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Symmetric Positive Solutions for Integral Boundary Value Problems with ϕ-Laplacian Operator

Yan Luo

Department of Mathematics, Hunan University of Science and Technology,
Xiangtan, Hunan, 411201, China
Corresponding author's E-mail: luoyan2527@126.com (Yan Luo)

Abstract

The aim of the paper is to study the existence, multiplicity, and nonexistence of symmetric positive solutions for integral boundary value problems with ϕ-Laplacian operator. We generalize and improve some previous results. Moreover, examples are given to illustrate the applicability of our results. Our analysis mainly relies on the fixed point theorem of cone expansion and compression of norm type.

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1 Introduction

The existence of symmetric positive solutions of boundary value problems has been studied by several authors in the literature, see [1-3,5-7,10] and the references therein.

In [8], Luo consider the boundary value problem

$$\begin{cases} (\phi(u''(t)))'' = w(t)f(t, u(t), u'(t)), \ t \in (0, 1), \\ u(0) = u(1) = \int_0^1 g(s)u(s)ds, \\ \phi(u''(0)) = \phi(u''(1)) = \int_0^1 h(s)\phi(u''(s))ds, \end{cases} \tag{1.1}$$

where $0 < \int_0^1 g(s)ds \leq \frac{2}{3}$.

In this paper, we still study (1.1), where $\frac{2}{3} < \int_0^1 g(s)ds < 1$. By applying the fixed point theorem (Lemma 1.1), we establish sufficient conditions for the existence, multiplicity and nonexistence of symmetric positive solutions of (1.1).

Lemma 1.1([4]). *Let P be a cone of real Banach space E, Ω_1 and Ω_2 be two bounded open sets in E such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Let operator $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be completely continuous. Suppose that one of the two conditions*
(i) $\|Tu\| \leq \|u\|, \forall\, u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|, \forall\, u \in P \cap \partial\Omega_2$,
(ii) $\|Tu\| \geq \|u\|, \forall\, u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|, \forall\, u \in P \cap \partial\Omega_2$
is satisfied. Then T has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Throughout the paper, we assume the following conditions hold.
(H1) ϕ is an odd, increasing homeomorphism from R onto R and there exist two increasing homeomorphisms ψ_1 and ψ_2 of $(0, \infty)$ onto $(0, \infty)$ such that

$$\psi_1(u)\phi(v) \leq \phi(uv) \leq \psi_2(u)\phi(v) \text{ for all } u,\ v > 0.$$

Moreover, $\phi,\ \phi^{-1} \in C^1(R)$, where ϕ^{-1} denotes the inverse of ϕ.
(H2) $w \in L^1[0, 1]$ is nonnegative, symmetric on $[0, 1]$ and $w(t) \not\equiv 0$ on any subinterval of $[0, 1]$.
(H3) $f : [0, 1] \times D \to R^+$ is continuous with $D = R^+ \times R$, $R^+ = [0, \infty)$. For $(t, u, v) \in [0, 1] \times D$, $f(t, u, v)$ is symmetric in t and even in v, i.e., f satisfies $f(1 - t, u, v) = f(t, u, v)$ and $f(t, u, -v) = f(t, u, v)$.
(H4) $g,\ h \in L^1[0, 1]$ are nonnegative, symmetric on $[0, 1]$, and $\frac{2}{3} < \mu = \int_0^1 g(s)ds < 1$, $0 < \nu = \int_0^1 h(s)ds < 1$.
Remark 1.1. Zhang [12] and Ma [9] discussed $\phi(u) = |u|^{p-2}u$ $(p > 1)$ and $\phi(u) = u$, respectively, so our paper improves and genera1izes the results of [12, 9] to some degree.

2 Preliminary Notes

Let the space $E = C^1[0, 1]$ endowed with the norm $\|u\| = \max\{\|u\|_0,\ \|u'\|_0\}$, where $\|u\|_0 = \max_{t\in[0,1]} |u(t)|$, be our Banach space. Define P to be a cone in E by $P = \{u \in E : u(t) \geq 0,\ u \text{ is concave, symmetric on } [0, 1]\}$. Also, define, for $0 < r < R$ two positive numbers, Ω_r and $\overline{\Omega}_{r,R}$ by $\Omega_r = \{u \in E : \|u\| < r\}$, $\overline{\Omega}_{r,R} = \{u \in E : r \leq \|u\| \leq R\}$. Note that $\partial\Omega_r = \{u \in E : \|u\| = r\}$.

We introduce the integral operator $T : E \to E$ by

$$Tu(t) = \int_0^1 H(t, s)\phi^{-1}\Big(\int_0^1 H_1(s, \tau)w(\tau)f(\tau, u(\tau), u'(\tau))d\tau\Big)ds, \tag{2.1}$$

where

$$H(t, s) = G(t, s) + \frac{1}{1 - \mu}\int_0^1 G(s, \tau)g(\tau)d\tau, \tag{2.2}$$

$$H_1(t, s) = G(t, s) + \frac{1}{1 - \nu}\int_0^1 G(s, \tau)h(\tau)d\tau, \tag{2.3}$$

$$G(t,s)=\begin{cases} t(1-s), & 0\le t\le s\le 1,\\ s(1-t), & 0\le s\le t\le 1.\end{cases}$$

From Lemma 2.1, 2.2 and 2.4 in [8], we have the following result.

Lemma 2.1. *Assume* (H1)-(H4) *hold. Then* $u\in E$ *is a solution of* (1.1) *if and only if* u *is a fixed point of the operator* T.

The operator $T:P\to P$ is completely continuous, the proof can be found in [8], Lemma 2.6.

Lemma 2.2([12]). *If* (H4) *holds, then* $\forall\ t,\ s\in[0,1]$, *the following results are true.*

(i) $G(t,s)\ge 0,\ H(t,s)\ge 0,\ H_1(t,s)\ge 0$;

(ii) $G(1-t,1-s)=G(t,s),\ H(1-t,1-s)=H(t,s),\ H_1(1-t,1-s)=H_1(t,s)$;

(iii) $\rho e(s)\le H(t,s)\le \gamma e(s),\ \rho_1 e(s)\le H_1(t,s)\le \gamma_1 e(s)$

with

$$\rho=\frac{\int_0^1 e(s)g(s)ds}{1-\mu},\ \rho_1=\frac{\int_0^1 e(s)h(s)ds}{1-\nu},\ e(s)=s(1-s),\ \gamma=\frac{1}{1-\mu},\ \gamma_1=\frac{1}{1-\nu},$$

where $H(t,s)$, $G(t,s)$ *and* $H_1(t,s)$ *are defined by* (2.2) *and* (2.3), *respectively.*

Lemma 2.3([11]). *Assume* (H1) *holds. Then,* $\forall\ u,\ v\in(0,\infty)$,

$$\psi_2^{-1}(u)v\le\phi^{-1}(u\phi(v))\le\psi_1^{-1}(u)v,$$

where ψ_1^{-1} *and* ψ_2^{-1} *denote the inverse of* ψ_1 *and* ψ_2, *respectively.*

So, now we have

$$\begin{aligned}(Tu)'(t)\ &=\int_t^1(1-s)\phi^{-1}\Big(\int_0^1 H_1(s,\tau)w(\tau)f(\tau,u(\tau),u'(\tau))d\tau\Big)ds\\ &\quad-\int_0^t s\phi^{-1}\Big(\int_0^1 H_1(s,\tau)w(\tau)f(\tau,u(\tau),u'(\tau))d\tau\Big)ds,\end{aligned}$$

$$(Tu)''(t)=-\phi^{-1}\Big(\int_0^1 H_1(t,s)w(s)f(s,u(s),u'(s))ds\Big).$$

For $u\in P$, $(Tu)''(t)\le 0$, which implies $(Tu)'(t)$ is non-increasing on $[0,1]$. Thus,

$$\begin{aligned}&|(Tu)'(t)|\le\max\{-(Tu)'(1),\ (Tu)'(0)\}\\ &\le\max\Big\{\int_0^1 sds,\ \int_0^1(1-s)ds\Big\}\phi^{-1}\Big(\int_0^1\gamma_1 e(\tau)w(\tau)f(\tau,u(\tau),u'(\tau))d\tau\Big)\\ &=\int_0^1 sds\phi^{-1}\Big(\int_0^1\gamma_1 e(\tau)w(\tau)f(\tau,u(\tau),u'(\tau))d\tau\Big).\end{aligned}\tag{2.4}$$

3 The existence of one symmetric positive solution

In order to state the following results we need to introduce the new notations:

$$f^{\beta} = \limsup_{\|u\|_0+\|v\|_0\to\beta} \max_{t\in[0,1]} \frac{f(t,u,v)}{\phi(\|u\|_0+\|v\|_0)},\ f_{\beta} = \liminf_{\|u\|_0+\|v\|_0\to\beta} \min_{t\in[0,1]} \frac{f(t,u,v)}{\phi(\|u\|_0+\|v\|_0)},$$

$$\delta = \frac{\rho\psi_2^{-1}\Big(\int_0^1 \rho_1 e(\tau)w(\tau)d\tau\Big)}{\gamma\psi_1^{-1}\Big(\int_0^1 \gamma_1 e(\tau)w(\tau)d\tau\Big)},$$

$$\sigma = \gamma\int_0^1 e(s)ds\psi_1^{-1}\Big(\int_0^1 \gamma_1 e(\tau)w(\tau)d\tau\Big),\ \xi = \int_0^1 sds\psi_1^{-1}\Big(\int_0^1 \gamma_1 e(\tau)w(\tau)d\tau\Big),$$

where β denotes 0 or ∞, and ρ, γ, ρ_1, γ_1 are defined in Lemma 2.2.

Remark 3.1. According to (H1), (H4) and the definitions of γ, δ, we have $\gamma > 3$ which implies $0 < \xi < \sigma$, $0 < \delta < 1$.

Theorem 3.1. *Assume* (H1)-(H4) *hold. In addition, suppose one of the following conditions is satisfied:*

(i) *There exist two constants* r, R *with* $0 < r \leq \delta R$ *such that* $f(t,u,v) \geq \phi(\frac{1}{\delta\sigma}r)$ *for* $(t,u,v) \in [0,1]\times[0,r]\times[-r,r]$, *and* $f(t,u,v) \leq \phi(\frac{1}{\sigma}R)$ *for* $(t,u,v) \in [0,1]\times[0,R]\times[-R,R]$;

(ii) $f_0 > \psi_2((\int_0^1 \rho e(s)ds)^{-1})(\int_0^1 \rho_1 e(\tau)w(\tau)d\tau)^{-1}$ *and* $f^{\infty} < \psi_1((2\int_0^1 \gamma e(s)ds)^{-1})$ $(\int_0^1 \gamma_1 e(\tau)w(\tau)d\tau)^{-1}$ *(particularly,* $f_0 = \infty$ *and* $f^{\infty} = 0$*).*

Then problem (1.1) *has at least one symmetric positive solution.*

Proof. Let the operator T be defined by (2.1).

(i) For $u \in P\cap\partial\Omega_r$, we obtain $u \in [0,r]$ and $u' \in [-r,r]$, which implies $f(t,u,u') \geq \phi(\frac{1}{\delta\sigma}r)$. Hence for $t \in [0,1]$, by Lemma 2.2 and Lemma 2.3,

$$\begin{aligned} Tu(t) &\geq \int_0^1 \rho e(s)\phi^{-1}\Big(\int_0^1 \rho_1 e(\tau)w(\tau)d\tau\phi\Big(\frac{1}{\delta\sigma}r\Big)\Big)ds \\ &\geq \rho\int_0^1 e(s)ds\psi_2^{-1}\Big(\int_0^1 \rho_1 e(\tau)w(\tau)d\tau\Big)\frac{1}{\delta\sigma}r = \delta\sigma\frac{1}{\delta\sigma}r = \|u\|, \end{aligned}$$

i.e., $u \in P\cap\partial\Omega_r$ implies

$$\|Tu\| \geq \|u\|. \tag{3.1}$$

Next, for $u \in P\cap\partial\Omega_R$, $u \in [0,R]$ and $u' \in [-R,R]$, which implies $f(t,u,u') \leq \phi(\frac{1}{\sigma}R)$. Thus for $t \in [0,1]$, by Lemma 2.2, Lemma 2.3,

$$\begin{aligned} Tu(t) &\leq \int_0^1 \gamma e(s)\phi^{-1}\Big(\int_0^1 \gamma_1 e(\tau)w(\tau)d\tau\phi\Big(\frac{1}{\sigma}R\Big)\Big)ds \\ &\leq \gamma\int_0^1 e(s)ds\psi_1^{-1}\Big(\int_0^1 \gamma_1 e(\tau)w(\tau)d\tau\Big)\frac{1}{\sigma}R = \sigma\frac{1}{\sigma}R = \|u\|, \end{aligned}$$

i.e., $u \in P \cap \partial\Omega_R$ implies

$$\|Tu\|_0 \leq \|u\|. \tag{3.2}$$

From (2.4), and noting that $0 < \xi < \sigma$,

$$\begin{aligned} |(Tu)'(t)| &\leq \int_0^1 sds\phi^{-1}\Big(\int_0^1 \gamma_1 e(\tau)w(\tau)d\tau\phi\Big(\frac{1}{\sigma}R\Big)\Big) \\ &\leq \int_0^1 sds\psi_1^{-1}\Big(\int_0^1 \gamma_1 e(\tau)w(\tau)d\tau\Big)\frac{1}{\sigma}R = \xi\frac{1}{\sigma}R < \|u\|, \end{aligned}$$

which implies that for $u \in P \cap \partial\Omega_R$

$$\|(Tu)'\|_0 \leq \|u\|. \tag{3.3}$$

By (3.2) and (3.3), we obtain that

$$\|Tu\| \leq \|u\|, \text{ for } u \in P \cap \partial\Omega_R. \tag{3.4}$$

(ii) Considering $f_0 > \psi_2((\int_0^1 \rho e(s)ds)^{-1})(\int_0^1 \rho_1 e(\tau)w(\tau)d\tau)^{-1}$, there exists $r > 0$ such that $f(t,u,v) \geq (f_0 - \varepsilon_1)\phi(\|u\|_0 + \|v\|_0)$, for $t \in [0,1]$, $\|u\|_0 + \|v\|_0 \in [0, 2r]$, where $\varepsilon_1 > 0$ satisfies $\rho\int_0^1 e(s)ds\psi_2^{-1}(\int_0^1 \rho_1 e(\tau)w(\tau)d\tau(f_0 - \varepsilon_1)) \geq 1$. Then, for $t \in [0,1]$, $u \in P \cap \partial\Omega_r$, which implies $\|u\|_0 + \|u'\|_0 \leq 2r$, we have

$$\begin{aligned} Tu(t) &\geq \int_0^1 \rho e(s)\phi^{-1}\Big(\int_0^1 \rho_1 e(\tau)w(\tau)(f_0 - \varepsilon_1)\phi(\|u\|_0 + \|u'\|_0)d\tau\Big)ds \\ &\geq \int_0^1 \rho e(s)\phi^{-1}\Big(\int_0^1 \rho_1 e(\tau)w(\tau)d\tau(f_0 - \varepsilon_1)\phi(\|u\|)\Big)ds \\ &\geq \rho\int_0^1 e(s)ds\psi_2^{-1}\Big(\int_0^1 \rho_1 e(\tau)w(\tau)d\tau(f_0 - \varepsilon_1)\Big)\|u\| \geq \|u\|, \end{aligned}$$

which implies for $u \in P \cap \partial\Omega_r$

$$\|Tu\| \geq \|u\|. \tag{3.5}$$

Next, turning to $f^\infty < \psi_1((2\int_0^1 \gamma e(s)ds)^{-1})(\int_0^1 \gamma_1 e(\tau)w(\tau)d\tau)^{-1}$, there exist $\overline{R} > 0$ such that $f(t,u,v) \leq (f^\infty + \varepsilon_2)\phi(\|u\|_0 + \|v\|_0)$, for $t \in [0,1]$, $\|u\|_0 + \|v\|_0 \in (\overline{R}, \infty)$, where $\varepsilon_2 > 0$ satisfies $\psi_1((2\int_0^1 \gamma e(s)ds)^{-1})(\int_0^1 \gamma_1 e(\tau)w(\tau)d\tau)^{-1} - f^\infty - \varepsilon_2 > 0$. Set $M = \max_{\|u\|_0 + \|v\|_0 \leq \overline{R},\ t\in[0,1]} f(t,u,v)$. Then $f(t,u,v) \leq M + (f^\infty + \varepsilon_2)\phi(\|u\|_0 + \|v\|_0)$. Choose $R > \max\{r, \overline{R}, \frac{1}{2}\phi^{-1}(M[\psi_1((2\int_0^1 \gamma e(s)ds)^{-1})$ $(\int_0^1 \gamma_1 e(\tau)w(\tau)d\tau)^{-1} - f^\infty - \varepsilon_2]^{-1})\}$. Hence for $u \in P \cap \partial\Omega_R$, we have

$$\begin{aligned} Tu(t) &\leq \int_0^1 \gamma e(s)\phi^{-1}\Big(\int_0^1 \gamma_1 e(\tau)w(\tau)[M + (f^\infty + \varepsilon_2)\phi(\|u\|_0 + \|u'\|_0)]d\tau\Big)ds \\ &\leq \int_0^1 \gamma e(s)ds\phi^{-1}\Big(\int_0^1 \gamma_1 e(\tau)w(\tau)d\tau\Big(\frac{M}{\phi(2R)} + f^\infty + \varepsilon_2\Big)\phi(2R)\Big) \\ &\leq \int_0^1 \gamma e(s)ds\psi_1^{-1}\Big(\int_0^1 \gamma_1 e(\tau)w(\tau)d\tau\Big(\frac{M}{\phi(2R)} + f^\infty + \varepsilon_2\Big)\Big)2R \leq \frac{1}{2}2R = \|u\|, \end{aligned}$$

i.e., $u \in P \cap \partial\Omega_R$ implies
$$\|Tu\|_0 \leq \|u\|. \tag{3.6}$$
By (2.4), and noting that $0 < \xi < \sigma$,
$$\begin{aligned} |(Tu)'(t)| &\leq \int_0^1 sds\phi^{-1}\Big(\int_0^1 \gamma_1 e(\tau)w(\tau)d\tau\Big(\frac{M}{\phi(2R)} + f^\infty + \varepsilon_2\Big)\phi(2R)\Big) \\ &\leq \frac{\xi}{\sigma}\int_0^1 \gamma e(s)ds\psi_1^{-1}\Big(\int_0^1 \gamma_1 e(\tau)w(\tau)d\tau\Big(\frac{M}{\phi(2R)} + f^\infty + \varepsilon_2\Big)\Big)2R < \|u\|, \end{aligned}$$

which implies that for $u \in P \cap \partial\Omega_R$
$$\|(Tu)'\|_0 \leq \|u\|. \tag{3.7}$$
By (3.6) and (3.7), we obtain that
$$\|Tu\| \leq \|u\|, \text{ for } u \in P \cap \partial\Omega_R. \tag{3.8}$$

Applying Lemma 1.1 to (3.1) and (3.4), or (3.5) and (3.8) yields that T has a fixed point $u \in P \cap \overline{\Omega}_{r,R}$ with $0 < r \leq \|u\| \leq R$. It follows from Lemma 2.1 that problem (1.1) has at least one symmetric positive solution u.
Theorem 3.2. *Assume* (H1)-(H4) *hold. And suppose* $f^0 < \psi_1((2\int_0^1 \gamma e(s)ds)^{-1})(\int_0^1 \gamma_1 e(\tau)w(\tau)d\tau)^{-1}$, $f_\infty > \psi_2((\int_0^1 \rho e(s)ds)^{-1})(\int_0^1 \rho_1 e(\tau)w(\tau)d\tau)^{-1}$ *(particularly,* $f^0 = 0$ *and* $f_\infty = \infty$*) are satisfied, then problem* (1.1) *has at least one symmetric positive solution.*

4 The existence of multiple symmetric positive solutions

Theorem 4.1. *Assume* (H1)-(H4) *hold, as do the following two conditions:*
(i) $f_0 > \psi_2((\int_0^1 \rho e(s)ds)^{-1})(\int_0^1 \rho_1 e(\tau)w(\tau)d\tau)^{-1}$ *and*
$f_\infty > \psi_2((\int_0^1 \rho e(s)ds)^{-1})(\int_0^1 \rho_1 e(\tau)w(\tau)d\tau)^{-1}$,
(ii) *There exists* $b > 0$ *satisfying* $f(t,u,v) < \phi(\frac{1}{\sigma}b)$, $(t,u,v) \in [0,1] \times [0,b] \times [-b,b]$.
Then problem (1.1) *has at least two symmetric positive solutions* $u_1(t),\ u_2(t)$, *which satisfy* $0 < \|u_1\| < b < \|u_2\|$.
Proof. Consider (i). If $f_0 > \psi_2((\int_0^1 \rho e(s)ds)^{-1})(\int_0^1 \rho_1 e(\tau)w(\tau)d\tau)^{-1}$, it follows from the proof of (3.5) that we can choose r with $0 < r < b$ such that
$$\|Tu\| \geq \|u\|, \text{ for } u \in P \cap \partial\Omega_r. \tag{4.1}$$
If $f_\infty > \psi_2((\int_0^1 \rho e(s)ds)^{-1})(\int_0^1 \rho_1 e(\tau)w(\tau)d\tau)^{-1}$, then like in the proof of (3.5), we can choose R with $b < R$ such that
$$\|Tu\| \geq \|u\|, \text{ for } u \in P \cap \partial\Omega_R. \tag{4.2}$$

Next, for $u \in P \cap \partial\Omega_b$, we have $u \in [0, b]$ and $u' \in [-b, b]$, then from (ii), we obtain $f(t, u, u') < \phi(\frac{1}{\sigma}b)$. Thus for $t \in [0, 1]$, like in the proof of (3.4), we have

$$\|Tu\| < \|u\|, \text{ for } u \in P \cap \partial\Omega_b. \tag{4.3}$$

Applying Lemma 1.1 to (4.1) and (4.3), or (4.2) and (4.3) yields that T has a fixed point $u_1 \in P \cap \overline{\Omega}_{r,b}$, and a fixed point $u_2 \in P \cap \overline{\Omega}_{b,R}$. It follows from Lemma 2.1 that problem (1.1) has at least two symmetric positive solutions u_1 and u_2. Noticing (4.3), we have $\|u_1\| \neq b$ and $\|u_2\| \neq b$, so $0 < \|u_1\| < b < \|u_2\|$.

Theorem 4.2. *Assume* (H1)-(H4) *hold, as do the following two conditions:*
(i) $f^0 < \psi_1((2\int_0^1 \gamma e(s)ds)^{-1})(\int_0^1 \gamma_1 e(\tau)w(\tau)d\tau)^{-1}$ *and*
$f^\infty < \psi_1((2\int_0^1 \gamma e(s)ds)^{-1})(\int_0^1 \gamma_1 e(\tau)w(\tau)d\tau)^{-1}$,
(ii) *There exists* $d > 0$ *satisfying* $f(t, u, v) > \phi(\frac{1}{\delta\sigma}d)$, $(t, u, v) \in [0, 1] \times [0, d] \times [-d, d]$.
Then problem (1.1) *has at least two symmetric positive solutions* $u_1(t)$, $u_2(t)$, *which satisfy* $0 < \|u_1\| < d < \|u_2\|$.

5 The nonexistence of positive solution

Theorem 5.1. *Assume* (H1)-(H4) *hold. If* $f(t, u, v) < \phi(\frac{1}{2\sigma}(\|u\|_0 + \|v\|_0))$ *for all* $(t, u, v) \in [0, 1] \times D$, *then problem* (1.1) *has no positive solution.*
Proof. Assume $u(t)$ is a positive solution of (1.1), we have $\|u\| = \|Tu\|$. From (2.1) and (2.4), it is easy to prove that $\|Tu\|_0 < \|u\|$ and $\|(Tu)'\|_0 < \|u\|$. So $\|Tu\| < \|u\|$, which is a contradiction.
Theorem 5.2. *Assume* (H1)-(H4) *hold. If* $f(t, u, v) > \phi(\frac{1}{\delta\sigma}(\|u\|_0 + \|v\|_0))$ *for all* $(t, u, v) \in [0, 1] \times D$, *then problem* (1.1) *has no positive solution.*

6 Applications

Example 6.1 Let $\phi(u) = |u|u$, $g(t) = \frac{3}{4}$, $h(t) = \frac{1}{2}$ in (1.1). Now we consider the boundary value problems

$$\begin{cases} (\phi(u''(t)))'' = w(t)f(t, u(t), u'(t)), \ t \in (0, 1), \\ u(0) = u(1) = \dfrac{3}{4}\displaystyle\int_0^1 u(s)ds, \\ \phi(u''(0)) = \phi(u''(1)) = \dfrac{1}{2}\displaystyle\int_0^1 \phi(u''(s))ds, \end{cases} \tag{6.1}$$

where $w(t) = 6$, $f(t, u, v) = (1 + \sin \pi t)(12^2 + u)(6 + \cos v)$ for $(t, u, v) \in [0, 1] \times [0, \infty) \times (-\infty, \infty)$.

Let $\psi_1(u) = \psi_2(u) = u^2$, $u > 0$. Then, by calculations we obtain that

$$\mu = \frac{3}{4}, \ \nu = \frac{1}{2}, \ \rho = \frac{1}{2}, \ \rho_1 = \frac{1}{6}, \ \gamma = 4, \ \gamma_1 = 2, \ \delta = \frac{\sqrt{3}}{48}, \ \sigma = \frac{2\sqrt{2}}{3}, \ \xi = \frac{\sqrt{2}}{2}.$$

Clearly, the conditions (H1)-(H4) hold.

Corollary 6.1 *The problem* (6.1) *has at least one symmetric positive solution.*

In fact, choosing $r = 1$, $R = 50$, we have $r = 1 < \frac{50\sqrt{3}}{48} = \delta R$. For $(t, u, v) \in [0, 1] \times [0, 1] \times [-1, 1]$,

$$f(t, u, v) = (1 + \sin \pi t)(12^2 + u)(6 + \cos v) \geq 12^2 \times 6 = \phi(\frac{1}{\delta\sigma} r),$$

and for $(t, u, v) \in [0, 1] \times [0, 50] \times [-50, 50]$,

$$f(t, u, v) \leq 2 \times (12^2 + 50) \times 7 = 2716 < 2812.5 = \phi(\frac{1}{\sigma} R).$$

So, it follows from the condition (i) of Theorem 3.1 that (6.1) has at least one symmetric positive solution.

Example 6.2 Let $\phi(u) = u,\ g(t) = \frac{3}{4}$, $h(t) = \frac{1}{2}$ in (1.1). Now we consider the boundary value problems

$$\begin{cases} (\phi(u''(t)))'' = w(t) f(t, u(t), u'(t)),\ t \in (0, 1), \\ u(0) = u(1) = \frac{3}{4} \int_0^1 u(s) ds, \\ \phi(u''(0)) = \phi(u''(1)) = \frac{1}{2} \int_0^1 \phi(u''(s)) ds, \end{cases} \tag{6.2}$$

where $w(t) = 6,\ f(t, u, v) = [\frac{1}{4} + t(1 - t)](1 + u)(\frac{1}{10} + v^2)[1 + (\|u\|_0 + \|v\|_0)^2]$ for $(t, u, v) \in [0, 1] \times [0, \infty) \times (-\infty, \infty)$.

Let $\psi_1(u) = \psi_2(u) = u,\ u > 0$. Then, by calculations we obtain that

$$\mu = \frac{3}{4},\ \nu = \frac{1}{2},\ \rho = \frac{1}{2},\ \rho_1 = \frac{1}{6},\ \gamma = 4,\ \gamma_1 = 2,\ \delta = \frac{1}{96},\ \sigma = \frac{4}{3},\ \xi = 1.$$

Clearly, the conditions (H1)-(H4) hold.

Corollary 6.2 *The problem* (6.2) *has at least two symmetric positive solutions.*

In fact,

$$f_0 = f_\infty = \infty > \psi_2((\int_0^1 \rho e(s) ds)^{-1})(\int_0^1 \rho_1 e(\tau) w(\tau) d\tau)^{-1} = 72,$$

and choosing $b = \frac{1}{5}$, for $(t, u, v) \in [0, 1] \times [0, \frac{1}{5}] \times [-\frac{1}{5}, \frac{1}{5}]$, we have that

$$\begin{aligned} f(t, u, v) &= [\frac{1}{4} + t(1 - t)](1 + u)(\frac{1}{10} + v^2)[1 + (\|u\|_0 + \|v\|_0)^2] \\ &\leq \frac{2}{4} \times (1 + \frac{1}{5}) \times (\frac{1}{10} + \frac{1}{25}) \times (1 + \frac{4}{25}) = \frac{1218}{12500} < \frac{3}{20} = \phi(\frac{1}{\sigma} b). \end{aligned}$$

So, it follows from Theorem 4.1 that (6.2) has at least two symmetric positive solutions $u_1(t),\ u_2(t)$ satisfying

$$0 < \|u_1\| < \frac{1}{5} < \|u_2\|.$$

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