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On The Special Curves in Minkowski 4 Spacetime

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Abstract

In [1], we gave a method for constructing Bertrand curves from the spherical curves in 3 dimensional Minkowski space. In this work, we construct the Bertrand curves corresponding to a spacelike geodesic and a null helix in Minkowski 4 spacetime.

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1 Preliminary Notes

In this section, we give basic notions of spacelike and null curves in Minkowski 4-space (see [2] , [3] and [6]). Let $\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4) : x_1, x_2, x_3, x_4 \in \mathbb{R}\}$ be a 4-dimensional vector space. For any vectors $x = (x_1, x_2, x_3, x_4), y = (y_1, y_2, y_3, y_4)$ in $\mathbb{R}^4$, the pseudo scalar product of x and y is defined to be $\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$. We call $(\mathbb{R}^4, \langle , \rangle)$ a Minkowski 4-space. We write $\mathbb{R}^4_1$ instead of $(\mathbb{R}^4, \langle , \rangle)$. We say that a non-zero vector $x \in \mathbb{R}^4_1$ is spacelike, lightlike (null) or timelike if $\langle x, x\rangle > 0$, $\langle x, x\rangle = 0$ or $\langle x, x\rangle < 0$ respectively. The norm of the vector $x \in \mathbb{R}^4_1$ is defined by $\|x\| = \sqrt{| \langle x, x\rangle |}$. For a vector $v \in \mathbb{R}^4_1$ and a real number c, we define a hyperplane with pseudo normal v by

$HP(v,c) = \{x \in \mathbb{R}^4_1 : \langle x, v\rangle = c\}$. We call $HP(v,c)$ a spacelike hyperplane, a timelike hyperplane or a lightlike hyperplane if v is timelike, spacelike or lightlike respectively. We also define de Sitter 3-space by $S^3_1 = \{x \in \mathbb{R}^4_1 : \langle x, x\rangle = 1\}$. For any $x = (x_1, x_2, x_3, x_4), y = (y_1, y_2, y_3, y_4), z = (z_1, z_2, z_3, z_4)$ in $\mathbb{R}^4_1$, we define a vector

$$x \wedge y \wedge z = \begin{vmatrix} -e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}$$

where (e_1, e_2, e_3, e_4) is the canonical basis of $\mathbb{R}^4_1$. We can easily show that $\langle a, (x \wedge y \wedge z)\rangle = \det(a, x, y, z)$.

Let $\gamma : I \longrightarrow S^3_1$ be a regular curve. We say that a regular curve γ is spacelike, timelike or null respectively, if $\gamma'(t)$ is spacelike, timelike or null at any $t \in I$, where $\gamma' = d\gamma/dt$. Now we describe the explicit differential geometry on spacelike and null curves in S^3_1.

Let γ be a spacelike regular curve, we can reparametrise γ by the arclength $s = s(t)$. Hence, we may assume that $\gamma(s)$ is a unit speed curve. So we have the tangent vector $t(s) = \gamma'(s)$ with $\|t(s)\| = 1$. In the case when $\langle t'(s), t'(s)\rangle \neq 1$, we have a unit vector $n(s) = \dfrac{t'(s) - \gamma(s)}{\|t'(s) - \gamma(s)\|}$. Moreover, define $e(s) = \gamma(s) \wedge t(s) \wedge n(s)$, then we have a pseudo orthonormal frame $\{\gamma(s), t(s), n(s), e(s)\}$ of $\mathbb{R}^4_1$ along γ. By the standard arguments, we can show the following Frenet-Serret type formulae: Under the assumption that $\langle t'(s), t'(s)\rangle \neq 1$,

$$\begin{aligned} \gamma'(s) &= t(s) \\ t'(s) &= -\gamma(s) + \kappa_g(s)\, n(s) \\ n'(s) &= \kappa_g(s)\, \delta(\gamma(s))\, t(s) + \tau_g(s)\, e(s) \\ e'(s) &= \tau_g(s)\, n(s) \end{aligned} \tag{1}$$

where $\delta(\gamma(s)) = -sign(n(s))$,

$$\begin{aligned} \kappa_g(s) &= \|t'(s) + \gamma(s)\| \\ \tau_g(s) &= \frac{\delta(\gamma(s))}{\kappa^2{}_g(s)} \det(\gamma(s), \gamma'(s), \gamma''(s), \gamma'''(s)) \end{aligned}$$

Now let $\gamma : I \longrightarrow S^3_1$ be a null curve. We will assume, in the sequel, that the null curve we consider has no points at which the acceleration vector is null. Hence $\langle \gamma''(t), \gamma''(t)\rangle$ is never zero. We say that a null curve $\gamma(t)$ in $\mathbb{R}^4_1$ is parametrized by the pseudo-arc if $\langle \gamma''(t), \gamma''(t)\rangle = 1$. If a null curve satisfies $\langle \gamma''(t), \gamma''(t)\rangle \neq 0$, then $\langle \gamma''(t), \gamma''(t)\rangle > 0$, and

$$u(t) = \int_{t_0}^{t} \langle \gamma''(t), \gamma''(t)\rangle^{1/4}\, dt$$

becomes the pseudo-arc parameter.

A null curve $\gamma(t)$ in $\mathbb{R}_1^4$ with $\langle \gamma''(t), \gamma''(t) \rangle \neq 0$ is a Cartan curve if $\{\gamma'(t), \gamma''(t), \gamma'''(t)\}$ is linearly independent for any t . For a Cartan curve $\gamma(t)$ in $\mathbb{R}_1^4$ with pseudo-arc parameter t , there exists a pseudo orthonormal basis $\{L, N, W_1, W_2\}$ such that

$$\begin{aligned} L &= \gamma' \\ L' &= W_1 \\ N' &= -\gamma + k_1 W_1 + k_2 W_2 \\ W_1' &= -k_1 L - N \\ W_2' &= -k_2 L \end{aligned} \tag{2}$$

where $\langle L, N \rangle = 1, \langle L, W_1 \rangle = \langle L, W_2 \rangle = \langle N, W_1 \rangle = \langle N, W_2 \rangle = \langle W_1, W_2 \rangle = 0$. We call $\{L, N, W_1, W_2\}$ as the Cartan frame and $\{k_1, k_2\}$ as the Cartan curvatures of γ. Since the Cartan frame is unique up to orientation, the number of the Cartan curvatures is minimum and the Cartan curvatures are invariant under Lorentz transformations, the set $\{L, N, W_1, W_2, k_1, k_2\}$ corresponds to the Frenet apparatus of a space curve. A direct computation shows that the values of the Cartan curvatures are

$$\begin{aligned} k_1 &= \frac{1}{2a^2} \left(\langle \gamma''', \gamma''' \rangle + 2aa'' - 4 (a')^2 \right) \\ k_2 &= -\frac{1}{a^4} \det(\gamma', \gamma'', \gamma''', \gamma^{(4)}) \end{aligned} \tag{3}$$

Theorem 1.1 *Let $\gamma(t)$ in $\mathbb{R}_1^4$ be a Cartan curve. Then γ is a pseudo-spherical curve iff k_2 is a nonzero constant.*

Theorem 1.2 *A Cartan curve $\gamma(t)$ in $\mathbb{R}_1^4$ fully lies on a pseudo-sphere iff there exists a fixed point A such that for each $t \in I$, $\langle A - \gamma(t), \gamma'(t) \rangle = 0$.*

2 Bertrand Curve Corresponding to A Space-like Geodesic on $\mathbf{S}_1^3$

Theorem 2.1 *Let γ be a spacelike geodesic curve on S_1^3. Then,*

$$\tilde{\gamma}(s) = a \int \gamma(v)\, dv + a \coth\theta \int e(v)\, dv + c$$

is a Bertrand curve where a and θ are constant numbers, c is a constant vector.

Proof. We will use the frame $\{\gamma(s), t(s), n(s), e(s)\}$ of γ given in the previous section. In this frame, let we choose $e(s)$ as a timelike vector (If $e(s)$ is a spacelike vector, the proof is similar). Hence $n(s)$ is spacelike and $\delta(\gamma(s)) = -1$. Using the equation (1), we can easily calculate that

$$\begin{aligned}
\tilde{\gamma}'(s) &= a\left[\gamma(s) + \coth\theta e(s)\right] \\
\tilde{\gamma}''(s) &= a\left[t(s) + \coth\theta\tau_g(s)\, n(s)\right] \\
\tilde{\gamma}'''(s) &= a[-\gamma(s) + \delta(\gamma(s))\kappa_g(s)\,\tau_g(s)\, t(s) \\
&\quad + \left(\kappa_g(s) + \coth\theta\tau_g'(s)\right) n(s) + \coth\theta\tau_g^2(s)\, e(s)]
\end{aligned}$$

Since $\langle\tilde{\gamma}'(s), \tilde{\gamma}'(s)\rangle = -\dfrac{a^2}{\sinh^2\theta}$, the curve $\tilde{\gamma}$ is timelike. If we calculate the first and second curvatures of $\tilde{\gamma}$ by using the equations in [8], we have

$$\begin{aligned}
\kappa(s) &= \frac{\sinh^2\theta\sqrt{1 + \coth^2\theta\tau_g^2}}{a} \\
\tau(s) &= \frac{A\sinh\theta}{a\sqrt{1 + \coth^2\theta\tau_g^2}}
\end{aligned}$$

where $A = \sqrt{\cosh^2\theta\left(\tau_g^2 + 1\right)^2 - \kappa_g^2\left(1 + \coth^2\theta\tau_g^2\right)}$. Since τ_g and κ_g are constants, we can choose $\beta = \dfrac{-a\sinh\theta\sqrt{1 + \coth^2\theta\tau_g^2}}{A}$ and $\alpha = \dfrac{a\coth^2\theta}{\sqrt{1 + \coth^2\theta\tau_g^2}}$, then we have $\alpha\kappa + \beta\tau = 1$. Hence $\tilde{\gamma}$ is a Bertrand curve.

3 Bertrand Curve Corresponding to A Null Helix on $\mathbf{S}_1^3$

Theorem 3.1 *Let γ be a null helix on S_1^3. Then,*

$$\tilde{\gamma}(s) = a\int L(v)\, dv + a\coth\theta\int W_2(v)\, dv + c$$

is a Bertrand curve where a and θ are constant numbers, c is a constant vector.

Proof.

$$\begin{aligned}
\tilde{\gamma}'(t) &= a\left[L(s) + \coth\theta W_2(t)\right] \\
\tilde{\gamma}''(t) &= a\left[1 - \coth\theta k_2\right] W_1(t) \\
\tilde{\gamma}'''(t) &= a\left[k_1(\coth\theta - 1)\, L(t) - (1 - \coth\theta k_2)\, N(t)\right]
\end{aligned}$$

Since $\langle\tilde{\gamma}'(t), \tilde{\gamma}'(t)\rangle = a^2\coth^2\theta$, the curve $\tilde{\gamma}$ is spacelike. If we calculate the first and second curvatures of $\tilde{\gamma}$, we have

$$\begin{aligned}
\kappa(t) &= \frac{(1 - \coth\theta k_2)}{a\coth^2\theta} \\
\tau(t) &= \frac{\sqrt{k_1^2\cosh^2\theta - 1}}{\cosh\theta}
\end{aligned}$$

Since k_1 and k_2 are constants, we can choose $\beta = -\dfrac{\cosh^3\theta}{\sqrt{k_1^2\cosh^2\theta - 1}}$ and $\alpha = \dfrac{a\cosh^2\theta}{(1-\coth\theta k_2)}$, then we have $\alpha\kappa + \beta\tau = 1$. Hence $\tilde{\gamma}$ is a Bertrand curve.

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