Mathematica Aeterna, Vol. 4, 2014, no. 8, 979 - 987

Jensen and Hermite-Hadamard inequalities for strongly convex set-valued maps

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Abstract

Counterparts of the classical integral and discrete Jensen inequalities and the Hermite-Hadamard inequalities for strongly convex set-valued maps are presented.

Mathematics Subject Classification: Primary 26A51. Secondary 39B62, 54C60

Keywords: Convex set-valued map, strongly convex set-valued map, Jensen inequality, Hermite-Hadamard inequality

1 Introduction

Let $I \subset \mathbf{R}$ be an interval and c be a positive number. Following Polyak [16] a function $f : I \to \mathbf{R}$ is called *strongly convex with modulus c* if

$$f(tx_1 + (1-t)x_2) \le tf(x_1) + (1-t)f(x_2) - ct(1-t)(x_1 - x_2)^2 \qquad (1)$$

for all $x_1, x_2 \in I$ and $t \in [0,1]$. f is called strongly concave with modulus c if $-f$ is strongly convex with modulus c. Many properties and applications of strongly convex functions can be found in the literature (see, for instance, [9], [12], [17], [15], [22]). Recently Huang [5], extended the definition (1) of strongly convex function to set-valued maps. He used such maps to investigate error bounds for some inclusion problems with set constraints. Some further properties of strongly convex set-valued maps can be found in [6]. Strongly concave set-valued maps were investigated in [8].

The aim of this paper is to present counterparts of the integral and discrete Jensen inequalities and the Hermite-Hadamard double inequalities for strongly convex set-valued maps.

2 Preliminaries

Throughout this paper Y be a Banach space, B be a closed unit ball in Y, $I \subset \mathbf{R}$ be an open interval and c be a positive constant.

Denote by $n(Y)$ the family all nonempty subsets of Y and by $cl(Y)$ the family of all closed nonempty subsets of Y. A set-valued map $F : I \to n(Y)$ is called *strongly convex with modulus c* if

$$tF(x_1) + (1-t)F(x_2) + ct(1-t)(x_1 - x_2)^2 B \subset F(tx_1 + (1-t)x_2) \qquad (2)$$

for all $x_1, x_2 \in I$ and $t \in [0,1]$ (see [5], [6]). The usual notion of convex set-valued maps corresponds to relation (2) with $c = 0$ (cf. e.g. [2], [3], [11], [20], [21]).

Clearly, the definition of strongly convex set-valued maps is motivated by that of strongly convex functions. The following lemma characterizes strongly convex set-valued maps with values in $cl(\mathbf{R})$ and shows connections between conditions (1) and (2) (cf. [7] where analogous result for convex set-valued maps is given).

Lemma 2.1 *A set-valued map $F : I \to cl(\mathbf{R})$ is strongly convex with modulus c if and only if it has one of the following forms:*
a) $F(x) = [f_1(x), f_2(x)], \quad x \in I,$
b) $F(x) = [f_1(x), +\infty), \quad x \in I,$
c) $F(x) = (-\infty, f_2(x)], \quad x \in I,$

d) $F(x) = (-\infty, +\infty), \qquad x \in I,$
where $f_1 : I \to \mathbf{R}$ *is strongly convex with modulus* c *and* $f_2 : I \to \mathbf{R}$ *is strongly concave with modulus* c*.*

Proof. The "if" part is clear. To prove the "only if" part note first that by (2) the values of F are convex. Moreover, if $F(x_0)$ is bounded from above (from below) for some $x_0 \in I$, then $F(x)$ is bounded from above (from below) for every $x \in I$. Define

$$f_1(x) = \inf F(x), \quad \text{if} \quad F(x) \quad \text{is bounded from below}$$

and

$$f_2(x) = \sup F(x), \quad \text{if} \quad F(x) \quad \text{is bounded from above.}$$

Then by the strong convexity of F its follows that f_1 is strongly convex with modulus c and f_2 is strongly concave with modulus c. Since the values of F are closed and convex, the result follows. □

3 The Jensen inequalities

It is well know that if a function $f : I \to \mathbf{R}$ is convex, then if satisfies the integral Jensen inequalities

$$f\left(\int_X \varphi(x)d\mu\right) \le \int_X f(\varphi(x))d\mu \tag{3}$$

for each probability measure space (X, Σ, μ) and all μ-integrable functions $\varphi : X \to I$.

In [9] the following version of the Jensen inequality for strongly convex functions was proved:

$$f\left(\int_X \varphi(x)d\mu\right) \le \int_X f(\varphi(x))\, d\mu - c\int_X (\varphi(x) - m)^2\, d\mu \tag{4}$$

where $m = \int_X \varphi(x)d\mu$. A counterpart of (3) for set-valued maps was obtained in [7]. The next Theorem gives a counterpart of (4) for set-valued maps.

Throughout this paper the integral of a set-valued map is understood in the sense of Aumann, i.e. it is the set of integrals of all integrable selections of this map.

Theorem 3.1 *Let* (X, Σ, μ) *be a probability measure space. If* $F : I \to cl(Y)$ *is strongly convex with modulus c, then for each square-integrable function* $\varphi : X \to I$

$$\int_X F(\varphi(x))d\mu + c\int_X (\varphi(x) - m)^2\, d\mu\, B \subset F\left(\int_X \varphi(x)d\mu\right), \tag{5}$$

where $m = \int_X \varphi(x)d\mu$*.*

Proof. The proof is divided into two steps. First, we assume that $Y = \mathbf{R}$. Then, by Lemma 2.1, F has one of the forms a)- d). Assume that $F(x) = [f_1(x), f_2(x)]$, $x \in I$ (the proof in the remaining cases is similar). Let $h : X \to \mathbf{R}$ be a μ-integrable selection of $F \circ \varphi$. Then, by the Jensen inequality for strongly convex function (4), we have

$$\begin{aligned} f_1\left(\int_X \varphi(x)d\mu\right) &\leq \int_X f_1(\varphi(x))d\mu - c\int_X (\varphi(x)-m)^2\, d\mu \\ &\leq \int_X (h(x))d\mu - c\int_X (\varphi(x)-m)^2\, d\mu \end{aligned}$$

and

$$\begin{aligned} f_2\left(\int_X \varphi(x)d\mu\right) &\geq \int_X f_2(\varphi(x))d\mu + c\int_X (\varphi(x)-m)^2\, d\mu \\ &\geq \int_X (h(x))d\mu + c\int_X (\varphi(x)-m)^2\, d\mu. \end{aligned}$$

Hence

$$\int_X (h(x))d\mu + c\int_X (\varphi(x)-m)^2\, d\mu\, [-1,1] \subset F\left(\int_X \varphi(x)d\mu\right).$$

Consequently

$$\int_X F(\varphi(x))d\mu + c\int_X (\varphi(x)-m)^2\, d\mu\, [-1,1] \subset F\left(\int_X \varphi(x)d\mu\right),$$

which finishes the proof in the case $Y = \mathbf{R}$.

Now, assume that Y is an arbitrary Banach space. Take a nonzero continuous linear functional $y^* \in Y^*$ and considerer the set-valued map $x \mapsto \overline{y^*(F(x))}$, $x \in I$. This set-valued map is strongly convex with modulus $c||y^*||$ and has closed values in $\mathbf{R}$. Therefore, by the previous step,

$$\int_X \overline{y^*(F(\varphi(x)))}d\mu + c||y^*||\int_X (\varphi(x)-m)^2\, d\mu\, [-1,1] \subset \overline{y^*\left(F\left(\int_X \varphi(x)d\mu\right)\right)}. \quad (6)$$

Fix a point $b \in B$ and take an arbitrary μ-integrable selection h of $F \circ \varphi$. Then, by (6) and the fact that

$$\int_X y^*(h(x))d\mu = y^*\left(\int_X h(x)d\mu\right),$$

we get

$$\begin{aligned} & y^*\left(\int_X h(x)d\mu + c\int_X (\varphi(x)-m)^2\, d\mu\, b\right) \\ \in\ & \int_X y^*(h(x))d\mu + c||y^*||\int_X (\varphi(x)-m)^2\, d\mu\, [-1,1] \\ \subset\ & \overline{y^*\left(F\left(\int_X \varphi(x)d\mu\right)\right)}. \end{aligned}$$

Since this condition holds for arbitrary $y^* \in Y^*$ and the set $\overline{y^*(F\left(\int_X(\varphi(x)d\mu\right))}$ is convex closed, by the separation theorem (see [18], Corollary 2.5.11) we obtain

$$\int_X h(x)d\mu + c\int_X (\varphi(x)-m)^2\,d\mu\, b \in F\left(\int_X \varphi(x)d\mu\right)$$

Thus

$$\int_X F(\varphi(x))d\mu + c\int_X (\varphi(x)-m)^2\,d\mu\, B \subset F\left(\int_X \varphi(x)d\mu\right),$$

which was to be proved. □

Now, assume that $X = I$, $\varphi(x) = x$ for $x \in I$, and $x_1, \ldots, x_n \in I$ are distinct points. Moreover, assume that μ is a probability measure concentrate at $x_1, \ldots, x_n$, that is $\mu(x_1) = t_i > 0$, $i = 1, \ldots, n$ and $t_1 + \cdots + t_n = 1$. Then

$$m = \int_X \varphi(x)d\mu = \sum_{i=1}^n t_i x_i, \qquad \int_X (\varphi(x)-m)^2\,d\mu = \sum_{i=1}^n t_i(x_i-m)^2$$

and

$$\int_X F(\varphi(x))d\mu = \sum_{i=1}^n t_i F(x_i).$$

Therefore, as the consequence of Theorem 3.1, we get the following discrete Jensen inequality for strongly convex set-valued maps.

Corollary 3.2 *If $f : I \to cl(Y)$ is strongly convex with modulus c, then*

$$\sum_{i=1}^n t_i F(x_i) + c\sum_{i=1}^n t_i(x_i-m)^2 B \subset F\left(\sum_{i=1}^n t_i x_i\right)$$

for all $n \in \mathbf{N}$, $x_1, \ldots, x_n \in I$, $t_1, \ldots, t_n > 0$ with $t_1 + \cdots + t_n = 1$ and $m = t_1x_1 + \cdots + t_nx_n$.

4 The Hermite-Hadamard inequality

It is known that if a function $f : I \to \mathbf{R}$ is convex then it satisfies the Hermite-Hadamard double inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}, \quad a, b \in I, \quad a < b. \tag{7}$$

The following version of the Hermite-Hadamard inequality for strongly convex functions was recently proved in [9]:

$$f\left(\frac{a+b}{2}\right)+\frac{c}{12}(a-b)^2\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{2}-\frac{c}{6}(a-b)^2, \quad (8)$$

for all $a,b\in I,\ a<b$.

In this section we present a counterpart of the above inequality (8) for strongly convex set-valued maps. The Hermite-Hadamard inequality for convex set-valued maps was obtained in [19] (cf. also [14], [10]) .

Theorem 4.1 *If a set-valued map $F:I\to cl(Y)$ is strongly convex with modulus c, then*

$$\frac{1}{b-a}\int_a^b F(x)dx+\frac{c}{12}(a-b)^2B\subset F\left(\frac{a+b}{2}\right) \quad (9)$$

and

$$\frac{F(a)+F(b)}{2}+\frac{c}{6}(a-b)^2B\subset\frac{1}{b-a}\int_a^b F(x)dx \quad (10)$$

for all $a,b\in I,\ a<b$.

Proof. Condition (9) follows from Theorem 3.1. To show this take $X=[a,b],\varphi(x)=x,x\in[a,b]$ and $\mu=\frac{1}{b-a}\lambda$, where λ is the Lebesgue measure on $\mathbf{R}$. Then

$$m=\int_X\varphi(x)d\mu=\frac{a+b}{2},\qquad F\left(\int_X\varphi(x)d\mu\right)=F\left(\frac{a+b}{2}\right),$$

$$\int_X(\varphi(x)-m)^2d\mu=\frac{1}{2}(a-b)^2\quad\text{and}\quad\int_X F(\varphi(x))\,d\mu=\frac{1}{b-a}\int_a^b F(x)dx.$$

Substituting these equalities to (5) we get (9).

To prove condition (10) take arbitrary $z=\frac{u+v}{2}+\frac{c}{6}(a-b)^2\beta$, where $u\in F(a),v\in F(b)$ and $\beta\in B$. Considerer the function $f:[a,b]\to Y$ defined by

$$f(x)=\frac{b-x}{b-a}u+\frac{x-a}{b-a}v+c(b-x)(x-a)\beta.$$

By the strong convexity of F we get

$$f(x)\in\frac{b-x}{x-a}F(a)+\frac{x-a}{b-a}F(b)+c\frac{b-x}{b-a}\frac{x-a}{b-a}(b-a)^2B\subset F\left(\frac{b-x}{b-a}a+\frac{x-a}{b-a}b\right)=F(x),$$

which means that f is a selection of F.

Simple calculations gives

$$\int_a^b f(x)dx = (b-a)\left[\frac{u+v}{2} + \frac{1}{6}c\beta(a-b)^2\right] = (b-a)z.$$

Hence

$$z = \frac{1}{b-a}\int_a^b f(x)dx \in \frac{1}{b-a}\int_a^b F(x)dx,$$

which finishes the proof. □

5 The converse of Hermite-Hadamard theorem

It is known that if a continuous function $f : I \to \mathbf{R}$ satisfies the left or the right-hand side inequality in (7), then it is convex (cf. e.g. [2], [4], [13]). An analogous result holds also for strong convexity: If $f : I \to \mathbf{R}$ is continuous and satisfies the left or the right-hand side inequality in (8), then it is strongly convex with modulus c (see [9]). In this section we present a set-valued counterpart of that result. Recall that a set-valued map $F : I \to n(Y)$ is said to be *continuous* at a point x_0 if for every neighbourhood V of zero in Y there exist a neighbourhood U of zero in $\mathbf{R}$ such that

$$F(x) \subset F(x_0) + V \quad \text{and} \quad F(x_0) \subset F(x) + V$$

for all $x \in (x_0 + U) \cap I$.

In what follows we assume that Y is a separable Banach space and denote by $bccl(Y)$ the family of all bounded convex closed and non-empty subsets of Y.

Theorem 5.1 *If $F : I \to bccl(Y)$ is continuous and satisfies*

$$\frac{1}{b-a}\int_a^b F(x)dx + \frac{c}{12}(a-b)^2 B \subset F\Big(\frac{a+b}{2}\Big), \quad a, b \in I,\ a < b. \tag{11}$$

or

$$\frac{F(a)+F(b)}{2} + \frac{c}{6}(a-b)^2 B \subset \frac{1}{b-a}\int_a^b F(x)dx, \quad a, b \in I,\ a < b, \tag{12}$$

then F is strongly convex with modulus c.

Proof. Assume that F satisfies (11) (if F satisfies (12) the proof is analogous). Define $G(x) = F(x) + cx^2B$, $x \in I$. Then

$$\begin{aligned}
\frac{1}{b-a}\int_a^b G(x)dx &= \frac{1}{b-a}\int_a^b F(x)dx + \frac{1}{b-a}\int_a^b cx^2Bdx \\
&= \frac{1}{b-a}\int_a^b F(x)dx + c\frac{a^2+ab+b^2}{3}B \\
&= \frac{1}{b-a}\int_a^b F(x)dx + c\frac{(a-b)^2}{12}B + c\Big(\frac{a+b}{2}\Big)^2 B \\
&\subset F\Big(\frac{a+b}{2}\Big) + c\Big(\frac{a+b}{2}\Big)^2 B = G\Big(\frac{a+b}{2}\Big).
\end{aligned}$$

Thus G satisfies the Hermite-Hadamard-type inclusion and it is also continuous. Therefore, by [10, Theorem 8], G is convex. Hence, using the definition of G and the characterization of strongly convex set-valued maps given in [6], we obtain that F is strongly convex with modulus c. This finished the proof. □

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Received: November, 2014