Mathematica Aeterna, Vol. 4, 2014, no. 8, 959 - 966

Positive Frequency of Sequences Arising from Neutral Difference Equations

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Abstract

By making use of frequency measures, in this paper we consider the positive frequency of sequences, which is produced by a class of neutral difference equations. The last example shows that our results are feasible.

Mathematics Subject Classification: 39A11, 39A10.

Keywords: Frequency measures; Positive frequency; Estimate; Superior limit; Neutral difference equation.

1 Introduction

To start with, we introduce some symbols as follows. Let $\mathbb{Z}[a,\infty)$ denote the integer set $\{a, a+1, a+2, \ldots\}$ and $\mathbb{Z}[a,b]$ the set $\{a, a+1, a+2, \ldots, b\}$. For any two sets A and B, their union, intersection, difference will be denoted by $A+B$, $A \cdot B$ and $A - B$, respectively. For a sequence $\{x(n)\}_{n\geq a}$ and a real number r, we denote the set $\{n \in \mathbb{Z}[a,\infty) : x(n) \geq r\}$ by $(x \geq r)$. Others such as $(x > r)$, $(x < r)$ etc, can be defined accordingly. Specially, when $x(n) \neq 0$ for all n, we denote the set $\{n \in \mathbb{Z}[a,\infty) : \frac{1}{x(n)} < r\}$ by $(x^{-1} < r)$. For the set $(x > r)$ (or others) of integers, the notation $|(x > r)|$ indicates the number of elements in $(x > r)$, and $(x > r)^{(n)}$ will denote the set $\{k \in (x > r) : k \leq n\}$.

Recall that in 1951, Niven [2] had introduced the concept of asymptotic density to study the properties of sequences of positive integers. In 2003 or so, Cheng et al. [1, Chapter 2] extended the idea of asymptotic density and introduced the concept of frequency measures to deal with the more general sequences of real numbers (or real vectors [3]). Precisely speaking, for a sequence

$\{x(n)\}_{n\geq a}$ of real numbers, we call the number ω_1 defined by

$$\omega_1 = \limsup_{n\to\infty} \frac{|(x \leq r)^{(n)}|}{n}$$

the upper frequency measure of $x \leq r$, and the number ω_2 defined by

$$\omega_2 = \liminf_{n\to\infty} \frac{|(x \leq r)^{(n)}|}{n}$$

the lower frequency measure of $x \leq r$. If $\omega_1 = \omega_2$, then the common limit will be called the frequency measure of $x \leq r$. The frequency measure of $x < r$ (or $x \geq r$, and so on) can be defined similarly. As usual, we denote the upper frequency measure of $x \leq r$ by $\mu^*(x \leq r)$ and the lower frequency measure of $x \leq r$ by $\mu_*(x \leq r)$.

We note that the frequency measures can be used to consider the properties of consequences, including oscillation and stability, see, e.g., the papers [3, 4, 5, 6] and their references. In the present paper we will impose the frequency measures to estimate the positive frequency of sequences, which stems from the following neutral difference equation

$$\triangle(x(n) + c(n)x(n-k)) + f(n, x(n-l)) = 0, \quad n \in \mathbb{N}, \tag{1}$$

where $\mathbb{N}$ stands for the set of nonnegative integers, $k \geq 1$ and $l \geq 0$ are integer, c maps $\mathbb{N}$ into $\mathbb{R}$ and $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$.

Let $\rho = \max\{k, l\}$. A sequence $\{x(n)\}_{n\geq -\rho}$ ($\{x(n)\}$ for short) is said to be a solution of (1) if it renders (1) into an identity for all $n \in \mathbb{N}$. The existence of solutions of (1) is clear. Indeed, for the given initial values $\{x(-\rho), x(-\rho + 1), \ldots, x(0)\}$, one can readily calculates from (1)

$$x(1), x(2), x(3), \ldots$$

in a unique manner.

For any integer m, let the set $\{n + m : n \in \Omega \subseteq \mathbb{Z}[a, \infty)\}$ be denoted by $E^m\Omega$. Before entering our main results, we recall some standard conclusions as follows:

Lemma 1.1 *[1, Chapter 2] Let Ω and Γ be subsets of $\mathbb{Z}[a, \infty)$ Then*

(i) $\mu_*(\Omega) + \mu^*(\Gamma) - \mu^*(\Omega \cdot \Gamma) \leq \mu^*(\Omega + \Gamma) \leq \mu^*(\Omega) + \mu^*(\Gamma) - \mu_*(\Omega \cdot \Gamma)$;

(ii) $\mu_*(\Omega) + \mu_*(\Gamma) - \mu^*(\Omega \cdot \Gamma) \leq \mu_*(\Omega + \Gamma) \leq \mu_*(\Omega) + \mu^*(\Gamma) - \mu_*(\Omega \cdot \Gamma)$;

(iii) if $\mu^*(\Omega) + \mu_*(\Gamma) > 1$, *then* $\Omega \cdot \Gamma$ *is infinite;*

(iv) if $n \in \mathbb{Z}[a, \infty) - \sum_{m=\alpha}^{\beta} E^m\Omega$ *and* $n - \alpha \geq a$, *then* $n - m \in \mathbb{Z}[a, \infty) - \Omega$ *for* $m \in \mathbb{Z}[\alpha, \beta]$;

(v) $\mu^*\left(\sum_{m=\alpha}^{\beta} E^m\Omega\right) \leq (\beta - \alpha + 1)\mu^*(\Omega)$ *and* $\mu_*\left(\sum_{m=\alpha}^{\beta} E^m\Omega\right) \leq (\beta - \alpha + 1)\mu_*(\Omega)$.

We remark that Lemma 1.1 (ii) implies that, for the case $\Omega + \Gamma = \mathbb{Z}[a, \infty)$ and $\Omega \cdot \Gamma = \phi$,

$$\mu_*(\Omega) + \mu^*(\Gamma) = 1.$$

Another fact is similar to [4, Lemma 5] (or [5, Lemma 2.5]).

Lemma 1.2 *Let A_s be the subset of $\mathbf{Z}[a, \infty)$ for $s = 1, 2, ..., n$. Then it follows that*

$$\mu^* \left(\sum_{s=1}^{n} A_s \right) \le \sum_{s=1}^{n} \mu^* (A_s) - (n-1)\mu_* \left(\prod_{s=1}^{n} A_s \right).$$

2 Main Results

Let $\{x(n)\}_{n \ge -\rho}$ be any solution of (1). In this section we devote to make estimates for the frequency of $x > 0$. Note that the symbol ρ defined by

$$\rho = \max\{k, l\}.$$

For the sake of convenience, we define

$$z(n) = x(n) + c(n)x(n-k) \quad \text{and} \quad q(n) = c(n-l)p(n) \quad \text{for} \quad n \in \mathbb{N},$$

where p verifies that

$$vf(n, v) \le p(n)v^2 \quad \text{for all} \quad (n, v) \in \mathbb{N} \times \mathbb{R}. \tag{A1}$$

Theorem 2.1 *Suppose that assumption (A1) holds. Suppose further that $\omega \in (0, 1)$ and*

$$\mu^*(p > 0) = \omega_p, \;\; \mu^*(c^{-1} < 1) = \omega_c, \;\; \mu^*(q \ge -1) = \omega_q,$$

$$\mu_* \left\{ (p > 0) \cdot (c^{-1} < 1) \cdot (q \ge -1) \right\} = \omega_0$$

as well as

$$(2k + 2l + 1)(\omega_p + \omega_c + \omega_q + \omega - 2\omega_0) < 1. \tag{2}$$

Then any nontrivial solution $\{x(n)\}$ of (1) has an estimate of positive frequency: $\omega < \mu^*(x > 0) < 1$.

Proof. We need only to prove that the frequency of $x > 0$ is neither $\mu^*(x > 0) \le \omega$ nor $\mu^*(x > 0) = 1$. Note that Lemma 1.2 amounts to

$$\begin{aligned} & \mu^* \left\{ \sum_{m=0}^{2k+2l} E^m[(p > 0) + (c^{-1} < 1) + (q \ge -1)] \right\} \\ \le \;\; & (2k + 2l + 1) \left\{ \mu^*(p > 0) + \mu^*(c^{-1} < 1) + \mu^*(q \ge -1) \right\} \\ & -2\mu_* \left\{ (p > 0) \cdot (c^{-1} < 1) \cdot (q \ge -1) \right\}. \end{aligned} \tag{3}$$

(i) In case $\mu^*(x>0)\le\omega$, by Lemma 1.1 it follows that

$$\begin{aligned} & \mu_*\left\{\mathbb{Z}[-\rho,\infty)-\sum_{m=0}^{2k+2l}E^m[(p>0)+(c^{-1}<1)+(q\ge -1)]\right\} \\ & +\mu^*\left\{\mathbb{Z}[-\rho,\infty)-\sum_{m=0}^{2k+2l}E^m(x>0)\right\} \\ = & \ 2-\mu^*\left\{\sum_{m=0}^{2k+2l}E^m[(p>0)+(c^{-1}<1)+(q\ge -1)]\right\}-\mu_*\left\{\sum_{m=0}^{2k+2l}E^m(x>0)\right\} \\ \ge & \ 2-(2k+2l+1)(\omega_p+\omega_c+\omega_q+\omega-2\omega_0) \\ > & \ 1, \end{aligned}$$

where we have used the conditions (2)–(3) for the above inequalities.

Now by Lemma 1.1(iii) we obtain an infinite set

$$\begin{aligned} & \left\{\mathbb{Z}[-\rho,\infty)-\sum_{m=0}^{2k+2l}E^m[(p>0)+(c^{-1}<1)+(q\ge -1)]\right\}\cdot \\ & \left\{\mathbb{Z}[-\rho,\infty)-\sum_{m=0}^{2k+2l}E^m(x>0)\right\}. \end{aligned} \tag{4}$$

Hence, from Lemma 1.1(iv) and (4) there exists an N satisfying $N-(2k+2l)\in\mathbb{N}$ so that

$$p(n)\le 0,\ c^{-1}(n)\ge 1,\ q(n)<-1,\ x(n)\le 0 \ \text{ for } \ n\in\mathbb{Z}[N-(2k+2l),N]. \tag{5}$$

Invoking the symbol $z(n)=x(n)+c(n)x(n-k)$ and (5) we have

$$z(n)\le x(n)\le 0 \ \text{ for } \ n\in\mathbb{Z}[N-(k+2l),N]. \tag{6}$$

Note that assumption (A1) and (5) implies that

$$f(n,x(n-l))\ge p(n)x(n-l) \ \text{ for } \ n\in\mathbb{Z}[N-(2k+l),N]. \tag{7}$$

Hence, by (1) it holds that

$$\triangle z(n)\le 0 \ \text{ for } \ n\in\mathbb{Z}[N-(k+l),N]. \tag{8}$$

Now combining (6)–(7) we have

$$\begin{aligned} 0 & = \triangle z(N)+f(N,x(N-l)) \\ & \ge \triangle z(N)+p(N)(Z(N-l)-c(N-l)x(N-k-l)) \\ & \ge \triangle z(N)+c(N-l)p(N)\left(\frac{Z(N-l)}{c(N-l)}-z(N-k-l)\right) \end{aligned}$$

$$\begin{aligned}
&\geq \triangle z(N) + c(N-l)p(N)\left(z(N-l) - z(N-k-l)\right) \\
&\geq \triangle z(N) + c(N-l)p(N) \sum_{n=N-k-l}^{N-l-1} \triangle z(n) \\
&\geq (1 + c(N-l)p(N)) \sum_{n=N-k-l}^{N} \triangle z(n),
\end{aligned}$$

which, together with (8), infers that

$$q(N) = c(N-l)p(N) \geq -1$$

and conflicts with (5) for q.

(ii) In case $\mu^*(x > 0) = 1$, we have $\mu_*(x \leq 0) = 0$. In a similar manners as above we arrive at the infinite set

$$\left\{\mathbb{Z}[-\rho,\infty) - \sum_{m=0}^{2k+2l} E^m[(p>0)+(c^{-1}<1)+(q\geq -1)]\right\} \cdot \left\{\mathbb{Z}[-\rho,\infty) - \sum_{m=0}^{2k+2l} E^m(x \leq 0)\right\}$$

and the relations (5) will be replaced by

$$p(n) \leq 0,\ c^{-1}(n) \geq 1,\ q(n) < -1,\ x(n) > 0 \ \text{ for } \ n \in \mathbb{Z}[N-(2k+2l), N]. \quad (9)$$

Consequently, in a similar discussion as above we will be led to a contradiction with $q(N) < -1$ in (9). The proof is complete.

In general, it is difficult to estimate the frequency for the following set

$$(p > 0) \cdot (c^{-1} < 1) \cdot (q \geq -1).$$

Fortunately, we can amplifies (3) as follows

$$\begin{aligned}
&\mu^* \left\{ \sum_{m=0}^{2k+2l} E^m[(p>0)+(c^{-1}<1)+(q\geq -1)] \right\} \\
\leq\ & (2k+2l+1)\left\{\mu^*(p>0) + \mu^*(c^{-1}<1) + \mu^*(q \geq -1)\right\}.
\end{aligned}$$

In other word, we can choose $\omega_0 = 0$ in Theorem 2.1. Hence, the following is clear.

Corollary 2.2 *Suppose that assumption (A1) holds. Suppose further that* $\omega \in (0,1)$ *and*

$$\mu^*(p>0) = \omega_p,\ \mu^*(c^{-1}<1) = \omega_c,\ \mu^*(q \geq -1) = \omega_q$$

as well as

$$(2k+2l+1)(\omega_p+\omega_c+\omega_q+\omega)<1.$$

Then any nontrivial solution $\{x(n)\}$ *of (1) has an estimate of positive frequency:* $\omega<\mu^*(x>0)<1$.

Theorem 2.3 *Suppose that assumption (A1) holds. Suppose further that* $\omega\in(0,1)$ *and*

$$\mu^*(p>0)=\omega_p,\ \mu^*(c^{-1}<1)=\omega_c,\ \mu_*[(p>0)\cdot(c^{-1}<1)]=\omega_{pc}$$

as well as

$$\mu^*(q<-1)>(2k+2l+1)(\omega_p+\omega_c+\omega-\omega_{pc}).$$

Then any nontrivial solution $\{x(n)\}$ *of (1) has an estimate of positive frequency:* $\omega<\mu^*(x>0)<1$.

Proof. Suppose to the contrary that $\mu^*(x>0)\le\omega$. Then, in view of Lemma 1.1 we have

$$\begin{aligned}
1 &= \mu_*\left\{\mathbb{Z}[-\rho,\infty)-\sum_{m=0}^{2k+2l}E^m[(p>0)+(c^{-1}<1)+(x>0)]\right\}\\
&\quad+\mu^*\left\{\sum_{m=0}^{2k+2l}E^m[(p>0)+(c^{-1}<1)+(x>0)]\right\}\\
&\le \mu_*\left\{\mathbb{Z}[-\rho,\infty)-\sum_{m=0}^{2k+2l}E^m[(p>0)+(c^{-1}<1)+(x>0)]\right\}\\
&\quad+(2k+2l+1)(\omega_p+\omega_c+\omega-\omega_{pc})\\
&< \mu_*\left\{\mathbb{Z}[-\rho,\infty)-\sum_{m=0}^{2k+2l}E^m[(p>0)+(c^{-1}<1)+(x>0)]\right\}\\
&\quad+\mu^*(q<-1),
\end{aligned}$$

which, with the help of Lemma 1.1(iii), derives that

$$\left\{\mathbb{Z}[-\rho,\infty)-\sum_{m=0}^{2k+2l}E^m[(p>0)+(c^{-1}<1)+(x>0)]\right\}\cdot(q<-1)$$

is infinite. Therefore, there exists an N satisfying $N-(2k+2l)\in\mathbb{N}$ such that

$$q(N)<-1$$

and

$$p(n)\le 0,\ c^{-1}(n)\ge 1,\ x(n)\le 0\ \text{ for }\ n\in\mathbb{Z}[N-(2k+2l),N].$$

The remainder is similar to the part in the proof of Theorem 2.1. As thus we have shown that $\mu^*(x > 0) \leq \omega$ is infeasible.

Likewise we can prove that $\mu^*(x > 0) < 1$. The proof is complete.

Next we end up this paper with an example.

Example 2.4 *Consider the following equation*

$$\Delta\left(x(n) + \frac{3}{4}x(n-2)\right) - 3x(n-1) = 0, \ n \in \mathbb{N}. \tag{10}$$

Then

$$c(n) = \frac{3}{4}, \ p(n) = -3 \quad \text{and} \quad q(n) = -\frac{9}{4}$$

and hence,

$$\mu^*(p > 0) = \mu^*(c^{-1} < 1) = \mu^*(q \geq -1) = 0$$

and

$$\mu^*(q < -1) = 1, \ \mu_*[(p > 0) \cdot (c^{-1} < 1)] = 0.$$

Now we take $\omega = \frac{1}{8}$. Then, by Corollary 2.2 or Theorem 2.3 we learn that, any nontrivial solution $\{x(n)\}$ of (10) has an estimate of positive frequency: $\frac{1}{8} < \mu^*(x > 0) < 1$. Indeed, $\left\{(-\frac{1}{2})^n\right\}_{n \geq -2}$ is such a solutioon, with frequency $\mu^*(x > 0) = \frac{1}{2}$.

ACKNOWLEDGEMENTS. The second (corresponding) author thanks the NNSF of China (no. 11271379) for his support.

The second author also wants to take this opportunity to thank Professor Sui Shun Cheng of Tsing Hua University for his assistance on this subject between 2005 and 2008.

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Received: October, 2014