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Rota Baxter operators of the simple 3-Lie algebra I

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Abstract

This paper studies Rota-Baxter operators on the simple 3-Lie algebras over the complex field. It is proved that there does not exist Rota-Baxter operators of weight zero with rank 3 on the simple 3-Lie algebras. And it provides the Rote-Baxter operators of weight zero with rank $1, 2, 4$, respectively .

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1 Introduction

We know that 3-Lie algebras [1] have wide applications in many fields of mathematics and mathematical physics[2]. In recent years, kinds of multiple algebras are studied [3-5]. For example, Rota Baxter 3-Lie algebra was introduced in the paper [5], and the structure of Rota Baxter 3-Lie algebras is discussed. Rota-Baxter (associative) algebras, originated from the work of G. Baxter [6] in probability and populated by the work of Cartier and Rota [7] have connections with many areas of mathematics and physics, including combinatorics, number theory, operads and quantum field theory. In particular Rota-Baxter algebras have played an important role in the Hopf algebra approach of renormalization of perturbative quantum field theory of Connes and Kreimer [7], as well as in the application of the renormalization method in solving divergent problems in number theory [8].

In this paper we investigate the existence of Rota-Baxter operators of the weight zero on the simple 3-Lie algebras over the complex field. First we introduce some basic notions.

A 3-Lie algebra is a vector space A over a field F endowed with a 3-ary multi-linear skew-symmetric operation $[\ ,\ ,\ ]$ satisfying the 3-Jacobi identity, $\forall x_1, x_2, x_3, y_2, y_3 \in A$,

$$[[x_1, x_2, x_3], y_2, y_3] = \sum_{i=1}^{3}[x_1, \cdots, [x_i, y_2, y_3], \cdots, x_3],\ \forall x_1, x_2, x_3 \in L. \quad (1)$$

Let A be a 3-Lie algebra, $\lambda \in F$, if a linear mapping $P : A \to A$ satisfies

$$\begin{aligned}[P(x_1), P(x_2), P(x_3)] = P([P(x_1), P(x_2), x_3] + [P(x_1), x_2, P(x_3)] \\ +[x_1, P(x_2), P(x_3)] + \lambda[P(x_1), x_2, x_3] + \lambda[x_1, P(x_2), x_3] \\ +\lambda[x_1, x_2, P(x_3)] + \lambda^2[x_1, x_2, x_3])\end{aligned}$$

P is called a Rota-Baxter operator of weight λ, and $(A, [, ,], P)$ is called a Rota-Baxter 3-Lie algebra. When $\lambda = 0$, we have

$$\begin{aligned}&[P(x_1), P(x_2), P(x_3)] \\ &= P([P(x_1), P(x_2), x_3] + [P(x_1), x_2, P(x_3)] + [x_1, P(x_2), P(x_3)]). \quad (2)\end{aligned}$$

2 Main results

In this section we study the Rota-Baxter operators on the simple 3-Lie algebras over the complex field F. From paper [9], there exists only one simple 3-Lie algebra, that is, 4-dimensional 3-Lie algebra A in the following multiplication

$$\begin{cases} [e_1, e_2, e_3] = e_4, \\ [e_1, e_2, e_4] = e_3, \\ [e_1, e_3, e_4] = e_2, \\ [e_2, e_3, e_4] = e_1, \end{cases} \quad (3)$$

where e_1, e_2, e_3, e_4 is a basis of the 3-Lie algebra A.

Let $P : A \to A$ be a linear mapping. Set $P(e_i) = \sum_{j=1}^{4} a_{ij}e_j,\ a_{ij} \in F, 1 \leq i, j \leq 4$. Then the matrix form of P in the basis e_1, e_2, e_3, e_4 is

$$M(P) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}.$$

The rank of the matrix $M(P)$ is called the rank of P and is denoted by $R(P)$.

Theorem *There does not exist Rota-Baxter operators P with $R(P) = 3$ of weight zero on the simple 3-Lie algebra.*

Proof By Eq.(2) and (3), for $1 \le l < m < n \le 4$, we have

$$
\begin{aligned}
&[P(e_l), P(e_m), P(e_n)]\\
&= [\sum_{j=1}^{4} a_{lj}e_j, \sum_{j=1}^{4} a_{mj}e_j, \sum_{j=1}^{4} a_{nj}e_j]\\
&= (a_{l1}a_{m2}a_{n3} - a_{l1}a_{m3}a_{n2})e_4 + (a_{l1}a_{m2}a_{n4} - a_{l1}a_{m4}a_{n2})e_3 + (a_{l1}a_{m3}a_{n4} - a_{l1}a_{m4}a_{n3})e_2\\
&+(a_{l2}a_{m3}a_{n4} - a_{l2}a_{m4}a_{n3})e_1 + (a_{l2}a_{m4}a_{n1} - a_{l2}a_{m1}a_{n4})e_3 + (a_{l2}a_{m3}a_{n1} - a_{l2}a_{m1}a_{n3})e_4\\
&+(a_{l3}a_{m4}a_{n2} - a_{l3}a_{m2}a_{n4})e_1 + (a_{l3}a_{m4}a_{n1} - a_{l3}a_{m1}a_{n4})e_2 + (-a_{l3}a_{m2}a_{n1} + a_{l3}a_{m1}a_{n2})e_4\\
&+(a_{l4}a_{m2}a_{n3} - a_{l4}a_{m3}a_{n2})e_1 + (a_{l4}a_{m1}a_{n3} - a_{l4}a_{m3}a_{n1})e_2 + (-a_{l4}a_{m2}a_{n1} + a_{l4}a_{m1}a_{n2})e_3\\
&= [a_{l2}(a_{m3}a_{n4} - a_{m4}a_{n3}) + a_{l3}(a_{m4}a_{n2} - a_{m2}a_{n4}) + a_{l4}(a_{m2}a_{n3} - a_{m3}a_{n2})]e_1\\
&+[a_{l1}(a_{m3}a_{n4} - a_{m4}a_{n3}) + a_{l3}(a_{m4}a_{n1} - a_{m1}a_{n4}) + a_{l4}(a_{m1}a_{n3} - a_{m3}a_{n1})]e_2\\
&+[a_{l1}(a_{m2}a_{n4} - a_{m4}a_{n2}) + a_{l2}(a_{m4}a_{n1} - a_{m1}a_{n4}) + a_{l4}(a_{m1}a_{n2} - a_{m2}a_{n1})]e_3\\
&+[a_{l1}(a_{m2}a_{n3} - a_{m3}a_{n2}) + a_{l2}(a_{m3}a_{n1} - a_{m1}a_{n3}) + a_{l3}(a_{m1}a_{n2} - a_{m2}a_{n1})]e_4.
\end{aligned}
$$

$$
\begin{aligned}
&P([P(e_l), P(e_m), e_n] + [P(e_l), e_m, P(e_n)] + [e_l, P(e_m), P(e_n)])\\
&=P([\sum_{j=1}^{4} a_{lj}e_j, \sum_{j=1}^{4} a_{mj}e_j, e_n] + [\sum_{j=1}^{4} a_{lj}e_j, e_m, \sum_{j=1}^{4} a_{nj}e_j])\\
&+P([e_l, \sum_{j=1}^{4} a_{mj}e_j, \sum_{j=1}^{4} a_{nj}e_j])\\
&= P((a_{l4}a_{m2} - a_{l2}a_{m4} - a_{l3}a_{n4} + a_{l4}a_{n3})e_1 + (a_{l4}a_{m1} - a_{l1}a_{m4} + a_{m3}a_{n4} - a_{m4}a_{n3})e_2\\
&+(a_{l1}a_{n4} - a_{l4}a_{n1} + a_{m2}a_{n4} - a_{m4}a_{n2})e_3 + (a_{l1}a_{m2} - a_{l2}a_{m1} + a_{l1}a_{m3}\\
&-a_{l3}a_{n1} + a_{m2}a_{n3}e_4 - a_{m3}a_{n2})e_4)\\
&=(a_{l4}a_{m2} - a_{l2}a_{m4} - a_{l3}a_{n4} + a_{l4}a_{n3}) \sum_{j=1}^{4} a_{1j}e_j + (a_{l4}a_{m1} - a_{l1}a_{n4} + a_{m3}a_{n4}\\
&-a_{m4}a_{n3}) \sum_{j=1}^{4} a_{2j}e_j + (a_{l1}a_{n4} - a_{l4}a_{n1} + a_{m2}a_{n4} - a_{m4}a_{n2}) \sum_{j=1}^{4} a_{3j}e_j\\
&+(a_{l1}a_{m2} - a_{l2}a_{m1} + a_{l1}a_{n3} - a_{l3}a_{n1} + a_{m2}a_{n3}e_4 - a_{m3}a_{n2}) \sum_{j=1}^{4} a_{4j}e_j.
\end{aligned}
$$

Since $R(P) = 3$, without loss of generality, we may suppose $P(e_1)$, $P(e_2)$, $P(e_3)$ are linearly independent. Then vectors

$$\alpha_1 = (a_{11}, a_{12}, a_{13}, a_{14}), \alpha_2 = (a_{21}, a_{22}, a_{23}, a_{24}), \alpha_3 = (a_{31}, a_{32}, a_{33}, a_{34}),$$

are linearly independent.

If $P(e_4) = 0$.

Then $[P(e_1), P(e_2), P(e_4)] = P([P(e_1), P(e_2), e_4]) = 0$, we obtain

$$P([P(e_1), P(e_2), e_4]) = P(\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} e_3 + \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} e_2 + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} e_1) = 0.$$

From $[P(e_1), P(e_3), P(e_4)] = [P(e_2), P(e_3), P(e_4)] = 0$, we get

$$P([P(e_1), P(e_3), e_4]) = P(\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} e_3 + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} e_2 + \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} e_1) = 0,$$

$$P([P(e_1), P(e_2), e_4]) = P(\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} e_3 + \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} e_2 + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} e_1) = 0.$$

Since $P(e_1), P(e_2), P(e_3)$ are linearly independent, we get

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0, \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = 0, \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = 0,$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = 0, \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = 0, \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = 0,$$

$$\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = 0, \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = 0, \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = 0.$$

Therefore, vectors $\alpha_1, \alpha_2, \alpha_3$ are linearly dependent. Contradiction.

Therefore, $P(e_4) \neq 0$. So we might assume that

$$P(e_4) = P(e_1) + \lambda P(e_2) + \mu P(e_3), \lambda, \mu \in F.$$

Denotes $e_4' = e_4 - e_1 - \lambda e_2 - \mu e_3$ then $P(e_4') = 0$. Then by Eq.(2),

$[P(e_1), P(e_2), P(e_4)] = \mu[P(e_1), P(e_2), P(e_3)]$
$= P([P(e_1), P(e_2), e_4] + \lambda[P(e_1), e_2, P(e_2)] + \mu[P(e_1), e_2, P(e_3)]$
$+[e_1, P(e_2), P(e_1)] + \mu[e_1, P(e_2), P(e_3)])$.
We obtain $P([P(e_1), P(e_2), e_4']) = 0$. It follows

$$[P(e_1), P(e_2), e_4'] = \kappa_1 e_4',\ \kappa_1 \in F.$$

Similarly, by the direct computation from $[P(e_1), P(e_3), P(e_4)]$ and $[P(e_2), P(e_3), P(e_4)]$, we get

$$[P(e_1), P(e_3), e_4'] = \kappa_2 e_4',\ [P(e_2), P(e_3), e_4'] = \kappa_3 e_4', \kappa_2, \kappa_3 \in F.$$

Summarizing above discussion, we get that the dimension of the subalgebra generated by the vectors $\{P(e_1), P(e_2), P(e_3), e_4'\}$ is less than 3. Therefore, vectors $e_4', P(e_1), P(e_2), P(e_3)$ are linearly dependent then

$$e_4' = \lambda_1 P(e_1) + \lambda_2 P(e_2) + \lambda_3 P(e_3).$$

So e_4' is contained in the image of P. From $P(e_4') = 0$, $e_4' \in KerP$. It contradicts to $e_4' \neq 0$.

Therefore, $R(P) \neq 3$. The result follows.

Remark There exist Rota-Baxter operators P of weight zero with $R(P) = 1, 2, 3$ on the simple 3-Lie algebra, respectively. For example. Define $P_1, P_2, P_3 : A \to A$ by

$P_1(e_1) = e_1 + e_2 + e_3 + e_4, P_1(e_2) = P_1(e_3) = P_1(e_4) = 0.$

$P_2(e_1) = e_1 + e_2, P_2(e_2) = e_3 + e_4, P_2(e_3) = P_2(e_4) = 0.$

$P_3(e_1) = e_2, P_3(e_2) = -e_1, P_3(e_3) = e_4, P_3(e_4) = e_3.$

By the direct computation, P_1, P_2, P_3 are Rota-Baxter operators, with $R(P_1) = 1$, $R(P_2) = 2$, $R(P_3) = 4$, respectively.

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