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n-Lie algebras with idempotent derivations

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Abstract

The identities on idempotent derivations of n-Lie algebras are provided, and the structure of n-Lie algebras which have idempotent derivations is discussed. The method of constructing n-Lie algebras which with idempotent derivations by n-Lie algebras and modules over the complex field is introduced.

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1 Fundamental notion

In this paper we investigate the structure of n-Lie algebras [1] with idempotent derivations over the complex field for $n \geq 3$. First we introduce some basic notions [2, 3].

An n-Lie algebra is a vector space A endowed with an n-ary multi-linear skew-symmetric operation $[\ ,\cdots,\ ]$ satisfying the n-Jacobi identity, that is, for all $x_1, \cdots, x_n,\ y_2, \cdots, y_n \in A$,

$$[[x_1, \cdots, x_n], y_2, \cdots, y_n] = \sum_{i=1}^{n}[x_1, \cdots, [x_i, y_2, \cdots, y_n], \cdots, x_n]. \tag{1}$$

Let I be a subspace of n-Lie algebra A, if I satisfies $[I, \cdots, I] \subseteq I$ ($[I, A, \cdots, A] \subseteq I$) then I is called a subalgebra (an ideal) of A. If an ideal I satisfies $[I, I, A, \cdots, A] = 0$, then I is called an abelian ideal.

A derivation of A is a linear map D of A satisfying for all $x_1, \cdots, x_n \in A$,

$$D([x_1, \cdots, x_n]) = \sum_{i=1}^{n} [x_1, \cdots, D(x_i), \cdots, x_n]. \tag{2}$$

If a derivation D satisfies $D^2 = D$, then D is called an idempotent derivation of A.

2 Main results

In the following, we suppose A is a finite dimensional n-Lie algebra over the complex field F ($n \geq 3$). We first prove some identities on idempotent derivations.

Lemma 1 *Let A be an n-Lie algebra, and D be an idempotent derivation, then for all $x_1, \cdots, x_n \in A$, we have*

1) $\sum\limits_{1 \leq i < j \leq n} [x_1, \cdots, D(x_i), \cdots, D(x_j), \cdots, x_n] = 0.$

2) $D([x_1, \cdots, D(x_i), \cdots, x_n]) = [x_1, \cdots, D(x_i), \cdots, x_n].$

3) $\sum\limits_{i=1}^{n} [D(x_1), \cdots, D(x_{i-1}), x_i, D(x_{i+1}), \cdots, D(x_n)] = 0.$

4) $[D(x_1), \cdots, D(x_n)] = 0.$

5) $[x_1, \cdots, x_i, D(x_{i+1}), \cdots, D(x_n)] = 0, 1 \leq i \leq n-2.$

Proof By identity (2) and $D = D^2$, for all $x_1, \cdots, x_n \in A$,

$$\begin{aligned} & D([x_1, \cdots, x_n]) = D^2([x_1, \cdots, x_n]) \\ = & \sum_{i=1}^{n} [x_1, \cdots, D^2(x_i), \cdots, x_n] + \sum_{1 \leq i < j \leq n} [x_1, \cdots, D(x_i), \cdots, D(x_j), \cdots, x_n] \\ = & D([x_1, \cdots, x_n]) + \sum_{1 \leq i < j \leq n} [x_1, \cdots, D(x_i), \cdots, D(x_j), \cdots, x_n]. \end{aligned}$$

It follows the result 1).

Thanks to the result 1) and

$$\begin{aligned} & D([x_1, \cdots, D(x_i), \cdots, x_n]) \\ = & [x_1, \cdots, D^2(x_i), \cdots, x_n] + \sum_{1 \leq i < j \leq n} [x_1, \cdots, D(x_i), \cdots, D(x_j), \cdots, x_n] \\ = & [x_1, \cdots, D^2(x_i), \cdots, x_n] = [x_1, \cdots, D(x_i), \cdots, x_n], \end{aligned}$$

we obtain the result 2).

By the result 1), we have

$$\begin{aligned} & \sum_{1 \leq i < j \leq n} D([x_1, \cdots, D(x_i), \cdots, D(x_j), \cdots, x_n]) \\ = & 2 \sum_{1 \leq i < j \leq n} [x_1, \cdots, D(x_i), \cdots, D(x_j), \cdots, x_n] \end{aligned}$$

$+\sum\limits_{1\le i<j<k\le n}[x_1,\cdots,D(x_i),\cdots,D(x_j),\cdots,D(x_k),\cdots,x_n]=0.$

Similarly, we have

$\sum\limits_{1\le i<j<k\le n}[x_1,\cdots,D(x_i),\cdots,D(x_j),\cdots,D(x_k),\cdots,x_n]=0.$

The result 3) follows from the induction.

By the identity (2) and the above discussion,

$$\begin{aligned}
&[D(x_1),\cdots,D(x_n)]\\
&=[x_1,D(x_2),\cdots,D(x_n)]-(n-1)[x_1,D(x_2),\cdots,D(x_n)]\\
&=D([D(x_1),x_2,D(x_3),\cdots,D(x_n)])-(n-1)[D(x_1),x_2,D(x_3),\cdots,D(x_n)]\\
&=D([D(x_1),\cdots,D(x_{i-1}),x_i,D(x_{i+1}),\cdots,D(x_n)])\\
&\quad-(n-1)[D(x_1),\cdots,D(x_{i-1}),x_i,D(x_{i+1}),\cdots,D(x_n)]\\
&=D([D(x_1),\cdots,D(x_{n-1}),x_n])-(n-1)[D(x_1),\cdots,D(x_{n-1}),x_n].
\end{aligned}$$

Again by the result 3), we have

$$\begin{aligned}
&n[D(x_1),\cdots,D(x_n)]\\
&=D(\sum_{i=1}^{n}[D(x_1),\cdots,D(x_{i-1}),x_i,D(x_{i+1}),\cdots,D(x_n)])\\
&-(n-1)\sum_{i=1}^{n}[D(x_1),\cdots,D(x_{i-1}),x_i,D(x_{i+1}),\cdots,D(x_n)]=0.
\end{aligned}$$

The result 4) holds.

Thanks to the result 4) and result 2),

$$\begin{aligned}
&D([x_1,D(x_2),\cdots,D(x_n)])\\
&=[D(x_1),\cdots,D(x_n)]+(n-1)[x_1,D(x_2),\cdots,D(x_n)]\\
&=(n-1)[x_1,D(x_2),\cdots,D(x_n)].
\end{aligned}$$

Thanks to $chF=0$, $[x_1,D(x_2),\cdots,D(x_n)]=0$.

Now suppose $[x_1,\cdots,x_{i-1},D(x_i),\cdots,D(x_n)]=0$ holds for i, where $1\le i\le n-1$. Then

$$\begin{aligned}
&D([x_1,\cdots,x_{i-1},x_i,D(x_{i+1}),\cdots,D(x_n)])\\
&=\sum_{j=1}^{i}[x_1,\cdots,D(x_j),\cdots,x_i,D(x_{i+1}),\cdots,D(x_n)]\\
&\quad+(n-i)[x_1,\cdots,x_{i-1},x_i,D(x_{i+1}),\cdots,D(x_n)]\\
&=(n-i)[x_1,\cdots,x_{i-1},x_i,D(x_{i+1}),\cdots,D(x_n)].
\end{aligned}$$

We obtain $[x_1,\cdots,x_{i-1},x_i,D(x_{i+1}),\cdots,D(x_n)]=0$ for all $1\le i\le n-2$. The proof is completed.

Lemma 2 *Let A be an n-Lie algebra, and D be an idempotent derivation, then there exists a basis $\{v_1,\cdots,v_r,u_1,\cdots,u_s\}$ of A such that*

$$\begin{cases} D(v_i)=v_i, 1\le i\le r,\\ D(u_j)=0, 1\le j\le s;\end{cases}\tag{3}$$

and the image $D(A)=I=\sum\limits_{i=1}^{r}Fv_i$ is an ideal, and the kernel $K=KerD=$

$\sum\limits_{j=1}^{s} Fu_j$ *is a subalgebra of* A.

Proof By properties of linear idempotent map on a finite dimensional vector space, there exists a basis $\{v_1, \cdots, v_r, u_1, \cdots, u_s\}$ of A such that D satisfies (3). Therefore, the restriction $D|_I : I \to I$ is identity, and $D|_K : K \to K$ is zero. By Lemma 1, for all $x_1, \cdots, x_n, y_1, \cdots, y_n \in A$,

$$[x_1, \cdots, D(x_i), \cdots, x_n] = D([x_1, \cdots, Dx_i, \cdots, x_n]) \in D(A) = I,$$

then $[DA, A, \cdots, A] \subseteq DA$, it implies that I is an ideal of A. For all $y_1, \cdots, y_n \in A$,

$$D([y_1, \cdots, y_n]) = \sum_{i=1}^{n} [y_1, \cdots, D(y_i), \cdots, y_n] = 0,$$

that is, $[y_1, \cdots, y_n] \in K$. Then K is a subalgebra of A.

Theorem 1 *Let* A *be an* n*-Lie algebra. Then there exists an idempotent derivation on* A *if and only if* $A = I \oplus K$*, where* I *is an abelian ideal and* K *is a subalgebra.*

Proof If there exists an idempotent derivation D on the n-Lie algebra A, thanks to Lemma 2, $A = I \oplus K$, where $I = D(A)$ is an ideal and $K = KerD$ is a subalgebra. For all $x_1, \cdots, x_r \in I$, $y_1, \cdots, y_{n-r} \in K$, where $r \geq 2$, since $[x_1, \cdots, x_r, y_1, \cdots, y_{n-r}] \in I$, by identities (2) and (3) and Lemma 2,

$$\begin{aligned} &[x_1, \cdots, x_r, y_1, \cdots, y_{n-r}] = D([x_1, \cdots, x_r, y_1, \cdots, y_{n-r}]) \\ &= \sum_{i=1}^{r} [x_1, \cdots, D(x_i), \cdots, x_r, y_1, \cdots, y_{n-r}] \\ &+ \sum_{j=1}^{n-r} [x_1, \cdots, x_r, y_1, \cdots, D(y_j), \cdots, y_{n-r}] \\ &= r[x_1, \cdots, x_r, y_1, \cdots, y_{n-r}]. \end{aligned}$$

Since $r \geq 2$ and $chF = 0$, we have $[x_1, \cdots, x_r, y_1, \cdots, y_{n-r}] = 0$. Therefore, $I = D(A)$ is an abelian ideal.

Conversely, define $D : A \to A$ as follows,

$$D(x) = x, \quad D(y) = 0, \quad \text{for all} \quad x \in I, \ y \in K.$$

Obviously, D satisfies $D^2 = D$, and for all $x_1, \cdots, x_n \in I, y_1, \cdots, y_n \in K$, and $2 \leq r \leq n-1$,

$$D([y_1, \cdots, y_n]) = \sum_{i=1}^{n} [y_1, \cdots, D(y_i), \cdots, y_n] = 0,$$

$$D([x_1, \cdots, x_n]) = [x_1, \cdots, x_n] = n[x_1, \cdots, x_n] = \sum_{i=1}^{n} [x_1, \cdots, D(x_i), \cdots, x_n] = 0,$$

$$D([x_1, \cdots, x_r, y_1, \cdots, y_{n-r}]) = [x_1, \cdots, x_r, y_1, \cdots, y_{n-r}] = 0,$$

$$\sum_{i=1}^{r} [x_1, \cdots, D(x_i), \cdots, x_r, y_1, \cdots, y_{n-r}]$$

$$+\sum_{j=1}^{n-r}[x_1,\cdots,x_r,y_1,\cdots,D(y_j),\cdots,y_{n-r}]$$
$$=r[x_1,\cdots,x_r,y_1,\cdots,y_{n-r}]=0.$$

Therefore, D is an idempotent derivation of A. The proof is completed.

From Theorem 1, for all n-Lie algebra $(A,[,\cdots,]_A)$ and an A-module (M,ρ). We can construct an n-Lie algebra B such that B with an idempotent derivation. Set $B=A\oplus M$. By paper [4], B is an n-Lie algebra such that $(A,[,\cdots,]_A)$ is a subalgebra of B and M is an abelian ideal, and for all $x_1,\cdots,x_{n-1}\in A$, $m\in M$, $[x_1,\cdots,x_{n-1},m]_B=\rho(x_1,\cdots,x_{n-1})(m)$. Then $D:B\to B$ defined by $D(x)=0$, $D(m)=m$ for all $x\in A$ and $m\in M$, is an idempotent derivation of B.

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References

[1] V. FILIPPOV, n-Lie algebras, *Sib. Mat. Zh.,* 1985, 26 (6), 126-140.

[2] R. BAI, Q. Li, R. Cheng, 3-Lie Algebras and Cubic Matrices, *Mathematica Aeterna*, 2014, 4(1): 169 C 174.

[3] R. BAI, L. GUO, J. LI, and Y. WU, Rota-Baxter 3-Lie algebras, *J. Math. Phys* 2013, 54, 063504.

[4] S. KASYMOV, On a theory of n-Lie algebras, Algebra Logika, 1987, 26(3): 277-297

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