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DISTANCE SPECTRA AND DISTANCE ENERGY OF SOME CLUSTER GRAPHS

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Abstract

The D-eigenvalues (distance eigenvalues) of a connected graph G are the eigenvalues of the distance matrix $D = D(G)$ of G. The collection of D - eigenvalues is the D - spectrum (distance spectrum) of G. In this paper, the distance polynomial, distance spectra and distance energy of some edge deleted graphs called cluster graphs are obtained.

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1 Introduction

Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_p\}$. The distance matrix $D = D(G)$of G has $d_{ii} = 0$ and d_{ij}= the length of the shortest path between the vertices v_i and v_j of G. The distance polynomial of G is defined as $det|\mu I - D|$. Where I is the unit matrix of order p. The eigenvalues of $D(G)$ are said to be the D - eigenvalues of G and form the D - spectrum of G, denoted by $Spec_D(G)$.

Since the distance matrix is symmetric, all its eigenvalues $\mu_i, i = 1, 2, \ldots, p$, are real and can be labelled so that $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_p$. If $\mu_{i1} > \mu_{i2} > \ldots \mu_{ig}$ are the distinct D - eigenvalues, then the D - spectrum can be written as

$$Spec_D(G) = \begin{pmatrix} \mu_{i1} & \mu_{i2} & \ldots & \mu_{ig} \\ m_1 & m_2 & \ldots & m_g \end{pmatrix}$$

where m_j indicates the algebraic multiplicity of the eigenvalue μ_{ij}. Of course, $m_1 + m_2 + \ldots + m_g = p$.

The D - energy, $E_D(G)$ is defined as

$$E_D(G) = \sum_{i=1}^{p} |\mu_i| \tag{1}$$

The concept of distance energy was recently introduced [6]. This definition was motivated by the much older and nowadays extensively studied graph energy defined in a mannerfully analogous to Eq.(1.1), but in terms of the ordinary graph spectrum (eigenvalues of the adjacency matrix of a graph). For more details on distance spectra and distance energy of graphs, see [4] - [7].

Two graphs are said to be D - equienergetic graphs, if they have the same D - energy.

The spectral graph theoretic definitions in this paper follow the book [1]. All graphs considered in this paper are simple.

2 Some cluster graphs

I. Gutman and L. Pavlović [2] introduced four classes of graphs obtained from complete graph by deleting edges. For the sake of continuity, we produce these here.

Definition 2.1. *[2] Let v be a vertex of the complete graph K_p, $p \geq 3$ and let e_i, $i = 1, 2, \ldots, k$, $1 \leq k \leq p-1$, be its distinct edges, all being incident to v. The graph $Ka_p(k)$ or $Cl(p,k)$ is obtained by deleting e_i, $i = 1, 2, \ldots, k$ from K_p. In addition, $Ka_p(0) \cong K_p$.*

Definition 2.2. *[2] Let f_i, $i = 1, 2, \ldots, k$, $1 \leq k \leq \lfloor p/2 \rfloor$ be independent edges of the complete graph K_p, $p \geq 3$. The graph Kb_p is obtained by deleting f_i, $i = 1, 2, \ldots, k$ from K_p. The graph $Kb_p(0) \cong K_p$.*

Definition 2.3. *[2] Let V_k be a k- element subset of the vertex set of the complete graph K_p, $2 \leq k \leq p$, $p \geq 3$. The graph $Kc_p(k)$ is obtained by deleting from K_p all the edges connecting pairs of vertices from V_k. In addition, $Kc_p(0) \cong Kc_p(1) \cong K_p$.*

Definition 2.4. *[2] Let $3 \leq k \leq p$, $p \geq 3$. The graph $Kd_p(k)$ obtained by deleting from K_p, the edges belonging to a k- membered cycle.*

The characteristic polynomials of the adjacency matrix of the above class of graphs are obtained in [2],[9] - [11]. In this paper, we obtain distance polynomial, distance spectra and distance enrgy of the following class of graphs, these class of graphs are defined in [10].

3 Main Results

Definition 3.1. *Let $(K_m)_i$, $i = 1, 2, \ldots, k$, $1 \leq k \leq \lfloor p/m \rfloor$, $1 \leq m \leq p$, be independent complete subgraphs with m vertices of the complete graph K_p, $p \geq 3$. The graph $K_{c_p}(m,k)$ obtained from K_p by deleting all the edges of k independent complete subgraphs $(K_m)_i$, $i = 1, 2, \ldots, k$. In addition, $K_{c_p}(m,0) \cong K_{c_p}(0,k) \cong K_{c_p}(0,0) \cong K_p$.*

In this paper, for the sake of brevity, in place of $K_{c_p}(m,k)$ we use $K_C(p,m,k)$.

Theorem 3.2. *Let p, m, k be positive integers. For $p \geq 3$, $1 \leq k \leq \lfloor p/m \rfloor$, $1 \leq m \leq p$, the distance polynomial of $K_c(p,m,k)$ is,*

$$det|\mu I - D| = X^{p-mk-1}(1+X)^{(m-2)k+1}\left\{(p-\mu-m)X - (p-2X-mk)(m-1)\right\} \times \\ \times \left\{X^2 - (m-2)X - m + 1\right\}^{k-1}$$

where X=1+μ

Proof: Without loss of generality, we assume that the vertices of $(K_p)_i$, as $v_1, v_2, \ldots, v_m,$
$v_{m+1}, \ldots, v_{2m}, v_{2m+1}, \ldots p$. The distance polynomial of $K_c(p, m, k)$ is equal to $det|\mu I - D|$. For the sake of computation, we take $det|D - \mu I|$ instead of $det|\mu I - D|$, and then we multiply the result by $(-1)^p$ to get the result of the Theorem 2.1.

Now, $det|D - \mu I|$ is equal to

$$\begin{vmatrix}
-\mu & 2 & \cdots & 2 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
2 & -\mu & \cdots & 2 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
2 & 2 & \cdots & -\mu & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 & -\mu & 2 & \cdots & 2 & 1 & 1 & \cdots & -1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 & 2 & -\mu & \cdots & 2 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 1 & \cdots & 1 & 2 & 2 & \cdots & -\mu & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & -\mu & 2 & \cdots & 2 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 2 & -\mu & \cdots & 2 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 2 & 2 & \cdots & -\mu & 1 & \cdots & 1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & -\mu & \cdots & 2 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 2 & \cdots & -\mu & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & -\mu & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & -\mu
\end{vmatrix} \tag{3}$$

We now perform the elementary transformations on the determinant (2.2) as follows:

Subtracting first column from all other columns and setting $1 + \mu = X$, we get (2.3).

$$\begin{vmatrix}
-\mu & 1+X & \cdots & 1+X & X & X & \cdots & X & X & X & \cdots & X & X & \cdots & X & X & \cdots & X \\
2 & -(1+X) & \cdots & 0 & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
2 & 0 & \cdots & -(1+X) & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 \\
1 & 0 & \cdots & 0 & -X & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & 2 & -X & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & \cdots & 0 & 1 & 1 & \cdots & -X & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -X & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & -X & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & & & & \vdots & & & & \vdots & & & \vdots & & & \vdots & \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & -X & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -X & \cdots & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & -X & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -X & \cdots & 0 \\
\vdots & & \vdots & & & & \vdots & & & & \vdots & & & \vdots & & & \vdots & \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & -X
\end{vmatrix} \tag{4}$$

Multiplying the rows 2,3,…,m of (2.3) by $-X$ and hence multiplying the determinant by $(-X)^{1-m}$ and setting $a = (-X)^{1-m}$, we get (2.4).

$$a\begin{vmatrix}
-\mu & 1+X & \cdots & 1+X & X & X & \cdots & X & X & X & \cdots & X & X & \cdots & X & X & \cdots & X \\
-2X & X(1+X) & \cdots & 0 & X & X & \cdots & X & X & X & \cdots & X & X & \cdots & X & X & \cdots & X \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-2X & 0 & \cdots & X(1+X) & X & X & \cdots & X & X & X & \cdots & X & X & \cdots & X & X & \cdots & X \\
1 & 0 & \cdots & 0 & -X & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & 1 & -X & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & \cdots & 0 & 1 & 1 & \cdots & -X & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -X & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & -X & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & -X & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -X & \cdots & 1 & 0 & \cdots & 0 \\
\vdots & & \vdots & & & & \vdots & & & & \vdots & & & \vdots & & & \vdots & \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & -X & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -X & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & -X
\end{vmatrix} \quad (5)$$

Adding the rows $mk+1, mk+2, \ldots, p$ of (2.4) to the rows $1, 2, \ldots, m$ and setting $t = -\mu + p - mk, s = 2X - mk$, we get (2.5).

$$a\begin{vmatrix}
t & 1+X & \cdots & 1+X & X & X & \cdots & X & X & X & \cdots & X & X & \cdots & X & 0 & \cdots & 0 \\
s & X(1+X) & \cdots & 0 & X & X & \cdots & X & X & X & \cdots & X & X & \cdots & X & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
s & 0 & \cdots & X(1+X) & X & X & \cdots & X & X & X & \cdots & X & X & \cdots & X & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & -X & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & 1 & -X & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & \cdots & 0 & 1 & 1 & \cdots & -X & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -X & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & -X & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & -X & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -X & \cdots & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & -X & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -X & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & -X
\end{vmatrix} \quad (6)$$

Determinant (2.5) is in the form $\begin{vmatrix} M & N \\ P & Q \end{vmatrix}$.As N is a ZERO block, hence (2.5) can be written as

$$|D - \mu I| = a(-X)^{p-mk}|M| \qquad (7)$$

In expression (2.6), $|M|$ is a determinant of order $mk \times mk$ is as given below.

$$\begin{vmatrix} t & 1+X & \cdots & 1+X & X & X & \cdots & X & X & X & \cdots & X & X & \cdots & X \\ s & X(1+X) & \cdots & 0 & X & X & \cdots & X & X & X & \cdots & X & X & \cdots & X \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ s & 0 & \cdots & X(1+X) & X & X & \cdots & X & X & X & \cdots & X & X & \cdots & X \\ 1 & 0 & \cdots & 0 & -X & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 1 & -X & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 1 & 1 & \cdots & -X & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -X & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & -X & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & -X & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -X & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & -X \end{vmatrix} \tag{8}$$

Extracting $1+X$ from each of the columns $2, 3, \ldots, m$ of (2.7), we get (2.8).

$$(1+X)^{m-1}\begin{vmatrix} t & 1 & \cdots & 1 & X & X & \cdots & X & X & X & \cdots & X & X & \cdots & X \\ s & X & \cdots & 0 & X & X & \cdots & X & X & X & \cdots & X & X & \cdots & X \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ s & 0 & \cdots & X & X & X & \cdots & X & X & X & \cdots & X & X & \cdots & X \\ 1 & 0 & \cdots & 0 & -X & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 1 & -X & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 1 & 1 & \cdots & -X & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -X & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & -X & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & -X & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -X & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & -X \end{vmatrix} \tag{9}$$

Subtracting the columns $2, 3, \ldots, m$ from the columns $m+1, m+2, \ldots, mk$ and setting $Z = 1 - m - X$, we get (2.9).

$$(1+X)^{m-1}\begin{vmatrix} t & 1 & \cdots & 1 & Z & Z & \cdots & Z & Z & Z & \cdots & Z & Z & \cdots & Z \\ s & X & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ s & 0 & \cdots & X & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & -X & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 1 & -X & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 1 & 1 & \cdots & -X & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -X & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & -X & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & -X & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -X & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & -X \end{vmatrix} \tag{10}$$

Adding the rows $m+1, m+2, \ldots, mk$ to first row and setting $c = t + mk - m$, i.e $c = p - \mu - m$, we get (2.10).

$$(1+X)^{m-1}\begin{vmatrix}
c & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
s & X & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
s & 0 & \cdots & X & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & -X & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & 1 & -X & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & \cdots & 0 & 1 & 1 & \cdots & -X & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -X & 1 & \cdots & 1 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & -X & \cdots & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & -X & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -X & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & -X
\end{vmatrix} \tag{11}$$

Expression (2.10) is equal to

$$(1+X)^{m-1}\begin{vmatrix}
c & 1 & 1 & \cdots & 1 \\
s & X & 0 & \cdots & 0 \\
s & 0 & X & \cdots & 0 \\
\cdots & \cdots & \cdots & \ddots & \cdots \\
s & 0 & 0 & \cdots & X
\end{vmatrix}\begin{vmatrix}
-X & 1 & 1 & \cdots & 1 \\
1 & -X & 1 & \cdots & 1 \\
1 & 1 & -X & \cdots & 1 \\
\cdots & \cdots & \cdots & \ddots & \cdots \\
1 & 1 & 1 & \cdots & -X
\end{vmatrix}^{k-1} \tag{12}$$

Simplifying (2.11)with the values $c = p - \mu - m, s = 2X - mk, X = 1 + \mu$ and substituting the result in (2.6) and then multiplying the result by $(-1)^p$, we get the result of Theorem 2.1.

Corollary 3.3. *For $1 \le m \le p$, $k = 1$, (2.1) takes the form*

$$det|\mu I - D| = (\lambda+1)^{p-m-1}(\lambda+2)^{m-1}\left\{(p-\mu-m)(\lambda+1) - (p - 2(\lambda+1) - m)(m-1)\right\} \tag{13}$$

4 Distance spectra and distance enrgy of $K_c(p,m,k)$ graphs

Theorem 4.1. *For $p \geq 3$, $1 \leq k \leq \lfloor p/m \rfloor$, $1 \leq m \leq p$, $mk < p$, the distance spectra of $K_c(p,m,k)$ contains -1 (p-mk-1 times), -2 ((m-2)k+1 times), α_1, α_2, α_3 (k-1 times), α_4 (k-1 times).*
where

$$\alpha_1 = \frac{(p+m-3)+\sqrt{(p+m-3)^2+4\{mk(m-1)-(p-1)(m-2)\}}}{2}$$

$$\alpha_2 = \frac{(p+m-3)-\sqrt{(p+m-3)^2+4\{mk(m-1)-(p-1)(m-2)\}}}{2}$$

$$\alpha_3 = \frac{m-4+\sqrt{(m-2)^2+4(m-1)}}{2}$$

$$\alpha_4 = \frac{m-4-\sqrt{(m-2)^2+4(m-1)}}{2}$$

Proof: It is easy to compute the roots of the result of Theorem 2.1. These roots are -1 (p-mk-1 times), -2 ((m-2)k+1 times), $\alpha_1, \alpha_2, \alpha_3$ and α_4 as given above and the collection of these roots forms the distance spectra of $K_c(p,m,k)$.

Theorem 4.2. *For $p \geq 3$, $1 \leq k \leq \lfloor p/m \rfloor$, $1 \leq m \leq p$,$mk < p$, the distance energy of $K_c(p,m,k)$ is*

$$E_D K_c(p,m,k) = p-mk-1+(m-4)k+1+|\alpha_1|+|\alpha_2|+(k-1)(|\alpha_3|+|\alpha_4|)$$

where$\alpha_i, i = 1,2,3,4,$ are as given in Theorem 3.1.

Proof: The distance spectrum of $K_c(p,m,k)$ is

$$\text{Spec}_D(K_c(p,m,k)) = \begin{cases} -1 & -2 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ p-mk-1 & (m-2)k+1 & 1 & 1 & k-1 & k-1 \end{cases}$$

Using the definition of the distance energy given by eqn.(1) and the spectra of $K_c(p,m,k)$, we get the result.

Here we give some examples in Table 1.

TABLE 1: The distance polynomial and distance spectra of $K_c(p,m,k)$ graphs.

p,m,k	**Distance polynomial of $K_c(p,m,k)$**	**D - spectra of $K_c(p,m,k)$**
9,3,2	$(\mu+1)^2(\mu+2)^3(\mu^2-9\mu-4)(\mu^2+\mu-2)$	(-1)^2, $(-2)^4, -0.4244, 9.4244, 1$
9,3,1	$(\mu+1)^5(\mu+2)^2(\mu^2-9\mu+2)$	(-1)^5, $(-2)^2, 0.2280, 8.7720$
8,4,1	$(\mu+1)^3(\mu+2)^3(\mu^2-9\mu+2)$	(-1)^3, $(-2)^3, 0.2280, 8.7700$
6,2,2	$(\mu+1)(\mu+2)(\mu^2-5\mu-4)(\mu^2+2\mu)$	(-1), (-2)$^2, 0, 0.7016, 5.7016$

In above table μ^t denotes the distance eigenvalue μ with algebraic multiplicity t.

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