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On paracompact spaces, projectively inductively closed functors, and dimension

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Abstract

We prove that projectively inductively closed (p.i.c. -) functors of a finite degree satisfy the logarithmic law for dimension dim of all paracompact $\sum$ - spaces. We get also sufficient conditions for a preservation of weakly countable-dimensional spaces by p.i.c. - functors.

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1 Introduction

By *Tych* we denote the category of all Tychonoff spaces and all their continuous functions. A Hausdorff compact space we call a compact space or just a compactum. By *Comp* we denote the full subcategory of *Tych*, whose objects are compacta.

Recall that a covariant functor $\mathcal{F} : Comp \rightarrow Comp$ is said to be normal [21] if it satisfies the following properties:

1) *preserves the empty set and singletons* , i.e. $\mathcal{F}(\emptyset) = \emptyset$ and $\mathcal{F}(\{1\}) = \{1\}$, where $\{k\}$ $(k \geq 0)$ denotes the set $\{0, 1, ..., k-1\}$ of nonnegative integers smaller than k. In this notatuons $0 = \{\emptyset\}$.

2) is *monomorphic*, i.e for any (topological) embedding $f : A \rightarrow X$, the mapping $\mathcal{F}(f) : \mathcal{F}(A) \rightarrow \mathcal{F}(X)$ is also an embedding.

3) is *epimorphic*, i.e. for any surjective mapping $f : X \rightarrow Y$, $\mathcal{F}(f) : \mathcal{F}(A) \rightarrow \mathcal{F}(X)$ is also surjective

4) *continuous*, i.e. for any inverse spectrum $S = \left\{X_\alpha; \pi_\beta^\alpha : \alpha \in A\right\}$ of compact spaces, the limit $f : \mathcal{F}(\lim S) \to \lim \mathcal{F}(S)$ of the mappings $\mathcal{F}(\pi_\alpha)$, where $\pi_\alpha : \lim S \to X_\alpha$ are the limiting projections of the spectrum S, is a homeomorphism.

5) *preserves intersections*, i.e. for any family $\{A_\alpha, \alpha \in A\}$ of closed subsets of a compact space X, the mapping $F(i) : \cap\{F(A_\alpha) : \alpha \in A\} \to F(X)$ defined by $F(i)(x) = F(i_\alpha)(x)$, where $i : A_\alpha \to X$ is the identity embeddings for all $\alpha \in \mathcal{A}$, is an embedding.

6) *preserves preimages*, i.e. for any mapping $f : X \to Y$ and an arbitrary closed set $A \subset Y$, the mapping $\mathcal{F}(f|f^{-1}(A))(F^{-1}(A) \to \mathcal{F}(A)$ is a homeomorphism.

7) *preserves weight*, i.e. $\omega(\mathcal{F}(X)) = \omega(X)$ for any infinite compactum X.

In what follows we shall use bigger classes than classes of normal functors. But any functors from this article shall *preserve empty set, intersections and be monomorphic.* By exp we denote the well-known *hyperspace* functor of nonempty closed subsets. This functor takes every (nonempty) compact space X to the set of all its nonempty closed subsets endowed with the (finite) *Vietoris* topology, and a continuous mapping $f : X \to Y$ to the mapping $\exp(f) : \exp(X) \to \exp(Y)$, defined by $\mathcal{F}(f)(A) = A$.

For a functor $\mathcal{F}$ and an element $a \in \mathcal{F}(X)$, the *support* of a is defined as intersection of all closed sets $A \subset X$ such that $a \in \mathcal{F}(A)$ (recall that we consider only monomorphic functors preserving intersections). This support we denote by $\mathrm{supp}_{\mathcal{F}(X)}(a)$. When it is clear what functor and space are meant, we denote the support of a merely by $\mathrm{supp}(a)$.

A.Ch.Chigogidze [8] extended an arbitrary intersection-preserving monomorphic functor $\mathcal{F} : Comp \to Comp$ to the category *Tych* by setting

$$F_\beta(X) = \{\alpha \in \mathcal{F}(\beta X) : \mathrm{supp}(a) \subset X\}$$

for any Tychnoff space X. If $f : X \to Y$ is a continuous mapping of Thychnoff spaces and $\beta f : \beta X \to \beta Y$ is the (unique) extension of f over their Stone-Cech compactifications, then

$$\mathcal{F}(\beta f)(\mathcal{F}(\beta X)) \subset \mathcal{F}_\beta(X).$$

The last inclusion is a corollary of a trivial fact

$$f(\mathrm{supp}(a)) \supset supp(\mathcal{F}(f)(a)). \tag{1.1}$$

Therefore, we can define the mapping

$$\mathcal{F}_\beta(f) = \mathcal{F}(\beta f)|X,$$

which makes F_β into a functor.

A.Ch.Chigogidze proved [8] that if a functor $\mathcal{F}$ has certain normality property, then F_β has the same property (modified when necessary). In what follows by a covariant functor $\mathcal{F} : Tych \to Tych$ we shall mean a functor of type F_β. For such a functor $\mathcal{F}$ and any compact space X the space $\mathcal{F}(X)$ is a compact space.

For a set A by $|A|$ we denote the cardinality of A. For a subset A of a space X by $\overline{A}^X$ the closure of A in X. By dim$X$ we denote the Lebesgue dimension, defined by finite open coverings of a (normal) space X (see, for example, at [1]).

One of the main notions of this article is a notion of *projectively inductively closed* functor (p.i.c. - functor). This notion was introduced in [25], where we gave sufficient conditions for a functor to be a p.i.c. - functor. In particular, any finitary normal functor is a p.i.c. - functor. In [25] we proved also that every preserving weight p.i.c. - functor of a finite degree preserves the class of stratifiable spaces and the class of σ - paracompact spaces. The same is true (even if we omit a preservation of weight) for paracompact Σ- spaces and paracompact p-spaces.

The main result of the present paper is Theorem 3.2 stating that every p.i.c. - functor of a finite degree satisfies the logarithmic law for dimension dim of all paracompact Σ - spaces. For compacta a similar result was received by V.N.Basmanov [4]. Corollaries 3.6 and 3.7 state the assertion of Theorem 3.2 takes place for all paracompact p - spaces, all paracompact σ -spaces, all stratifiable, in particular, metrizable spaces. This assertion holds for every normal finitary functor (Corollaries 3.8 and 3.9). We prove also that every p.i.c. - functor (in particular, finitely open functor preserving preimages) of a finite degree transforming finite sets into finite-dimensional compacta preserves weakly countable - dimensional Σ - paracompact spaces (Theorems 3.14 and 3.15). Corollary 3.16 states that every normal finitary functor preserves weakly countable - dimensional -paracompact spaces.

All spaces are assumed to be Tychonoff, and all mappings, continuous. Any additional information on general topology and covariant functors one can find, for example, in ([10], [11], [21]).

2 Preliminaries

Recall some definitions and facts, concerning this paper.

Definition 2.1 [2]. A *network* for a space X is a collection $\mathcal{N}$ of subsets of X such that whenever $x \in U$. With U open, there exists $F \in N$ with $x \in F \subset U$. A family $\mathcal{A}$ of subsets of X is said to be σ *-locally finite* if it is a union of countably many families $\mathcal{A}_n$ which are locally finite in X.

Definition 2.2 [19]. A topological space X is called a σ -space, if it has a σ-locally finite network.

Remark 2.3. A rather simple observation of the Definition 2.2 shows us that every closed subset of a σ -space is a σ-space.

Theorem 2.4 [16]. *A countable product of paracompact σ -spaces is a paracompact σ-space.*

In 1969 K.Nagami [18] has invented more general class than the class of -spaces.

Definition 2.5. A space X is a (*strong*) - Σ - space if there exists a σ - discrete collection $\mathcal{N}$, and a cover c of X by closed countably compact (compact) sets such that, whenever $C \in c$ and $C \subset U$ with open U, then $C \subset F \subset U$ for some $F \in \mathcal{N}$.

Clearly, from Definitions 2.1 and 2.5 we have

Proposition 2.6. *Every perfect preimage of a σ- space is a strong Σ - space. In particular, every σ- space is a Σ - space.*

K.Nagami [18] has shown that the class of strong Σ - spaces is strictly larger than the class of perfect preimages of σ - spaces. On the other hand, the class of perfect preimages of σ - spaces is much larger than the class of σ-spaces. For example, every compact σ - space is metrizable.

Definition 2.7 [3]. A space X is called a p-space if there exists a countable family u_n such that:

1) u_n consists of open subsets of βX;

2) $X \subset \cup u_n$ for each n;

3) $\cap_n st\,(x, u_n) \subset X$ for every $x \in X$.

Here for a family ν of subsets of a space Y by $st\,(y, \nu)$ we denote the set $\cup \{V \in \nu : y \in V\}$.

Theorem 2.8 [3]. *The class of paracompact p spaces coincides with the class of perfect preimages of metrizable spaces.*

Corollary 2.9 *Every paracompact p space is a perfect preimages of a paracompact σ - space and, consequently, is a paracompact Σ - space.*

Theorem 2.8 also yields

Corollary 2.10 [3]. *Every countable product of paracompact p- spaces is again a paracompact p-space.*

Proposition 2.11 [3]. *Every closed subspace of a paracompact p space is again a paracompact p space.*

Let us recall some more notions and facts.

Definition 2.12 [7]. A space X is stratifiable if there is a function G which assigns to each $n \in \omega$ and closed set $H \subset X$ an open set $G(n, H)$ containing H such that

(1) If $H \subset K$ then $G\,(n, H) \subset G\,(n, K)$;

(2) $H = \cap_n \overline{G\,(n, H)}$.

This class of stratifiable spaces was defined in 1961 by J.Ceder [7]. But he called these spaces by M_3-spaces. The latter form was proposed by C.R.Borges [5] in 1966.

Theorem 2.13 [15]. *Every stratifiable space is a* σ *- space.*

Theorem 2.14 [5]. *Stratifiable spaces are preserved by closed mappings.*

From Theorem 2.14 we get.

Corollary 2.15. *An image of a metrizable space under closed mapping is stratifiable. In particular, every metrizable space is stratifiable.*

Going back to functors $\mathcal{F} : Comp \to Comp$, we, evidently, have

$$a \in \mathcal{F}(\mathrm{supp}(a)). \tag{2.1}$$

If a functor $\mathcal{F}$ preserves preimages, then $\mathcal{F}$ preserves supports [21], i.e.

$$f\left(\mathrm{supp}\left(a\right)\right) = \mathrm{supp}\left(\mathcal{F}\left(f\right)\left(a\right)\right). \tag{2.2}$$

The property (2.2) can be conversed.

Proposition 2.16 [21]. *Any monomorphic preserving intersections functor preserves supports if and only if it preserves preimages.*

Definition of the functor $\mathcal{F}$ and property (2.2) imply that

$$f(\mathrm{supp}_{\mathcal{F}(X)}\left(a\right)) = \mathrm{supp}_{\mathcal{F}_{\beta}\left(Y\right)} F_{\beta}\left(f\right)\left(a\right) \tag{2.3}$$

for any preimage preserving functor $\mathcal{F} : Comp \to Comp$, continuous mapping $f : X \to Y$, and $a \in \mathcal{F}_{\beta}\left(X\right)$.

Now we recall one construction given by V.N.Basmanov [4].

Let $\mathcal{F} : Comp \to Comp$ be a functor. By $C\left(X, Y\right)$ we denote the space of all continuous mappings from X to Y with compact-open topology.

In particular, $C\left(\{k\}, Y\right)$ is naturally homeomorphic to the k-th power Y^k of the space Y; the homeomorphism takes each mapping $\xi : \{k\} \to Y$ to the point $\left(\xi\left(0\right), ..., \xi\left(k-1\right)\right) \in Y^k$.

For a functor $\mathcal{F}$, compact space X, and a positive integer k, V.N.Basmanov [4] defined the mapping $\pi_{\ \mathcal{F},X,k} : C\left(\{k\}, X\right) \times \mathcal{F}\left(\{k\}\right) \to \mathcal{F}\left(X\right)$ by $\pi_{\mathcal{F},X,k}\left(\ \xi, a\right) = \mathcal{F}\left(\xi\right)\left(a\right)$ for any $\xi \in C\left(\{k\}, X\right)$ and $a \in \mathcal{F}\left(\{k\}\right)$.

When it is clear what functor $\mathcal{F}$ and what space X are meant, we omit the subscripts $\mathcal{F}$ and X and write $\pi_{X,k}$ or π_k instead of $\pi_{\ \mathcal{F},X,k}$.

According to Shchepin's theorem ([25], Theorem 3.1), the mapping $\mathcal{F} : C\left(Z, Y\right) \to C\left(\ \mathcal{F}\left(Z\right),\ \mathcal{F}\left(Y\right)\right)$ is continuous for any *continuous* functor $\mathcal{F}$ and compact spaces Z and Y. This implies the following assertion.

Proposition 2.17 [4]. *If* $\mathcal{F}$ *is a continuous functor,* X *is a compact space, and* k *is a positive integer, then the mapping* $\pi_{\ \mathcal{F},X,k}$ *is continuous.*

Let $\mathcal{F}_k$ be a subfunctor of a functor $\mathcal{F}$ defined as follows. For a compact space X, $\mathcal{F}_k(X)$ is the image of the mapping $\pi_{\ \mathcal{F},X,k}$ and for a mapping $f : X \to$

Y, $\mathcal{F}_k(f)$ is the restriction of $\mathcal{F}(f)$ to $\mathcal{F}_k(X)$. Denote by $\bar{f} : C(\{k\}, X) \to (C\{k\}, Y)$ the mapping which takes ξ to composition $f \circ \xi$. It is easy to see that

$$\pi_{Y,k} \circ \bar{f} \times id_{\mathcal{F}(\{k\})} = \mathcal{F}(f) \circ \pi_{X,k}. \tag{2.4}$$

Therefore, $\mathcal{F}(f)(\mathcal{F}_k(X)) \subset \mathcal{F}_k(Y)$. Hence, $\mathcal{F}_k$ is a functor. A functor $\mathcal{F}$ is called a functor of degree n, if $\mathcal{F}_n(X) = \mathcal{F}(X)$ for any compact space X, but $\mathcal{F}_{n-1}(X) \neq \mathcal{F}(X)$ for some X. The next assertion (Proposition 2.18) is Shchepin's definition of the functor $\mathcal{F}_k$. But using Basmanov's definition we should prove it. One can find the proof in [24].

Proposition 2.18. *For any continuous functor ? and a compact space X, we have*

$$\mathcal{F}_k(X) = \{a \in \mathcal{F}(X) : \ supp(a) \leq k\}.$$

Corollary 2.19. *For any compact space X, we have*

$$\exp_k(X) = \{a \in \exp(X) : |a| \leq k\}.$$

The definition of a support and the property (2.1) imply

Proposition 2.20. For a functor $\mathcal{F}$, a compact space X, and a closed subset A of X,

$$\mathcal{F}(A) = \{a \in \mathcal{F}(X) : \mathrm{supp}(a) \subset A\}.$$

For a Tychonoff space X, a functor $\mathcal{F} : Comp \to Comp$, and a positive integer k, we put

$$\mathcal{F}_k(X) = \pi_{\mathcal{F},\beta X,k}((C(\{k\}), X) \times \mathcal{F}(\{k\}))$$

and denote the restriction of $\pi_{\mathcal{F},\beta X,k}$ to $C(\{k\}) \times \mathcal{F}(\{k\})$ by $\pi_{\ \mathcal{F},X,k}$. If $f : X \to Y$ is a continuous mapping, then

$$(\beta f)(\mathcal{F}_k(X)) \subset \mathcal{F}_k(Y),$$

in view of the equality (2.4) for the mapping βf. Therefore, setting

$$\mathcal{F}_k(f) = \mathcal{F}_k(\beta f)\,|\mathcal{F}(X)$$

we obtained a mapping

$$\mathcal{F}_k(f) : \mathcal{F}_k(X) \to \mathcal{F}_k(Y).$$

Thus, we have defined the covariant functor

$$\mathcal{F}_k : Tych \to Tych$$

that extends the functor $\mathcal{F}_k : Comp \to Comp$ to the category Tych. Proposition 2.18 implies the following assertion.

Proposition 2.21 [24]. *If $\mathcal{F} : Comp \to Comp$ is a continuous functor, then $\mathcal{F}_k : Tych \to Tych$ is a subfunctor of the functor F_β, and*

$$\mathcal{F}_k(X) = \mathcal{F}_\beta(X) \cap \mathcal{F}_k(\beta X). \tag{2.5}$$

Proposition 2.22 [24]. *For a compact space X, a continuous functor $\mathcal{F}$ and a positive integer k, the set $\mathcal{F}_k(X)$ is closed in $\mathcal{F}_\beta(X)$.*

Propositions 2.21 and 2.22 imply

Proposition 2.23 [24]. *For a Tychonoff space X, a continuous functor $\mathcal{F}$, and a positive integer k, the set $\mathcal{F}_k(X)$ is closed in $\mathcal{F}_\beta(X)$.*

Proposition 2.24 [24]. *For a Tychonoff space X, a continuous functor $\mathcal{F}$, and a positive integer k, the set $\mathcal{F}_k(X)$ is closed in $\mathcal{F}_{k+1}(X)$.*

Recall that a functor $\mathcal{F}$ is said to be finitely open [24], if the set $\mathcal{F}_k(\{k+1\})$ is open in $\mathcal{F}(\{k+1\})$ for any positive integer k. The dual for this definition states that $\mathcal{F}(\{k+1\}) \setminus \mathcal{F}_k(\{k+1\})$ is closed in $\mathcal{F}(\{k+1\})$.

Remark 2.25. As an example of a finitely open functor one can take any finitary functor F, i.e. a functor F such that $F(\{k\})$ is finite for any positive integer k. In particular, the hyperspace functor exp and its subfunctors $\exp_m$ are finitary and, consequently, finitely open functors.

Lemma 2.26 [25] *For any continuous, preserving preimages functor F_β the mapping $\pi_{F_\beta,X,1}$ is a homeomorphism.*

Definition 2.27. An epimorphism $f : X \to Y$ is called *inductively closed* if there exists a closed subset A of X such that $f(A) = Y$ and $f|A$ is a closed mapping.

Definition 2.28 [25]. A functor F_β is said to be projectively inductively closed (p.i.c.) if the mapping $\pi_{F_\beta,X,k}$ is inductively closed for any Tychonoff space X and positive integer k.

The next theorem gives us sufficient conditions for a functor F_β to be projectively inductively closed (a p.i.c.-functor)

Theorem 2.29 [25]. *Every continuous, finitely open functor $F_\beta : Tych \to Tych$, that preserves preimages is a p.i.c.-functor*

From Remark 2.25 and Theorem 2.29 we get

Corollary 2.30. *Every finitary normal functor, in particular, the functor* $\exp_m$ *is a p.i.c.-functor.*

Theorem 2.31 [25]. *Let F_β be a p.i.c.-functor of a finite degree. Then F_β preserves the class of paracompact Σ-spaces and the class of paracompact p-spaces.*

3 Dimension

Remark 3.1. One of the main tools for the proof of Theorem 3.2. is the inequality

$$\dim(\prod_{i=1}^{m} X_i) \leq \sum_{i=1}^{m} \dim X_i. \qquad (*)$$

The most general result here belongs to B.A. Pasynkov. In ([20], 1975) he declared that the inequality (*) holds for any paracompact Σ-spaces X_i. But the proof is still unpublished, though there is a hand-written text of it. In ([13], 1983) V.V. Filippov proved that the inequality (*) holds for any paracompact p-spaces . This result was declared by V.V. Filippov in ([12], 1973). In 1989 the inequality (*) was proved for paracompact σ -spaces independently by I.M.Leibo [16] and T.F. Zhuraev (see [22], [23]).

Theorem 3.2. *Let F_β be a p.i.c.-functor of finite degree m, and let X be a paracompact Σ-space. Then*

$$\dim F\beta(X) \leq m\dim X + \dim F_\beta(\{m\}) \equiv d(m) \qquad (3.1)$$

Proof. If $\dim X = \infty$, the assertion is trivial. Now let $\dim X = n < \infty$. We prove our assertion by induction on m. If $m = 1$, the equality (3.1) holds, since $F_1(X)$ is homeomorphic to the product $X \times F(\{1\})$ by Lemma 2.26. Hence, $\dim(X \times F(\{1\}) \leq \dim X + \dim F(\{1\})$ according to Remark 3.1, because $F(\{1\})$ is a paracompact Σ-space being a compact space. Now let for all $1 < k \leq m$ the equality

$$\dim F_l(X) \leq l \dim X + \dim F_\beta(\{l\}) \equiv d(l) \qquad (3.1_l)$$

has been proven. Let us prove the equality (3.1_k). Recall, that for a positive integer p by S_p is denoted the symmetric group of all homeomorphisms $\sigma \in C(\{p\},\{p\})$. It consists exactly of $p!$ elements. Fix $b \in F_l(X) \backslash F_{l-1}(X)$. Let $\mathrm{supp}(b) = \{x_0, x_1, ..., x_{l-1}\}$, and let $b = \pi_l(\xi, a) = F(\xi)(a)$. Let $h_\xi : \{l\} \to \mathrm{supp}(b)$ be a bijection defined by: $h_\xi(i) = x_{\xi(i)}$. By $j : \mathrm{supp}(b) \to X$ we denote the identical injection, i.e. $j(x_i) = x_i$. Let $g : \mathrm{supp}(b) \to \{l\}$ be a bijection defined by: $g(x_i) = i$. The composition

$$\sigma_\xi \equiv g \circ J \circ h_\xi : \{l\} \to \{l\}. \qquad (3.2)$$

is a bijection by definitions of g, J, and h_ξ. It is clear that (3.2) implies

$$\xi = j \circ h_\xi. \qquad (3.3)$$

Therefore,

$$\sigma_\xi = g \circ \xi \text{ and } \xi = g^{-1} \circ \sigma_\xi \qquad (3.4)$$

Hence, ξ is a bijection from $\{k\}$ to $\mathrm{supp}(b)$ and an injection from $\{k\}$ to $F_k(X)$.

In what follows we need more definitions and facts. Let X be a topological space let G be a topological group, and let e be the neutral element of G. Let $\alpha : G \times X \to X$ be a continuous mapping. We denote $\alpha(g, x)$ *by* $g(x)$. The mapping α is called an action of the group G on the space X if:

$$e(x) = x \text{ and } g_2(g_1(x)) = g_2 g_1(x) \tag{3.5}$$

In this definition we assume that we consider only one fixed action α. Let $x \in X$. An orbit of x with respect to the action α is the set $\bar{x} = \{g(x) : g \in G\}$. Clearly, two orbits $\bar{x}_1$ and $\bar{x}_2$ either coincides or disjoint. So, we have decomposition $R(G)$ of the space X, whose members are orbits $\bar{x}$. A quotient set of X with respect to this decomposition we denote by X/G. A quotient mapping $X \to X/G$ we denote by $q - q(G)$. The set X/G is endowed by the quotient topology with respect to the mapping q. Thus, we have quotient mapping $q : X \to X/G$ defined by: $q(x) = \bar{x}$.

Theorem 3.3. [6] *The quotient mapping* $q : X \to X/G$ *is open.*

Now we are going back to the proof of Theorem 3.2. Since $b \in F_k(X) \backslash F_{k-1}(X)$ is fixed, we may assume that $g^{-1} : k \to \mathrm{supp}(b)$ is the identity mapping. Hence, in view of (3.4) we may identify ξ with σ_ξ or we may assume that $\xi = \xi_b \in S_k$.

Put $Z = \pi_k^{-1}(F_k(X) \backslash F_{k-1}(X))$. In the space Z we define an action α of the group S_k in the following way:

$$\alpha(\sigma, (\xi, \alpha)) = (\xi \circ \sigma^{-1}, F(\sigma)(\alpha))$$

or, what the same

$$\sigma(\xi, \alpha) = (\xi \circ \sigma^{-1}, F(\sigma)(\alpha)) \tag{3.6}$$

First of all we have to show that

$$\alpha(S_k \times Z) = Z \tag{3.7}$$

Since $e(\xi, \alpha) = (\xi, \alpha)$ we have to verify only that

$$\alpha(S_k \times Z) \subset Z \tag{3.8}$$

To do that, it suffices to check that if

$$\pi_k(\xi, a) = b \in F_k(X) \backslash F_{k-1}(X) \text{ then } \pi_k(\sigma(\xi, a)) = b.$$

We have

$$\pi_k(\sigma(\xi, a)) = (\text{in view of } (3.6)) =$$

$$= \pi_k(\xi \circ \sigma^{-1}, F(\sigma)(a)) = F(\xi \circ \sigma^{-1})(F(\sigma)(a) = F(\xi \circ \sigma^{-1} \circ \sigma)(a) = F(\xi)(a) = b.$$

Thus, inclusion (3.8) and, consequently, equality (3.7) hold.

Let $q : Z \to Z/S_k$ be the quotient mapping. We are going to show that

$$Z/S_k \text{ is homeomorphic to } F_k(X)\backslash F_{k-1}(X) \tag{3.9}$$

and

$$\text{mappings } q \text{ and } \pi_k \mid Z \text{ coincide as mapping of sets.} \tag{3.10}$$

From property (3.10) we shall get that decomposition of Z generated by mappings q and $\pi_k \mid Z$ coincide. The space Z is equipped with quotient topology. On the other hand, since F is a p.i.c.-functor, the mapping π_k is inductively closed and, consequently, is a quotient one. But $\pi_k \,|Z$ is a restriction of the quotient mapping π_k onto a preimage of open in $F_k(X)$ set $F_k(X)\backslash F_{k-1}(X)$. Therefore, $\pi_k \,|Z$ is a quotient mapping. Hence, from (4.10) we shall get that spaces Z/S_k and $F_k(X)\backslash F_{k-1}(X)$ are identically homeomorphic.

To prove (3.10), it suffices to show that

$$\overline{(\xi,\alpha)} = \pi_k^{-1}(b) \text{ for } b = \pi_k(\xi,a) \in F_k(X)\backslash F_{k-1}(X) \tag{3.11}$$

Let $\sigma(\xi,a) \in \overline{(\xi,a)}$. Then $\pi_k(\sigma(\xi,a))=$ by (3.6) $= \pi_k(\xi\sigma^{-1}, F(\sigma)(a)) = F(\xi\sigma^{-1})(F(\sigma)(a)) = F(\xi\sigma^{-1}\sigma)(a) = F(\xi)(a) = \pi_k(\xi,a) = b$. Hence, we proved that

$$\overline{(\xi,a)} \subset \pi_k^{-1}(b).$$

Now let $\pi_k^{-1}(b)= \{(\xi_i,a_i) : i = 1,...,s\}$. Then $b = \pi_k(\xi_i,a_i) = F(\xi_i)(a_i)$ for each $i \in \{1,...,s\}$. Hence, $F(\xi_i)(a_i) = b = F(\xi)(a)$ for each $i \in \{1,...,s\}$. But for each $i \in \{1,...,s\}$ there exists $\sigma_i \in S_k$ such that $\xi = \xi_i\sigma_i$. Hence, $\xi_i = \xi\sigma_i^{-1}$ for each $i \in \{1,...,s\}$. Then $\pi_k(\xi_i,a_i) = F(\xi\sigma_i^{-1})(a_i) = F(\xi)(F(\sigma_i^{-1})(a_i) = b = F(\xi)(a)$ for each $i \in \{1,...,s\}$. But ξ is an injection and F is monomorphic functor. Therefore, $F(\xi)$ is a monomorphism. Hence $a = F(\sigma_i^{-1})(a_i)$or $a_i = F(\sigma_i)(a)$ for each $i \in \{1,...,s\}$. Consequently, for each $i \in \{1,...,s\}$ we have $(\xi_i,a_i) = (\xi\sigma_i^{-1}, F(\sigma_i)(a)) \in \overline{(\xi,a)}$ So, $\pi_k^{-1}(b) \subset \overline{(\xi,a)}$. Equality (3.11) is proved. Hence, equality (3.10) holds.

Thus, q and $\pi_k \,|Z$ coincide like mappings of the topological space Z. This implies, in view of Theorem 3.3. the following statement:

$$\pi_k \,|Z \text{ is an opening mapping.} \tag{3.12}$$

Further, we have

$$\left|\pi_k^{-1}\ (b)\right| = k! \tag{3.13}$$

for every $b \in F_k(X)\backslash F_{k-1}(X)$.

Indeed, (3.11) implies that $\left|\pi_k^{-1}\ (b)\right| = \overline{|(\xi,\,a)|}$. On the other hand, $\overline{|(\xi,\,a)|} \geqslant k!$ according to (3.6), because $\sigma_1 \neq \sigma_2$ implies $(\xi\sigma_1^{-1}F(\sigma_1)(\alpha)) \neq (\xi\sigma_2^{-1}F(\sigma_2)(\alpha))$ for each $\sigma_1,\sigma_2 \in S_k$.

Lemma 3.4. *The mapping $\pi_k \,|Z$ is a local homeomorphism.*

Proof. Let $b \in F_k(X)\backslash F_{k-1}(X)$. Then $\left|\pi_k^{-1}(b)\right| = k!$ in view of (3.13). Let $\pi_k^{-1}(b) = \{z_i : i \in \{k!\}\}$. Let $\{U_i : i \in \{k!\}\}$ be a disjoint family of open neighborhoods of points z_i in the space Z.
Denote the set $\pi_k(U_i)$by V_i. Then according to (3.12), every V_i is an open neighborhood of the point b. Set $V = \cap\{V_i : i \in \{k!\}\}$ and $W_i = ((\pi_k\,|Z)^{-1}(V)) \cap U_i$. Let us show that

$$\pi_k(W_i) = V \tag{3.14}$$

for every $i \in \{k!\}$.

Inclusion $\subset$ is trivial. Now let $b_1 \in V$. Then, clearly, $\pi_k^{-1}(b_1) \subset (\pi_k\,|Z)^{-1}(V)$. On the other hand, $\pi_k^{-1}(b') \cap (U_i)$ is not empty for any $b' \in V_i$, since $V_i = \pi_k(U_i)$. Hence $\pi_k^{-1}(b') \cap (U_i)$ is not empty, because $b_1 \in V \subset V_i$. Thus, equality (3.14) is proved.

At last, $\pi_k\,|W_i$ is a one-to-one correspondence. Indeed, $\pi_k^{-1}(b_1) \cap W_i$ is a nonempty set in view of (3.14). On the other hand, if $\left|\pi_k^{-1}(b_1) \cap W_i\right| \geqslant 2$ then $\left|\pi_k^{-1}(b_1)\right| > k!$. But this contradicts to (3.13).

So, $\pi_k\,|W_i : W_i \to V$ is a homeomorphism being a one-to-one continuous open mapping. Lemma 3.4 is proved.

Remark 3.5. Clearly, we proved more than just the assertion of Lemma 3.4. Namely, we proved that every point $b \in F_k(X)\backslash F_{k-1}(X)$ has an open neighborhood $V = V_b$ such that $\pi_k^{-1}(V_b)$ is a union of a disjoint open family $\{W_i^b : i \in \{k!\}\}$ with property: $\pi_k\left|W_i^b\right. : W_i^b \to V_b$ is a homeomorphism for every $i \in \{k!\}$.

To prove inequality (3.1_k), in view of Dowker‘s theorem (see [1, Ch 4] or [9]) and an inductive assumption, it suffices to show that for every closed in $F_k(X)$ set $A \subset F_k(X)\backslash F_{k-1}(X)$ we have

$$\dim A \leqslant k \dim X + \dim F_\beta(\{k\}) \equiv d(k) \tag{3.15}$$

One the one hand, we have

$$\dim(X^k \times F_\beta(\{k\})) \leqslant k \dim X + \dim F_\beta(\{k\}) \tag{3.16}$$

according to Remark 3.1, since X and $F_\beta(\{k\})$ are paracompact $\sum$-spaces (the last is true view of Proposition 2.6). On the other hand, the space $F_k(X)\backslash F_{k-1}(X)$ is locally homeomorphic to an open subset of the product $X^k \times F_\beta(\{k\})$ in view of Lemma 3.4. Moreover, according to Remark 3.5, there is a open covering $\{U_\gamma : \gamma \in \Gamma\}$ of the set $F_k(X)\backslash F_{k-1}(X)$ such that every $\pi_k^{-1}(U_\gamma)$ is homeomorphic to the product $U_\gamma \times \{k!\} \subset X^k \times F_\beta(\{k\})$.

Denote by V_γ the intersection $U_\gamma \cap A$. The family $\nu \in \{V_\gamma : \gamma \in \Gamma\}$ is an open covering of the space A. But A is a paracompact space as a closed subset of the space $F_k(X)$ which is paracompact, because of Theorem 2.31. Hence, there exists an open in A locally finite refinement $\omega = \{W_\delta : \delta \in D\}$

of the covering υ. Again, every $\pi_k^{-1}(W_\delta)$ is homeomorphic to the product $W_\sigma \times \{k!\} \subset X^k \times F_\beta(\{k\})$.

There exists a closed in A (and, consequently, in $F_k(X)$) refinement $s = \{S_\delta : \delta \in D\}$ of ω such that $S_\delta \subset W_\delta$ (see, for instance, [1]). Every S_δ is homeomorphic to a closed subset of $X^k \times F_\beta(\{k\})$, because the closed in $X^k \times F_\beta(\{k\})$ set $\pi_k^{-1}(S_\delta)$ is homeomorphic to the product $S_\delta \times \{k!\}$ which is a discrete union of S_δ's. Hence,

$$\mathrm{dimS}_\delta \leqslant \dim(X^k \times F_\beta(\{k\})) \leqslant \text{(by (3.16))} \leqslant d(k)$$

The covering s is locally finite, since $S_\delta \subset W_\delta$. Consequently, $\dim A \leqslant d(k)$ according to locally finite sum theorem for dim (see [1, Ch 4] or [9]). Thus, inequality (3.15) is checked. Hence, we proved that inequality (3.1_k) holds. So, by induction, Theorem 3.2. is proved.

Proposition 2.6, Corollaries 2.9, 2.15 and Theorem 3.2 yield.

Corollary 3.6. *Let F be a p.i.c.-functor of finite degree m, and let X be either a paracompact σ - space or a paracompact p-space. Then*

$$\dim F_\beta(X) \leqslant m \dim X + \dim F_\beta(m)$$

Corollary 3.7. *Let F be a p.i.c.-functor of finite degree m, and let X be a stratifiable, in particular metrizable, space. Then*

$$\dim F_\beta(X) \leqslant m \dim X + \dim F_\beta(m)$$

Since every finitary functor is finitely open, in view of Remark 2.25, Theorems 2.29 and 3.2, and Corollaries 2.30, 3.6, and 3.7 imply two following statements.

Corollary 3.8. *Let F be a normal finitary functor of finite degree m, in particular, the functor $\exp_m$, and let X be a paracompact $\sum$-space (in particular, a paracompact σ - space or a paracompact p - space). Then*

$$\dim F_\beta(X) \leqslant m \,\mathrm{dimX}$$

Corollary 3.9. *Let F be a normal finitary functor of finite degree m, in particular, the functor $\exp_m$, and let X be a stratifiable, in particular metrizable space. Then*

$$\dim F_\beta(X) \leqslant m \,\mathrm{dimX}$$

Remark 3.10. If somebody doesn't like to apply statements with unpublished proofs, then he (she) can be satisfied with Corollary 3.6. In fact, this assertion can be proved like Theorem 3.2. Necessary changings are: 1) inequality (3.16) holds in view of Corollary 2.10 and Remark 3.1 for paracompact p-spaces, and according to Theorem 2.4, Remark 3.1, and Morita's theorem [17] on dimension of a product of a paracompact space and a compact space for paracompact σ - space; 2) as for a paracompactness of the set A, one can use (in addition to Theorem 2.31) Proposition 2.6 for a σ - space.

Let us recall

Definition 3.11. A space X is said to be *weakly countable-dimensional* if X is a union of a countable family of closed subsets X_i such that $\dim X_i < \infty$ for each i.

The next two statements are trivial.

Proposition 3.12. *Every closed subspace of a weakly countable-dimensional space is a weakly countable-dimensional space again.*

Proposition 3.13. *Let Y be a weakly countable-dimensional space and m be natural number. If every closed finite-dimensional subspace X of Y satisfies the inequality (*). Then Y^m is a weakly countable-dimensional space.*

Remark 3.1, Corollaries 3.6 and 3.7, and Propositions 3.12 and 3.13 imply.

Theorem 3.14. *Let F be a p.i.c-functor of a finite degree transforming finite sets into finite-dimensional compacta, let X be a weakly countable-dimensional space, and let X belong to one of the following classes:*

a) Σ - *paracompact space;*

b) *p-paracompact spaces;*

c) σ - *paracompact spaces;*

d) *stratifiable spaces;*

e) *metrizable spaces.*

Then $F_\beta(X)$ is a weakly countable-dimensional space.

Theorems 2.29 and 3.14 imply.

Theorem 3.15. *Let F be a continuous finitely open functor of a finite degree transforming finite sets into finite-dimensional compacta and preserving preimages, let X be a weakly countable-dimensional space, and let X belong to one of the following classes:*

a) Σ - *paracompact space;*

b) *p-paracompact spaces;*

c) σ - *paracompact spaces;*

d) *stratifiable spaces;*

e) *metrizable spaces.*

Then $F_\beta(X)$ is a weakly countable-dimensional space.

Remark 2.25, Corollary 2.30, and Theorem 3.14 yield.

Corollary 3.16. *Let F be a normal finitary functor of a finite degree, in particular, the functor* $\exp_m$, *let X be a weakly countable-dimensional space, and let X belong to one of the following classes:*

a) Σ - *paracompact space;*

b) *p-paracompact spaces;*

c) σ - *paracompact spaces;*

d) *stratifiable spaces;*

e) *metrizable spaces.*

Then $F_\beta(X)$ is a weakly countable-dimensional space.

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