rs+r-1)!}, \qquad (5)
\end{aligned}$$

for $k = 1, 2, \ldots$.

Next, when $k = rs + r - 1$, we note that

$$[\cosh(t/n)^{1/r}-1]^{rs+r-1}\sinh(t/n)^{1/r} = O(n^{-s-2+1/r})$$

and it follows that

$$\begin{aligned}
|I_1| &\le \frac{n^{s+1-1/r}}{r}\int_0^1 |t^{\frac{1-r}{r}}[\cosh(t/n)^{1/r}-1]^k\sinh(t/n)^{1/r}\rho^{(s)}(t)|\,dt \\
&= O(n^{-1}).
\end{aligned}$$

Hence, if $\psi(x)$ is an arbitrary continuous function, then

$$\lim_{n\to\infty}\int_0^1 x^{rs+r-1}\delta_n^{(s)}[\cosh^{-1}(x+1)]^r\psi(x)\,dx=0, \tag{6}$$

for $s=1,2,\ldots$.

Further,

$$\int_{-1}^0 x^{rs+r-1}\delta_n^{(s)}(0)\psi(x)\,dx \;=\; n^{s+1}\int_{-1}^0 x^{rs+r-1}\rho^{(s)}(0)\psi(x)\,dx$$

and it follows that

$$\mathrm{N}\!-\!\lim_{n\to\infty}\int_{-1}^0 x^{rs+r-1}\delta_n^{(s)}(0)\psi(x)\,dx=0. \tag{7}$$

Now let $\varphi(x)$ be an arbitrary function in $\mathcal{D}[-1,1]$. By Taylor's Theorem we have

$$\varphi(x)=\sum_{k=0}^{rs+r-2}\frac{x^k}{k!}\varphi^{(k)}(0)+\frac{x^{rs+r-1}}{(rs+r-1)!}\varphi^{(rs+r-1)}(\xi x),$$

where $0<\xi<1$. Then

$$\begin{aligned}
\mathrm{N}\!-\!\lim_{n\to\infty}\langle\delta_n^{(s)}[\cosh_+^{-1}(x+1)]^r,\varphi(x)\rangle &= \mathrm{N}\!-\!\lim_{n\to\infty}\int_{-1}^1\delta_n^{(s)}[\cosh_+^{-1}(x+1)]^r\varphi(x)\,dx\\
&= \mathrm{N}\!-\!\lim_{n\to\infty}\sum_{k=0}^{rs+r-2}\frac{\varphi^{(k)}(0)}{k!}\int_{-1}^1 x^k\delta_n^{(s)}[\cosh^{-1}(x+1)]^r\,dx\\
&+\mathrm{N}\!-\!\lim_{n\to\infty}\int_0^1\frac{x^{rs+r-1}}{(rs+r-1)!}\delta_n^{(s)}[\cosh^{-1}(x+1)]^r\varphi^{(rs+r-1)}(\xi x)\,dx\\
&+\mathrm{N}\!-\!\lim_{n\to\infty}\int_{-1}^0\frac{x^{rs+r-1}}{(rs+r-1)!}\delta_n^{(s}(0)\varphi^{(rs+r-1)}(\xi x)\,dx\\
&=\sum_{k=0}^{rs+r-2}\sum_{j=0}^{k}\sum_{i=0}^{j}\frac{(-1)^{s+k-j}s!}{r2^{j+2}}\binom{k}{j}\binom{j}{i}\\
&\times\frac{[(j-2i+1)^{rs+r-1}-(j-2i-1)^{rs+r-1}]}{(rs+r-1)!}\varphi^{(k)}(0)
\end{aligned}$$

on using equations (5), (6) and (7). This completes the proof of the theorem.

Replacing x by $-x$ in Theorem7, we get

Corollary 2.3 *The neutrix composition $\delta^{(s)}[\cosh_-^{-1}(|x|+1)]^r$ exists and*

$$\begin{aligned}
\delta^{(s)}[\cosh_-^{-1}(|x|+1)]^r &= \sum_{k=0}^{rs+r-2}\sum_{j=0}^{k}\sum_{i=0}^{j}\frac{(-1)^{s+k+j}s!}{r2^{j+2}}\binom{k}{j}\binom{j}{i}\\
\times\frac{[(j-2i+1)^{rs+r-1}-(j-2i-1)^{rs+r-1}]}{(rs+r-1)!}\delta^{(k)}(x),
\end{aligned} \tag{8}$$

for $r,s=1,2,\ldots$.

Corollary 2.4 *The neutrix composition* $\delta^{(s)}[\cosh^{-1}(|x|+1)]^r$ *exists and*

$$\delta^{(s)}[\cosh^{-1}(|x|+1)]^r = \sum_{k=0}^{rs+r-2}\sum_{j=0}^{k}\sum_{i=0}^{j}\frac{(-1)^{s+j}[1+(-1)^k]s!}{r2^{j+2}}\binom{k}{j}\binom{j}{i} \times\frac{[(j-2i+1)^{rs+r-1}-(j-2i-1)^{rs+r-1}]}{(rs+r-1)!}\delta^{(k)}(x), \qquad (9)$$

for $r, s = 1, 2, \ldots$.

Proof. Equation follows immediately on noting that

$$\delta^{(s)}[\cosh^{-1}(|x|+1)]^r = \delta^{(s)}[\cosh_+^{-1}(x+1)]^r + \delta^{(s)}[\cosh_-^{-1}(|x|+1)]^r.$$

References

[1] J.G. van der Corput, *Introduction to the neutrix calculus,* J. Analyse Math., **7**(1959), 291-398.

[2] B. Fisher, *On defining the change of variable in distributions,* Rostock. Math. Kolloq., **28**(1985), 75-86.

[3] B. Fisher, *The delta function and the composition of distributions*, Dem. Math. **35**(1)(2002), 117-123.

[4] B. Fisher, *The composition and neutrix composition of distributions*, Proc. Math. Methods Engineering, Springer, 2006, 59-69.

[5] B. Fisher, *Some results on the neutrix composition of the delta function*, Filomat, **26**(6)(2012), 1247-1256.

[6] B. Fisher and F. Al-Sirehy, *Some results on the Neutrix Composition of Distributions Involving the Delta Function and the Function* $\cosh_+^{-1}(x+1)$,

[7] B. Fisher and A. Kilicman,*On the composition and neutrix composition of the delta function and powers of the inverse hyperbolic sine function*, Integral Transforms Spec. Funct., **21**(12)(2010), 935-944.

[8] B. Fisher and A. Kilicman,*On the neutrix composition of the delta and inverse hyperbolic sine functions*, Journal of Applied Mathematics, **2011**, Article ID 612353, 12 pages, 2011.

[9] I.M. Gel'fand and G.E. Shilov, "Generalized Functions", Vol. I, Academic Press, 1964.

[10] T. Kraiweeradechachai, S. Orankitjaroen, B. Fisher and E. Özçağ, *Further results on the neutrix composition of the delta function*, East-West J. Math., **11**(2)(2009), 151-164.

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