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A New Approach on the Spaces of Generalized Fibonacci Difference Null and Convergent Sequences

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Abstract

In this current study, the most apparent aspect is to submit new sequence spaces $c_0(\widehat{F}(r,s))$ and $c(\widehat{F}(r,s))$ under the domain of the matrix $\widehat{F}(r,s)$ constituted by using Fibonacci sequence and non-zero real number r and s, of c_0 and c respectively. Here, we have studied some algebraic structures including some inclusion relations, linear isomorphism, solidity and some topological structures such as basis of the spaces $c_0(\widehat{F}(r,s))$ and $c(\widehat{F}(r,s))$. Eventually, we have presented the alpha-, beta-, gamma-duals of these spaces using available technique in the literature and characterization of the classes $(c_0(\widehat{F}(r,s)),X)$ and $(c(\widehat{F}(r,s)),X)$ for some sequence space X.

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1 Introduction

Let us begin giving some fundamental concepts which are going to be used in the rest of the article. They are fairly elementary and probably well-known to other researchers in this field. Let X be a set. A sequence in X may be viewed as a mapping $f:\mathbb{N}\to X$ where $\mathbb{N}=\{0,1,2,\ldots\}$. Thus we have associated to each integer n an element of X, namely $f(n)$. One often suppresses the fact that we have a function by simply considering a sequence as the image elements, say, $x_1,x_2,x_3,\ldots$ or alternatively, just writes "the sequence x_n" or $\{x_n\}_{n=0}^{\infty}$. As it is known to all of us, ω denotes the family of all real (or complex)-valued sequences, ω is a linear space, and any linear subspace of ω is known as a *sequence space*. A sequence (x_n) converges to limit a if each neighborhood of a contains almost all terms of the sequence. In this case we say that (x_n) converges to a as n goes to ∞. We denote by c, the set of all convergent sequences in fields $\mathbb{R}$ or $\mathbb{C}$. A sequence (x_n) is called a null sequence if it converges to zero. We denote the set of all null sequences by c_0. A sequence is bounded if the set of its terms is bounded. The set of all bounded sequences is denoted by ℓ_∞. It is clear that the sets c, c_0 and ℓ_∞ are the subspaces of the ω. Therefore, c, c_0 and ℓ_∞, equipped with a vector space structure, form a sequence space. Also by bs, cs, ℓ_1 and ℓ_p we denote the spaces of all bounded, convergent, absolutely and p-absolutely convergent series, respectively. As it is well known, we call a sequence space X with a linear topology a K-space if and only if each of the maps $p_n : X \to \mathbb{R}$ defined by $p_n(x) = x_n$ is continuous for all $n \in \mathbb{N}$. A K-space X is called an FK-space if and only if X is a complete linear metric space. In other words, we can say that an FK-space is a complete total paranormed space. Note here that some discussion of FK-spaces is given in [1]. An FK-space whose topology is normable is called a BK-space or a Banach coordinate space, so a BK-space is a normed FK-space. The space ℓ_p $(1 \leq p < \infty)$ is a BK-space with $\|x\|_p = (\sum_k |x_k|^p)^{\frac{1}{p}}$ and c_0, c and ℓ_∞ are BK-spaces with $\|x\|_\infty = sup_k |x_k|$. In addition to this, by $\mathcal{F}$, we denote the collection consisting of all non-empty and finite subsets of $\mathbb{N}$.

In this paragraph, we shall introduce the notion of a matrix mapping from a sequence space X to a sequence space Y. Given any infinite matrix $A = (a_{nk})$ of real numbers a_{nk}, where $n,\ k \in \mathbb{N}$, any sequence x, we write $Ax = \big((Ax)_n\big)$, the A-transform of x, if $(Ax)_n = \sum_k a_{nk}x_k$ converges for each $n \in \mathbb{N}$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. If $x \in X$ implies that $Ax \in Y$, then we say that A defines a *matrix mapping* from X into Y and denote it by $A : X \to Y$. By $(X : Y)$ we mean the class of all infinite matrices such that $A : X \to Y$.

Another concept we need is that of matrix domain. So, the concept is introduced in this paragraph. The X_A is said to be *matrix domain* of an infinite matrix A for any subspace X of the all real-valued sequence space ω and is described as

$$X_A := \big\{x = (x_k) \in \omega : Ax \in X\big\}. \tag{1}$$

The new sequence space X_A generated by the limitation matrix A from the space X either includes the space X or is included by the space X, in general, i.e., the space X_A is the expansion or the contraction of the original space X, see [2, p. 51] for more details.

In order to construct a new sequence space, a triangle matrix was previously used. To get detailed information about it, one must search through the articles

[3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], and [15]. From these articles it can be seen that future needs.

In 1981; the concept of difference operator $(\Delta x) = (\Delta(x_k)) = (x_k - x_{k+1})$ in the classical sequence spaces was coined by Kızmaz [16] as follows

$$X(\Delta) = \{x = (x_k) \in \omega : (x_k - x_{k+1}) \in X\}$$

for $X = \ell_\infty$, c and c_0. Since then, the study of generalized difference sequence has largely motivated many researcher.

Let λ be a normed or paranormed sequence space. In that case, the matrix domain λ_Δ is known the *difference sequence space.*

When p satisfied $0 < p < 1$, the difference space bv_p, consisting of all sequences (x_k) such that $(x_k - x_{k-1})$ is in the sequence space ℓ_p, was studied by Altay and Başar [17] and in the case $1 \leq p \leq \infty$ by Başar and Altay [18], and Çolak et al. [19]. More recently, Kirişçi and Başar [13] have introduced and examined the generalized difference sequence spaces

$$\widehat{X} = \{x = (x_k) \in \omega : B(r,s)x \in X\},$$

where $B(r,s) = \{b_{nk}(r,s)\}$ denotes generalized difference matrix as follows:

$$b_{nk}(r,s) := \begin{cases} r & , \quad (k = n), \\ s & , \quad (k = n-1), \\ 0 & , \quad (0 \leq k < n-1 \quad \text{or} \quad k > n), \end{cases}$$

X denotes any of the spaces ℓ_∞, ℓ_p, c and c_0, $1 \leq p < \infty$, and $B(r,s)x = (sx_{k-1} + rx_k)$ with $r, s \in \mathbb{R} \setminus \{0\}$. Following Kirişçi and Başar [13], Candan [3] has examined the sequence space $X(\widetilde{B})$ as the set of all sequences whose $\widetilde{B}(\tilde{r}, \tilde{s})$-transforms are in the space $X \in \{\ell_\infty, \ell_p, c, c_0\}$, where $\widetilde{B}(\tilde{r}, \tilde{s})$ denotes the double sequential band matrix $\widetilde{B}(\tilde{r}, \tilde{s}) = \{b_{nk}(r_n, s_n)\}$ defined by

$$b_{nk}(\widetilde{r}, \widetilde{s}) := \begin{cases} r_n & , \quad (k = n), \\ s_n & , \quad (k = n-1), \\ 0 & , \quad (0 \leq k < n-1 \quad \text{or} \quad k > n), \end{cases}$$

for all $k, n \in \mathbb{N}$, where $\tilde{r} = (r_n)_{n=0}^\infty$ and $\tilde{s} = (s_n)_{n=0}^\infty$ be given convergent sequences of positive real numbers and $1 \leq p < \infty$.

Also in [20], [21], [22], [23], [24], [25], [26], and [27] authors studied certain difference sequence spaces. Furthermore, quite recently, Kara [28] has defined the Fibonacci difference matrix $\widehat{F}$ by means of the Fibonacci sequence $(f_n)_{n\in\mathbb{N}}$ and introduced the new sequence spaces $\ell_p(\widehat{F})$ and $\ell_\infty(\widehat{F})$ which are derived by the matrix domain of $\widehat{F}$ in the sequence spaces ℓ_p and ℓ_∞, respectively; where $1 \leq p < \infty$.

The general frame of the remaining of the article can be given as follows:

In part 2, some fundamental notations and concepts are presented. In part 3, sequence spaces $c_0(\widehat{F}(r,s))$ and $c(\widehat{F}(r,s))$ are introduced and also some inclusion relations are given. Moreover, their bases of these spaces are determined. In part 4, the alpha-, beta-, gamma-duals of the spaces $c_0(\widehat{F}(r,s))$ and $c(\widehat{F}(r,s))$ are constructed and then the classes $(c_0(\widehat{F}(r,s)),X)$ and $(c(\widehat{F}(r,s)),X)$ of matrix transformations are characterized. Here X denotes any one of the spaces ℓ_∞, f, c, f_0, c_0, bs, fs and ℓ_1.

2 Preliminaries

In this section, after we give some historical information about the Fibonacci sequence, we introduce new brand sequence spaces, using both the Fibonacci sequence and non-zero real number r and s.

The widely-known Fibonacci sequence has wide applications in several other branches of sciences such as mathematics, arts, architecture, design, music, etc. for centuries. Let us start giving very brief historical knowledge about the Fibonacci sequence. The Fibonacci sequence first appeared in the book *"Liber Abaci"*, in 1202, which means *"The Book of Calculation"* written by Leonardo of Pisa, also known of Fibonacci. There are many ways to introduce the Fibonacci sequence, each of which is an equivalent way of defining the same thing. Here, let us explain this concept. A numeric sequence is a set of ordered numbers generated by well-defined algorithm. The easiest method of generating a number sequence is to use one or two kernel values and an suitable recursive equation. One of the most well-known number sequences is summation (Fibonacci sequences) sequence. This sequence is obtained by the following recursive formula

$$f_n = f_{n-1} + f_{n-2} \quad \text{with} \quad n \geq 2.$$

That is, each term in the sequence is equal to the sum of the previous two terms. This sequence requires the kernel values f_0 and f_1. Throughout our study, we will take f_0 and f_1 as 1.

Now, we are taking a look at some of the famous properties such as *Golden Ratio*, and *Cassini formula* of the Fibonacci sequence [29].

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2} = \varphi \quad \text{(Golden Ratio)},$$

$$\sum_{k=0}^{n} f_k = f_{n+2} - 1 \quad \text{for each} \quad n \in \mathbb{N},$$

$$\sum_k \frac{1}{f_k} \text{ converges},$$

$$f_{n-1}f_{n+1} - f_n^2 = (-1)^{n+1} \text{ for all } n \geq 1 \text{ (Cassini Formula)}.$$

It can easily be derived by replaceing f_{n+1} in Cassini's formula namely $f_{n-1}^2 + f_n f_{n-1} - f_n^2 = (-1)^{n+1}$.

3 Some Algebraic and Topological Properties of Generalized Fibonacci Difference Spaces of Null and Convergent Sequences

In present section, after we introduce the spaces $c_0(\widehat{F}(r,s))$ and $c(\widehat{F}(r,s))$, some algebraic and topological properties of them will be investigated.

When p satisfied $1 \leq p \leq \infty$, the sequence space $\ell_p(\widehat{F})$ has been defined by Kara [28], also Kara et al. in [30] have investigated some classes of compact operators on the spaces $\ell_p(\widehat{F})$ and $\ell_\infty(\widehat{F})$, where $1 \leq p \leq \infty$. After then Başarır et all in [31] have introduced Fibonacci null sequence space $c_0(\widehat{F})$ and Fibonacci convergent sequence space $c(\widehat{F})$, in recent years. More details can be found [28], [29], [30], [31], and [32].

Let us define the sets $c_0(\widehat{F}(r,s))$ and $c(\widehat{F}(r,s))$ as the sets of all sequences whose $\widehat{F}(r,s) = \{f_{nk}(r,s)\}$ transforms is in the well-know sequence spaces c_0 and c, respectively, namely,

$$\begin{aligned} c_0(\widehat{F}(r,s)) &= \left\{ x = (x_n) \in \omega : \lim_{n\to\infty} \left(r\frac{f_n}{f_{n+1}} x_n + s\frac{f_{n+1}}{f_n} x_{n-1} \right) = 0 \right\}, \\ c(\widehat{F}(r,s)) &= \left\{ x = (x_n) \in \omega : \exists l \in \mathbb{C} \ni \lim_{n\to\infty} \left(r\frac{f_n}{f_{n+1}} x_n + s\frac{f_{n+1}}{f_n} x_{n-1} \right) = l \right\}. \end{aligned}$$

Where $\widehat{F}(r,s) = \{f_{nk}(r,s)\}$ is the double generalized band matrix defined by the sequence (f_n) of Fibonacci numbers as follows

$$f_{nk}(r,s) = \begin{cases} s\frac{f_{n+1}}{f_n} & , \quad k = n-1, \\ r\frac{f_n}{f_{n+1}} & , \quad k = n, \\ 0 & , \quad 0 \leq k < n-1 \text{ or } k > n \end{cases} \tag{2}$$

for all $k, n \in \mathbb{N}$ where $r, s \in \mathbb{R} \setminus \{0\}$. With the help of the notation of (1), the spaces $c_0(\widehat{F}(r,s))$ and $c(\widehat{F}(r,s))$ can be rewritten as follows:

$$c_0(\widehat{F}(r,s)) = (c_0)_{\widehat{F}(r,s)} \text{ and } c(\widehat{F}(r,s)) = c_{\widehat{F}(r,s)}.$$

We have come up the inverse matrix $\widehat{F}^{-1}(r,s) = \{f_{nk}^{-1}(r,s)\}$ of the Fibonacci generalized matrix with a rudimentary calculation as follows

$$f_{nk}^{-1}(r,s) = \begin{cases} \frac{1}{r}\left(-\frac{s}{r}\right)^{n-k}\frac{f_{n+1}^2}{f_k f_{k+1}} & , \quad 0 \leq k \leq n, \\ 0 & , \quad k > n \end{cases}$$

for all $k, n \in \mathbb{N}$.

Additionally, define the sequence $y = (y_n)$ by the $\widehat{F}(r,s)$-transform of a sequence $x = (x_n)$, i.e.,

$$y_n = (\widehat{F}(r,s)(x))_n = \begin{cases} rx_0 & , \quad n = 0, \\ r\frac{f_n}{f_{n+1}}x_n + s\frac{f_{n+1}}{f_n}x_{n-1} & , \quad n \geq 1 \end{cases} \tag{3}$$

for all $n \in \mathbb{N}$. We note here that, the sequences $x = (x_k)$ and $y = (y_k)$ are connected by relation (3) everywhere in the text.

We should state here that the matrix $\widehat{F}(r,s)$ can be reduced to the matrix $\widehat{F}$ in case $r = 1$ and $s = -1$. Therefore, the results related to the spaces $c_0(\widehat{F}(r,s))$ and $c(\widehat{F}(r,s))$ are more general and more inclusive than the corresponding consequences of the spaces $c_0(\widehat{F})$ and $c(\widehat{F})$ more recently defined by Başarır et all in [31].

Let us begin with the theorem which is one of our principal objects of study.

Theorem 3.1. *The sets $c_0(\widehat{F}(r,s))$ and $c(\widehat{F}(r,s))$ are the linear spaces with the co-ordinatewise addition and scalar multiplication which are the $BK-$spaces with the following norm*

$$\|x\|_{c_0(\widehat{F}(r,s))} = \|x\|_{c(\widehat{F}(r,s))} = \|\widehat{F}(r,s)x\|_\infty.$$

Proof. Since the first assertion is simple and easy to prove, we ignore its proof in here. Widely known as one of the properties the spaces c_0 and c are $BK-$spaces with respect to their naturel norms, from prior to arguments. Also the matrix $\widehat{F}(r,s)$ introduced above is a triangle matrix, it is not difficult to say that $c_0(\widehat{F}(r,s))$ and $c(\widehat{F}(r,s))$ are $BK-$spaces with the given norms from Theorem 4.3.12 of Wilansky in [33], so the claim is proved. □

Now, for $A = (a_{nk})$ be an arbitrary infinite matrix, we are going to give some properties which are needed in the next lemma.

$$\sup_{n\in\mathbb{N}}\sum_k |a_{nk}| < \infty \tag{4}$$

$$\lim_{n\to\infty} a_{nk} = 0 \quad \text{for each} \quad k \in \mathbb{N} \tag{5}$$

$$\exists \alpha_k \in \mathbb{C} \ni \lim_{n\to\infty} a_{nk} = \alpha_k \quad \text{for each} \quad k \in \mathbb{N} \tag{6}$$

$$\lim_{n\to\infty}\sum_k a_{nk} = 0 \tag{7}$$

$$\exists \alpha \in \mathbb{C} \ni \lim_{n\to\infty}\sum_k a_{nk} = \alpha \tag{8}$$

$$\sup_{K\in\mathcal{F}}\sum_n \left|\sum_{k\in K} a_{nk}\right| < \infty, \tag{9}$$

where $\mathbb{C}$ and $\mathcal{F}$ denote the set of all complex numbers and the collection of all finite subsets of $\mathbb{N}$, respectively.

The following lemma [34] which is on the characterization of the matrix transformations between some classical sequence spaces is needed before going to the results related to inclusion relations stated in Theorem 3.3.

Lemma 3.2. *The following statements hold:*

(a) $A = (a_{nk}) \in (c_0, c_0)$ *if and only if (4) and (5) hold.*

(b) $A = (a_{nk}) \in (c_0, c)$ *if and only if (4) and (6) hold.*

(c) $A = (a_{nk}) \in (c, c_0)$ *if and only if (4), (5) and (7) hold.*

(d) $A = (a_{nk}) \in (c, c)$ *if and only if (4), (6) and (8) hold.*

(e) $A = (a_{nk}) \in (c_0, \ell_\infty) = (c, \ell_\infty)$ *if and only if condition (4) holds.*

(f) $A = (a_{nk}) \in (c_0, \ell_1) = (c, \ell_1)$ *if and only if condition (9) holds.*

Theorem 3.3. *(i) If* $|s/r| < 1/4$*, then* $c_0 = c_0(\widehat{F}(r,s))$ *and* $c = c(\widehat{F}(r,s))$.

(ii) If $|s/r| \geq 1/4$*, then the inclusions* $c_0 \subset c_0(\widehat{F}(r,s))$ *and* $c \subset c(\widehat{F}(r,s))$ *are strictly valid.*

Proof. Let us assume that $\lambda \in \{c_0, c\}$. Obviously, the matrix $\widehat{F}(r,s) = \{f_{nk}(r,s)\}$ satisfies the following conditions

$$\sup_{n\in\mathbb{N}}\sum_k |f_{nk}(r,s)| = \sup_{n\in\mathbb{N}}\left(|r|\frac{f_n}{f_{n+1}} + |s|\frac{f_{n+1}}{f_n}\right) \leq 2|r| + |s|,$$

$$\lim_{n\to\infty} f_{nk}(r,s)=0,$$

$$\lim_{n\to\infty}\sum_k f_{nk}(r,s)=\lim_{n\to\infty}\left(r\frac{f_n}{f_{n+1}}+s\frac{f_{n+1}}{f_n}\right)=\frac{r}{\varphi}+s\varphi.$$

Combining (a) and (c) of Lemma 3.2 give $\widehat{F}(r,s)\in(\lambda,\lambda)$, which means that $\widehat{F}(r,s)x\in\lambda$ for an arbitrary $x\in\lambda$. Therefore, $x\in\lambda_{\widehat{F}(r,s)}$. Eventually $\lambda\subset\lambda_{\widehat{F}(r,s)}$.

(i) Let $|s/r|<1/4$. Since the inverse matrix $\widehat{F}^{-1}(r,s)=\{f_{nk}^{-1}(r,s)\}$ of the matrix $\widehat{F}(r,s)$ also satisfies the conditions

$$\sup_{n\in\mathbb{N}}\sum_k\left|f_{nk}^{-1}(r,s)\right|\le\frac{1}{\inf_{n\in\mathbb{N}}\frac{rf_n}{f_{n+1}}}\sum_k\left(\frac{\sup_{n\in\mathbb{N}}\frac{sf_{n+2}}{f_{n+1}}}{\inf_{n\in\mathbb{N}}\frac{rf_n}{f_{n+1}}}\right)^k\le\frac{2}{r}\sum_k\left(\frac{4s}{r}\right)^k,$$

to prove $\lim_{n\to\infty} f_{nk}^{-1}(r,s)=\frac{1}{r}\left(\frac{-s}{r}\right)^{n-k}\frac{f_{n+1}^2}{f_kf_{k+1}}=0$, let us consider the following equation

$$0\le\left|\frac{f_{n+1}}{rf_n}\left(\frac{-s}{r}\right)^{n-k}\frac{\frac{f_{n+1}}{f_{k+1}}}{\frac{f_k}{f_n}}\right|=\left|\frac{f_{n+1}}{rf_n}\prod_{i=k}^{n-1}\frac{s\frac{f_{i+2}}{f_{i+1}}}{r\frac{f_i}{f_{i+1}}}\right|\le\frac{f_{n+1}}{|r|f_n}\prod_{i=k}^{n-1}\left|\frac{\sup_{i\in\mathbb{N}}s\frac{f_{i+2}}{f_{i+1}}}{\inf_{i\in\mathbb{N}}r\frac{f_i}{f_{i+1}}}\right|\le\frac{f_{n+1}}{|r|f_n}\left|\frac{4s}{r}\right|^{n-k}$$

and $\lim_{n\to\infty}\frac{f_{n+1}}{|r|f_n}\left|\frac{4s}{r}\right|^{n-k}=\frac{\varphi}{|r|}0=0$, therefore $\lim_{n\to\infty} f_{nk}^{-1}(r,s)=0$,

$$\lim_{n\to\infty}\sum_k f_{nk}^{-1}(r,s)\le\frac{1}{\inf_{n\in\mathbb{N}}\frac{rf_n}{f_{n+1}}}\lim_{n\to\infty}\sum_{k=0}^{n}\left(-\frac{\inf_{n\in\mathbb{N}}\frac{sf_{n+2}}{f_{n+1}}}{\inf_{n\in\mathbb{N}}\frac{rf_n}{f_{n+1}}}\right)^k\le\frac{2}{r}\lim_{n\to\infty}\sum_{k=0}^{n}\left(\frac{-s}{r}\right)^k.$$

That is $\widehat{F}^{-1}(r,s)\in(\lambda:\lambda)$. Hence, if $x\in\lambda_{\widehat{F}(r,s)}$, then $y=\widehat{F}(r,s)x\in\lambda$ and $x=\widehat{F}^{-1}(r,s)y\in\lambda$. Thus, the opposite inclusion $\lambda_{\widehat{F}(r,s)}\subset\lambda$ is just proved.

(ii) Let $|s/r|\ge 1/4$. Then, by taking into account the sequence $x=(x_k)=((1/r)(-s/r)^kf_{k+1}^2)$ one can easily seen $x\in\lambda_{\widehat{F}}(r,s)\setminus\lambda$. This shows that the inclusion $\lambda\subset\lambda_{\widehat{F}(r,s)}$ is strictly valid. This last step completes the proof. □

Remark 3.4. *It can be easily verified that the absolute property is invalid on the spaces $c_0(\widehat{F}(r,s))$ and $c(\widehat{F}(r,s))$; in other words, $\|x\|_{c_0(\widehat{F}(r,s))}\neq\||x|\|_{c_0(\widehat{F}(r,s))}$ and $\|x\|_{c(\widehat{F}(r,s))}\neq\||x|\|_{c(\widehat{F}(r,s))}$ for at least one sequence in the spaces $c_0(\widehat{F}(r,s))$ and $c(\widehat{F}(r,s))$. This briefly tells us that $c_0(\widehat{F}(r,s))$ and $c(\widehat{F}(r,s))$ are the sequence spaces of non-absolute type, in which $|x|=(|x_k|)$.*

Theorem 3.5. *The sequence spaces $c_0(\widehat{F}(r,s))$ and $c(\widehat{F}(r,s))$ of non-absolute type are linearly norm isomorphic to the spaces c_0 and c, respectively, i.e., $c_0(\widehat{F}(r,s))\cong c_0$ and $c(\widehat{F}(r,s))\cong c$.*

Proof. Since the proof of the second part of the theorem is similar to the first part of it. Here, we only focus on the first one. To confirm the first claim, we have to show the existence of a linear bijection between the spaces $c_0(\widehat{F}(r,s))$ and c_0. Let us evaluate the map T mentioned above, with the help of the notation of (3), from $c_0(\widehat{F}(r,s))$ to c_0 by $x \mapsto y = \widehat{F}(r,s)x$. It is easy to show that both T is linear and injective.

Now, take any $y = (y_k) \in c_0$ and later define the following sequence $x = (x_k)$ as follows

$$x_k = \sum_{j=0}^{k} \frac{1}{r}\left(-\frac{s}{r}\right)^{k-j} \frac{f_{k+1}^2}{f_j f_{j+1}} y_j \quad \text{for all} \quad k \in \mathbb{N}. \tag{10}$$

In that case, we get

$$\lim_{k\to\infty} [\widehat{F}(r,s)(x))]_k = \lim_{k\to\infty} [r\frac{f_k}{f_{k+1}} \sum_{j=0}^{k} \frac{1}{r}\left(\frac{-s}{r}\right)^{k-j} \frac{f_{k+1}^2}{f_j f_{j+1}} y_j$$
$$+ s\frac{f_{k+1}}{f_k} \sum_{j=0}^{k-1} \frac{1}{r}\left(\frac{-s}{r}\right)^{k-j-1} \frac{f_k^2}{f_j f_{j+1}} y_j] = \lim_{k\to\infty} y_k = 0$$

which shows that $x \in c_0(\widehat{F}(r,s))$ since $y = (y_k)$ lies c_0. That is, T is surjective. Moreover, one can easily see for every $x \in c_0(\widehat{F}(r,s))$ that

$$\|x\|_{c_0(\widehat{F}(r,s))} = \sup_{k\in\mathbb{N}} \left| r\frac{f_k}{f_{k+1}} x_k + s\frac{f_{k+1}}{f_k} x_{k-1} \right| = \sup_{k\in\mathbb{N}} |y_k| = \|y\|_\infty < \infty.$$

which means that T is norm preserving. Consequently T is a linear bijection which shows that the spaces $c_0(\widehat{F}(r,s))$ and c_0 are linearly isomorphic. This conclusion is what was sought for. $\square$

Now, let us give some basic algebraic properties such as inclusion relations of the spaces $c_0(\widehat{F}(r,s))$ and $c(\widehat{F}(r,s))$ newly defined above.

Theorem 3.6. *The inclusion $c_0(\widehat{F}(r,s)) \subset c(\widehat{F}(r,s))$ is strictly valid.*

Proof. To prove the theorem, we have to find an element which belongs to $c(\widehat{F}(r,s))$ but which does not belong to $c_0(\widehat{F}(r,s))$. Obviously, the inclusion $c_0(\widehat{F}(r,s)) \subset c(\widehat{F}(r,s))$ is valid. Now, let us show that this inclusion is strict. To do this, define the sequence $x = (x_k) = \left(\sum_{j=0}^{k}(1/r)(-s/r)^{k-j} f_{k+1}^2/f_j^2\right)$ for all $k \in \mathbb{N}$. Then, we get together with (3) for all $k \in \mathbb{N}$ that

$$(\widehat{F}(r,s)(x))_k = r\frac{f_k}{f_{k+1}} \sum_{j=0}^{k} \frac{1}{r}\left(-\frac{s}{r}\right)^{k-j} \frac{f_{k+1}^2}{f_j^2} + \frac{f_{k+1}}{f_k} \sum_{j=0}^{k-1} \frac{1}{r}\left(-\frac{s}{r}\right)^{k-1-j} \frac{f_k^2}{f_j^2} = \frac{f_{k+1}}{f_k}$$

for all $k \in \mathbb{N}$. Newly obtained equality above tells us that $(\widehat{F}(r,s)(x))_k = \frac{f_{k+1}}{f_k} \to \varphi$ $(k \to \infty)$, which is Golden ratio. That is, $\widehat{F}(r,s)(x) \in c \setminus c_0$. In other words, the sequence x lies in $c(\widehat{F}(r,s))$; but, it does not lie in $c_0(\widehat{F}(r,s))$. Therefore, the inclusion $c_0(\widehat{F}(r,s)) \subset c(\widehat{F}(r,s))$ is strictly valid.

□

Theorem 3.7. *If $|-s/r| \geq 1$, then the space ℓ_∞ does not include the spaces $c_0(\widehat{F}(r,s))$ and $c(\widehat{F}(r,s))$.*

Proof. Let $|-s/r| \geq 1$. We will do this by selecting the sequence $x = (x_k) = ((1/r)(-s/r)^k f_{k+1}^2)$. We know that both $f_{k+1}^2 \to \infty$ as $k \to \infty$ and $\widehat{F}(r,s)x = e^{(0)} = (1,0,0,\ldots)$. Therefore the sequence x lies in the space $c_0(\widehat{F}(r,s))$ but does not lie in the space ℓ_∞. In other words, this means that the space ℓ_∞ does not include both the space $c_0(\widehat{F}(r,s))$ and the space $c(\widehat{F}(r,s))$. And this is exactly what we expect to see. □

Before we state the next theorem, let us remember the definition the sequence space to be solid. The sequence space λ is said to be *solid* (cf. [35, p. 48]) if and only if

$$\widetilde{\lambda} := \{(u_k) \in \omega : \exists (x_k) \in \lambda \text{ such that } |u_k| \leq |x_k| \text{ for all } k \in \mathbb{N}\} \subset \lambda.$$

Theorem 3.8. *The spaces $c_0(\widehat{F}(r,s))$ and $c(\widehat{F}(r,s))$ are not solid when $|-s/r| \geq 1$.*

Proof. Let $|-s/r| \geq 1$. This proof can be seen by considering the sequences $u = (u_k)$ and $v = (v_k)$ defined by $u_k = (1/r)(-s/r)^k f_{k+1}^2$ and $v_k = (-1)^{k+1}$ for all $k \in \mathbb{N}$. Then, we obviously see that $u \in c_0(\widehat{F}(r,s))$ and $v \in \ell_\infty$. But, multiplication sequences $uv = \{(-1)^{k+1}(1/r)(-s/r)^k f_{k+1}^2\}$ does not lie in the space $c_0(\widehat{F}(r,s))$, since

$$\begin{aligned}(\widehat{F}(r,s)(uv))_k &= r\frac{f_k}{f_{k+1}}(-1)^{k+1}(1/r)(-s/r)^k f_{k+1}^2 + s\frac{f_{k+1}}{f_k}(-1)^k(1/r)(-s/r)^{k-1} f_k^2 \\ &= 2(-1)^{k+1}(-s/r)^k f_k f_{k+1}\end{aligned}$$

for all $k \in \mathbb{N}$. This indicates that the multiplication spaces $\ell_\infty c_0(\widehat{F}(r,s))$ of the spaces ℓ_∞ and $c_0(\widehat{F}(r,s))$ is not a subset of $c_0(\widehat{F}(r,s))$. In conclusion, the space $c_0(\widehat{F}(r,s))$ is not solid.

It is obvious that when the spaces $c_0(\widehat{F})$ is replaced by the one $c(\widehat{F})$, we result in the fact that $c(\widehat{F})$ is not solid. This terminates the proof. □

Now, let us give the definition of the Schauder basis. Any given sequence space X normed by $\|.\|_X$ contains a sequence (a_n) with the property that for every $x \in X$ there is a unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty} \left\| x - \sum_{k=0}^{n} \alpha_k a_k \right\| = 0$$

then (a_n) is called a *Schauder basis* (or *basis*) for X. The series $\sum_k \alpha_k a_k$ which has the sum x is then called the expansion of x with respect to (a_n) and written as $x = \sum_k \alpha_k a_k$.

One can obviously know from Theorem 2.3 of Jarrah and Malkowsky [36] that the domain λ_T of an infinite matrix $T = (t_{nk})$ in a normed sequence space λ has a basis iff λ has a basis, if T is a triangle. Thus, directly from this fact, we have:

Corollary 3.9. *Define the sequences* $c^{(-1)} = \left\{c_k^{(-1)}\right\}_{k\in\mathbb{N}}$ *and* $c^{(n)} = \left\{c_k^{(n)}\right\}_{k\in\mathbb{N}}$ *for every fixed* $n \in \mathbb{N}$ *by*

$$c_k^{(-1)} = \sum_{j=0}^{k} \frac{1}{r}\left(-\frac{s}{r}\right)^j \frac{f_{k+1}^2}{f_j f_{j+1}} \quad \text{and} \quad c_k^{(n)} = \begin{cases} 0 & , \quad 0 \le k \le n-1 \\ \frac{1}{r}\left(-\frac{s}{r}\right)^{n-k} \frac{f_{k+1}^2}{f_n f_{n+1}} & , \quad k \ge n. \end{cases}$$

Then, the following statements hold:

(a) The sequence $\left\{c^{(n)}\right\}_{n=0}^{\infty}$ *is a basis for the space* $c_0(\widehat{F}(r,s))$ *and every sequence* $x \in c_0(\widehat{F}(r,s))$ *has a unique representation* $x = \sum_n (\widehat{F}(r,s)(x))_n c^{(n)}$.

(b) The sequence $\left\{c^{(n)}\right\}_{n=-1}^{\infty}$ *is a basis for the space* $c(\widehat{F}(r,s))$ *and every sequence* $z = (z_n) \in c(\widehat{F}(r,s))$ *has a unique representation* $z = lc^{(-1)} + \sum_n \left[(\widehat{F}(r,s)(z))_n - l\right] c^{(n)}$, *where* $l = \lim_{n\to\infty}(\widehat{F}(r,s)(z))_n$.

4 The alpha-, beta- and gamma-duals of the spaces $c_0(\widehat{F}(r,s))$ and $c(\widehat{F}(r,s))$, and some matrix transformations

By using the sequence spaces ℓ_1, cs and bs, the alpha-, beta- and gamma-duals λ^α, λ^β and λ^γ of a sequence space λ are introduced as follows:

$$\begin{aligned} \lambda^\alpha &= \{a = (a_k) \in \omega : ax = (a_k x_k) \in \ell_1 \text{ for all } x = (x_k) \in \lambda\}, \\ \lambda^\beta &= \{a = (a_k) \in \omega : ax = (a_k x_k) \in cs \text{ for all } x = (x_k) \in \lambda\}, \\ \lambda^\gamma &= \{a = (a_k) \in \omega : ax = (a_k x_k) \in bs \text{ for all } x = (x_k) \in \lambda\}. \end{aligned}$$

respectively.

In this section, we determine the alpha-, beta- and gamma-duals of the spaces $c_0(\widehat{F}(r,s))$ and $c(\widehat{F}(r,s))$, and characterize the classes of infinite matrices from the spaces $c_0(\widehat{F}(r,s))$ and $c(\widehat{F}(r,s))$ to the spaces c_0, c, ℓ_∞, f, f_0, bs, fs, cs and ℓ_1, and from the space f to the spaces $c_0(\widehat{F}(r,s))$ and $c(\widehat{F}(r,s))$.

Now, we are going to use two lemmas necessary to prove the theorems related to the alpha-, beta- and gamma-duals of the spaces $c_0(\widehat{F})$ and $c(\widehat{F})$.

Lemma 4.1. *Let λ be any of the spaces c_0 or c and $a=(a_n)\in\omega$, and the matrix $B=(b_{nk})$ be defined by $B_n=a_n(\widehat{F}^{-1}(r,s))_n$, that is,*

$$b_{nk}=\begin{cases} a_n f_{nk}^{-1}(r,s) & , \quad 0\le k\le n, \\ 0 & , \quad k>n \end{cases}$$

for all $k,n\in\mathbb{N}$. Then, $a\in\lambda^{\beta}_{\widehat{F}(r,s)}$ iff $B\in(\lambda,\ell_1)$.

Proof. Let y be as introduced early in part3, that is the $y=\widehat{F}(r,s)x$ which is transform of a sequence $x=(x_n)\in\omega$. In that case, we get with the help of (10) that

$$a_nx_n=a_n(\widehat{F}^{-1}(r,s)(y))_n=(By)_n \quad \text{for all} \quad n\in\mathbb{N}. \tag{11}$$

Therefore, we obtained by (11) that $ax=(a_nx_n)\in\ell_1$ with $x\in\lambda_{\widehat{F}(r,s)}$ iff $By\in\ell_1$ with $y\in\lambda$. Finally, it is easily deducted that $a\in\lambda^{\beta}_{\widehat{F}(r,s)}$ iff $B\in(\lambda,\ell_1)$. This conclusion is what we would anticipate. ☐

Lemma 4.2. [37, Theorem 3.1] *Let $C=(c_{nk})$ be defined via a sequence $a=(a_k)\in\omega$ and the inverse matrix $V=(v_{nk})$ of the triangle matrix $U=(u_{nk})$ by*

$$c_{nk}=\begin{cases} \sum_{j=k}^{n} a_jv_{jk} & , \quad 0\le k\le n, \\ 0 & , \quad k>n \end{cases}$$

for all $k,n\in\mathbb{N}$. Then, for any sequence space λ,

$$\begin{aligned} \lambda_U^{\gamma} &= \{a=(a_k)\in\omega : C\in(\lambda,\ell_\infty)\}, \\ \lambda_U^{\beta} &= \{a=(a_k)\in\omega : C\in(\lambda,c)\}. \end{aligned}$$

When one combines Lemmas 2.1, 4.1 and 4.2, the following is obtained;

Corollary 4.3. *Consider the sets d_1, d_2, d_3 and d_4 defined as follows:*

$$
\begin{aligned}
d_1 &= \left\{a=(a_k)\in\omega : \sup_{K\in\mathcal{F}}\sum_{n}\left|\sum_{k\in K}\frac{1}{r}\Big(-\frac{s}{r}\Big)^{n-k}\frac{f_{n+1}^2}{f_kf_{k+1}}a_n\right|<\infty\right\},\\
d_2 &= \left\{a=(a_k)\in\omega : \sup_{n\in\mathbb{N}}\sum_{k=0}^{n}\left|\sum_{j=k}^{n}\frac{1}{r}\Big(-\frac{s}{r}\Big)^{j-k}\frac{f_{j+1}^2}{f_kf_{k+1}}a_j\right|<\infty\right\},\\
d_3 &= \left\{a=(a_k)\in\omega : \lim_{n\to\infty}\sum_{j=k}^{n}\frac{1}{r}\Big(-\frac{s}{r}\Big)^{j-k}\frac{f_{j+1}^2}{f_kf_{k+1}}a_j \text{ exists for each } k\in\mathbb{N}\right\},\\
d_4 &= \left\{a=(a_k)\in\omega : \lim_{n\to\infty}\sum_{k=0}^{n}\sum_{j=k}^{n}\frac{1}{r}\Big(-\frac{s}{r}\Big)^{j-k}\frac{f_{j+1}^2}{f_kf_{k+1}}a_j \text{ exists}\right\}.
\end{aligned}
$$

Then, the following statements hold:

(a) $\{c_0(\widehat{F})\}^\alpha=\{c(\widehat{F})\}^\alpha=d_1$.

(b) $\{c_0(\widehat{F})\}^\beta=d_2\cap d_3$ *and* $\{c(\widehat{F})\}^\beta=d_2\cap d_3\cap d_4$.

(c) $\{c_0(\widehat{F})\}^\gamma=\{c(\widehat{F})\}^\gamma=d_2$.

Theorem 4.4. *Let $\lambda=c_0$ or c and μ be an arbitrary subset of ω. Then, we have $A=(a_{nk})\in(\lambda_{\widehat{F}},\mu)$ if and only if*

$$D^{(m)}=\left(d_{nk}^{(m)}\right)\in(\lambda,c) \text{ for all } n\in\mathbb{N}, \tag{12}$$

$$D=(d_{nk})\in(\lambda,\mu), \tag{13}$$

where

$$d_{nk}^{(m)}=\begin{cases}\sum_{j=k}^{m}\frac{1}{r}\big(-\frac{s}{r}\big)^{j-k}\frac{f_{j+1}^2}{f_kf_{k+1}}a_{nj} & , \quad 0\le k\le m,\\ 0 & , \quad k>m\end{cases}$$

and $d_{nk}=\sum_{j=k}^{\infty}\frac{1}{r}\big(-\frac{s}{r}\big)^{j-k}\frac{f_{j+1}^2}{f_kf_{k+1}}a_{nj}$ *for all* $k,m,n\in\mathbb{N}$.

Proof. If we follow the same technique of thinking as applied by Kirişçi and Başar [13], we are going to prove the theorem. In proving the necessity part of the theorem, let us assume that $A=(a_{nk})\in(\lambda_{\widehat{F}(r,s)},\mu)$ and $x=(x_k)\in\lambda_{\widehat{F}(r,s)}$.

Following equalities can be directly obtained with the help of the relation (10)

$$\begin{aligned}\sum_{k=0}^{m} a_{nk}x_k &= \sum_{k=0}^{m} a_{nk}\sum_{j=0}^{k}\frac{1}{r}\Big(-\frac{s}{r}\Big)^{j-k}\frac{f_{k+1}^2}{f_j f_{j+1}}y_j \\ &= \sum_{k=0}^{m}\sum_{j=k}^{m}\frac{1}{r}\Big(-\frac{s}{r}\Big)^{j-k}\frac{f_{j+1}^2}{f_k f_{k+1}}a_{nj}y_k \qquad (14)\\ &= \sum_{k=0}^{m} d_{nk}^{(m)}y_k \\ &= D_n^{(m)}(y)\end{aligned}$$

for all $m,n\in\mathbb{N}$. Since the $A-$transform of every $x\in\lambda_{\widehat{F}(r,s)}$ exists and also lies in c according to the hypothesis, $D^{(m)}$ also lies in c for every fixed $m\in\mathbb{N}$, that is $D^{(m)}\in(\lambda,c)$. In that case, we can derive from (14) as $m\to\infty$ that $Ax=Dy$. From this, we understand that $D\in(\lambda,\mu)$, which gives the desired result.

For proving the sufficiency part of the theorem, let us assume that conditions (12), (13) hold and take an arbitrary $x\in\lambda_{\widehat{F}(r,s)}$. By our assumption, using Corollary 4.3, we obtain $(d_{nk})_{k\in\mathbb{N}}\in\lambda^{\beta}$ for every fixed $n\in\mathbb{N}$. This requires the existence of the $A-$transform of x, that is Ax exists because $A_n=(a_{nk})_{k\in\mathbb{N}}\in\lambda^{\beta}_{\widehat{F}(r,s)}$ for all $n\in\mathbb{N}$. Moreover, we can easily see from the equality (14) as $m\to\infty$ that $Ax=Dy$ and this results in that $A\in(\lambda_{\widehat{F}(rs)},\mu)$. This marks the end of the proof.

□

If we replace the roles of the spaces $\lambda_{\widehat{F}(r,s)}$ and λ with μ in Theorem 4.4, then we directly obtain the following theorem:

Theorem 4.5. *Suppose that the elements of the infinite matrices $A=(a_{nk})$ and $B=(b_{nk})$ are connected with the relation*

$$b_{nk}:=s\frac{f_{n+1}}{f_n}a_{n-1,k}+r\frac{f_n}{f_{n+1}}a_{nk} \qquad (15)$$

for all $k,n\in\mathbb{N}$ and μ be any given sequence space. Then, $A\in(\mu,\lambda_{\widehat{F}(r,s)})$ if and only if $B\in(\mu,\lambda)$.

Proof. To prove the theorem, let us suppose that $z=(z_k)\in\mu$ and the matrices A and B are connected with the relation (15) for all $k,n\in\mathbb{N}$. In that case, we can write following equality

$$\sum_{k=0}^{m} b_{nk}z_k=\sum_{k=0}^{m}\left(s\frac{f_{n+1}}{f_n}a_{n-1,k}+r\frac{f_n}{f_{n+1}}a_{nk}\right)z_k \quad \text{for all} \quad m,n\in\mathbb{N}. \qquad (16)$$

By passing to limit as $m \to \infty$ (16) it is easily to see that $(Bz)_n = [\widehat{F}(r,s)(Az)]_n$. Thus, in conclusion, it can be said that $Az \in \lambda_{\widehat{F}(r,s)}$ whenever $z \in \mu$ iff $Bz \in \lambda$ whenever $z \in \mu$. In fact, this is exactly what we want to prove. $\square$

Before presenting some results of Theorems 4.4 and 4.5, let us give very short historical knowledge and brief developments about the space of *almost convergent sequences*. There are two different notations, which are f and $\widehat{c}$, of the space of almost convergent sequences. Since the Fibonacci sequence is also denoted by f, in order to avoid any confusion, throughout the article notation $\widehat{c}$ will be used for the space of almost convergent sequence. The space $\widehat{c}$ of almost convergent sequences was introduced by G.G. Lorentz [38], using the idea of the Banach limits. The shift operator φ is defined on ω by $\varphi_n(x) = x_{n+1}$ for all $n \in \mathbb{N}$. A Banach limit L is defined on ℓ_∞, as a non–negative linear functional, such that $L(\varphi x) = L(x)$ and $L(e) = 1$. A sequence $x = (x_k) \in \ell_\infty$ is said to be almost convergent to the generalized limit α if all Banach limits of x is α [38] , and denoted by $\widehat{c} - \lim\ x_k = \alpha$. Let φ^j be the composition of φ with itself j times and define $t_{mn}(x)$ for a sequence $x = (x_k)$ by

$$t_{mn}(x) := \frac{1}{m+1}\sum_{j=0}^{m} \varphi_n^j(x) \quad \text{for all} \quad m, n \in \mathbb{N}.$$

It has been proved by Lorentz in [38] that $\widehat{c} - \lim\ x_k = \alpha$ iff $\lim_{m\to\infty} t_{mn}(x) = \alpha$ when it is uniformly in n. The fact that a convergent sequence is almost convergent and its both ordinary and generalized limits are equal is a widely known fact. By $\widehat{c}_0$ and $\widehat{c}$, we denote the space of all almost null and almost convergent sequences, in other words

$$\begin{aligned}
\widehat{c}_0 \quad &:= \quad \left\{ x = (x_k) \in \omega : \lim_{m\to\infty} \sum_{k=0}^{m} \frac{x_{n+k}}{m+1} = 0 \text{ uniformly in } n \right\}, \\
\widehat{c} \quad &:= \quad \left\{ x = (x_k) \in \omega : \exists \alpha \in \mathbb{C} \ni \lim_{m\to\infty} \sum_{k=0}^{m} \frac{x_{n+k}}{m+1} = \alpha \text{ uniformly in } n \right\}.
\end{aligned}$$

It is obvious that the following inclusion relations $c \subset \widehat{c} \subset \ell_\infty$ are valid.

Also, notations by $\widehat{c}_0$, $\widehat{c}$ and $\widehat{cs}$ denote the spaces of almost null and almost convergent sequences and series, respectively. Now, it is time to give the following two lemmas used to characterize the strongly and almost conservative matrices:

Lemma 4.6. [39] *$A = (a_{nk}) \in (f, c)$ if and only if (4), (6) and (8) hold, and*

$$\lim_{n\to\infty} \sum_k \Delta(a_{nk} - \alpha_k) = 0 \tag{17}$$

also holds, where $\Delta(a_{nk} - \alpha_k) = a_{nk} - \alpha_k - (a_{n,k+1} - \alpha_{k+1})$ for all $k, n \in \mathbb{N}$.

Lemma 4.7. [40] *$A = (a_{nk}) \in (c, f)$ if and only if (4) holds, and*

$$\exists \alpha_k \in \mathbb{C} \ni f - \lim \ a_{nk} = \alpha_k \ \ \textit{for each fixed} \ \ k \in \mathbb{N}; \tag{18}$$

$$\exists \alpha \in \mathbb{C} \ni f - \lim \sum_k a_{nk} = \alpha. \tag{19}$$

Now, we list the following conditions;

$$\sup_{m \in \mathbb{N}} \sum_{k=0}^{m} \left| d_{mk}^{(n)} \right| < \infty \tag{20}$$

$$\exists d_{nk} \in \mathbb{C} \ni \lim_{m \to \infty} d_{mk}^{(n)} = d_{nk} \text{ for each } k, n \in \mathbb{N} \tag{21}$$

$$\sup_{n \in \mathbb{N}} \sum_k |d_{nk}| < \infty \tag{22}$$

$$\exists \alpha_k \in \mathbb{C} \ni \lim_{n \to \infty} d_{nk} = \alpha_k \text{ for each } k \in \mathbb{N} \tag{23}$$

$$\sup_{N, K \in \mathcal{F}} \left| \sum_{n \in N} \sum_{k \in K} d_{nk} \right| < \infty \tag{24}$$

$$\exists \beta_n \in \mathbb{C} \ni \lim_{m \to \infty} \sum_{k=0}^{m} d_{mk}^{(n)} = \beta_n \text{ for each } n \in \mathbb{N} \tag{25}$$

$$\exists \alpha \in \mathbb{C} \ni \lim_{n \to \infty} \sum_k d_{nk} = \alpha \tag{26}$$

It is obvious that Theorem 4.4 and Theorem 4.5 result in several consequences. In fact, when we combine Theorems 4.4, 4.5 and Lemmas 3.2, 4.6 and 4.7, then we get the following conclusion:

Corollary 4.8. *Let $A = (a_{nk})$ be an infinite matrix and $a(n,k) = \sum_{j=0}^{n} a_{jn}$ for all $k, n \in \mathbb{N}$. Then, the following statements hold:*

(a) $A = (a_{nk}) \in (c_0(\widehat{F}(r,s)), c_0)$ if and only if (20), (21), (22) hold and (23) also holds with $\alpha_k = 0$ for all $k \in \mathbb{N}$.

(b) $A = (a_{nk}) \in (c_0(\widehat{F}(r,s)), cs_0)$ if and only if (20), (21), (22) hold and (23) also holds with $\alpha_k = 0$ for all $k \in \mathbb{N}$ with $a(n,k)$ instead of a_{nk}.

(c) $A = (a_{nk}) \in (c_0(\widehat{F}(r,s)), c)$ if and only if (20), (21), (22) and (23) hold.

(d) $A = (a_{nk}) \in (c_0(\widehat{F}(r,s)), cs)$ if and only if (20), (21), (22) and (23) hold with $a(n,k)$ instead of a_{nk}.

(e) $A = (a_{nk}) \in (c_0(\widehat{F}(r,s)), \ell_\infty)$ if and only if conditions (20), (21) and (22) hold.

(f) $A = (a_{nk}) \in (c_0(\widehat{F}(r,s)), bs)$ *if and only if conditions (20), (21) and (22) hold with* $a(n,k)$ *instead of* a_{nk}.

(g) $A = (a_{nk}) \in (c_0(\widehat{F}(r,s)), \ell_1)$ *if and only if (20), (21) and (24) hold.*

(h) $A = (a_{nk}) \in (c_0(\widehat{F}(r,s)), bv_1)$ *if and only if (20), (21) and (24) hold with* $a_{nk} - a_{n-1,k}$ *instead of* a_{nk}.

Corollary 4.9. *Let* $A = (a_{nk})$ *be an infinite matrix. Then, the following statements hold:*

(a) $A = (a_{nk}) \in (c(\widehat{F}(r,s)), \ell_\infty)$ *if and only if (20), (21), (22) and (25) hold.*

(b) $A = (a_{nk}) \in (c(\widehat{F}(r,s)), bs)$ *if and only if (20), (21), (22) and (25) hold with* $a(n,k)$ *instead of* a_{nk}.

(c) $A = (a_{nk}) \in (c(\widehat{F}(r,s)), c)$ *if and only if (20), (21), (22), (23), (25) and (26) hold.*

(d) $A = (a_{nk}) \in (c(\widehat{F}(r,s)), cs)$ *if and only if (20), (21), (22), (23), (25) and (26) hold with* $a(n,k)$ *instead of* a_{nk}.

(e) $A = (a_{nk}) \in (c(\widehat{F}(r,s)), c_0)$ *if and only if (20), (21), (22), (23) hold with* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$, *(25) and (26) also hold with* $\alpha = 0$.

(f) $A = (a_{nk}) \in (c(\widehat{F}(r,s)), cs_0)$ *if and only if (20), (21), (22), (23) hold with* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$, *(25) and (26) also hold with* $\alpha = 0$ *with* $a(n,k)$ *instead of* a_{nk}.

(g) $A = (a_{nk}) \in (c(\widehat{F}(r,s)), \ell_1)$ *if and only if (20), (21), (24) and (25) hold.*

(h) $A = (a_{nk}) \in (c(\widehat{F}(r,s)), bv_1)$ *if and only if (20), (21), (24) and (25) hold with* $a_{nk} - a_{n-1,k}$ *instead of* a_{nk}.

Corollary 4.10. $A = (a_{nk}) \in (c(\widehat{F}(r,s)), f)$ *if and only if (20), (21), (25) and (26) hold, and (22), (23) also hold with* d_{nk} *instead of* a_{nk}.

Corollary 4.11. $A = (a_{nk}) \in (c(\widehat{F}(r,s)), f_0)$ *if and only if (20), (21), (25) and (26) hold, and (22), (23) also hold with* d_{nk} *instead of* a_{nk} *and* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$.

Corollary 4.12. $A = (a_{nk}) \in (c(\widehat{F}(r,s)), fs)$ *if and only if (20), (21), (22),(23), (25) and (26) hold with* $a(n,k)$ *instead of* a_{nk} *and (22), (23) hold with* $d(n,k)$ *instead of* d_{nk}.

Corollary 4.13. $A = (a_{nk}) \in (f, c(\widehat{F}(r,s)))$ *if and only if (4), (6), (8) and (21) hold with* b_{nk} *instead of* a_{nk}, *where* b_{nk} *is defined by (15).*

Corollary 4.14. *$A = (a_{nk}) \in (f, c_0(\widehat{F}(r,s)))$ if and only if (4) and (8) hold, (6) and (21) also hold with b_{nk} instead of a_{nk} and $\alpha_k = 0$ for all $k \in \mathbb{N}$, where b_{nk} is defined by (15).*

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References

[1] A. Wilansky, Functional Analysis, Blaisdell, 1964.

[2] F. Başar, Summability Theory and Its Applications, Bentham Science Publishers, e-books, Monographs, xi+405 pp., İstanbul 2012, ISB:978-1-60805-252-3.

[3] M. Candan, Domain of the double sequential band matrix in the classical sequence spaces, J. Inequal. Appl. 2012, **281**(2012).

[4] M. Candan, Almost convergence and double sequential band matrix, Acta. Math. Sci. **34B**(2)(2014), 354–366.

[5] M. Candan, A new sequence space isomorphic to the space $\ell(p)$ and compact operators, J. Math. Comput. Sci. 4 (2014), No. 2, 306-334.

[6] M. Candan, Domain of the double sequential band matrix in the spaces of convergent and null sequences, Adv. Difference Equ.(2014), 2014:163.

[7] M. Candan, A. Güneş, Paranormed sequence space of non-absolute type founded using generalized difference matrix, Proc. Nat. Acad. Sci. India Sect. A, under press.

[8] M. Candan, Some new sequence spaces derived from the spaces of bounded, convergent and null sequences, Int. J. Mod. Math. Sci.,**12**(2)(2014),74-87.

[9] M. Candan, K. Kayaduman, Almost convergent sequence space derived by generalized Fibonacci matrix and Fibonacci core, Brithish J. Math. Comput. Sci.,**7**(2) (2015), 150-167.

[10] M. Candan, E.E. Kara, A study on topological and geometrical characteristics of new Banach sequence spaces, under communication.

[11] M. Candan, G. Kılınç, A different look for paranormed Riesz sequence space derived by Fibonacci matrix, under communication.

[12] E. Malkowsky, E. Savaş, Matrix transformations between sequence spaces of generalized weighted mean, Appl. Math. Comput. **147** (2004) 333–345.

[13] M. Kirişçi, F. Başar, Some new sequence spaces derived by the domain of generalized difference matrix, Comput. Math. Appl. **60** (2010) 1299-1309.

[14] M. Şengönül, F. Başar, Some new Cesàro sequence spaces of non-absolute type which include the spaces c_0 and c, Soochow J. Math. **31** (1) (2005) 107-119.

[15] M. Mursaleen, A.K. Noman, On the spaces of λ-convergent and bounded sequences, Thai J. Math. **8** (2)(2010) 311-329.

[16] H. Kızmaz, On certain sequence spaces, Canad. Math. Bull. **24** (2) (1981) 169–176.

[17] B. Altay, F. Başar, The fine spectrum and the matrix domain of the difference operator Δ on the sequence space ℓ_p, $(0 < p < 1)$, Commun. Math. Anal. **2** (2) (2007) 1–11.

[18] F. Başar, B. Altay, On the space of sequences of p-bounded variation and related matrix mappings, Ukrainian Math. J. **55** (2003) 136–147.

[19] R. Çolak, M. Et, E. Malkowsky, Some Topics of Sequence Spaces, in: Lecture Notes in Mathematics, Fırat Univ. Press (2004) 1-63 ISBN: 975-394-0386-6.

[20] B. Choudhary, S.K. Mishra, A note on Köthe–Toeplitz duals of certain sequence spaces and their matrix transformations, Internat. J. Math. Math. Sci. **18** (4) (1995) 681-688.

[21] M.A. Sarıgöl, On difference sequence spaces, J. Karadeniz Tech. Univ. Fac. Arts Sci. Ser. Math. -Phys. **10** (1987) 63–71.

[22] M. Et, On some difference sequence spaces, Turkish J. Math. **17** (1993) 18–24.

[23] M. Mursaleen, Generalized spaces of difference sequences, J. Math. Anal. Appl. **203** (3) (1996) 738–745.

[24] S.K. Mishra, Matrix maps involving certain sequence spaces, Indian J. Pure Appl. Math. **24** (2) (1993) 125–132.

[25] A.K. Gaur, M. Mursaleen, Difference sequence spaces, Int. J. Math. Math. Sci. **21** (4) (1998) 701–706.

[26] Ç.A. Bektaş, M. Et, R. Çolak, Generalized difference sequence spaces and their dual spaces, J. Math. Anal. Appl. **292** (2) (2004) 423-432.

[27] M. Et, A. Esi, On Köthe-Toeplitz duals of generalized difference sequence spaces, Bull. Malaysian Math. Sc. Soc. **23** (2000) 25-32.

[28] E.E. Kara, Some topological and geometrical properties of new Banach sequence spaces, J. Inequal. Appl. 2013, **38**(2013).

[29] T. Koshy, Fibonacci and Lucas Numbers with Applications, Wiley, 2001.

[30] E.E. Kara, M. Başarır, M.Mursaleen, Compact operators on the Fibonacci difference sequence spaces $\ell_p(\widehat{F})$ and $\ell_\infty(\widehat{F})$, 1st International Eurasian Conference on Mathematical Sciences and Applications, Prishtine-Kosovo, September 3-7, 2012.

[31] M. Başarır, F. Başar, E.E. Kara, On the spaces of Fibonacci difference null convergent sequences, arXiv:1309.0150v1, under press.

[32] S. Demiriz, E.E. Kara, M. Başarır, On the Fibonacci almost convergent sequence spaces and Fibonacci core, Kyungpook Mathematical Journal, (Accepted)

[33] Wilansky, A: Summability Through Functional Analysis, North-Holland Mathematics Studies, **85** (1984) Elsevier, Amsterdam.

[34] M. Stieglitz, H. Tietz, Matrix transformationen von folgenr*ä*umen eine ergebnisübersicht, Math. Z. **154**(1997), 1–16.

[35] P.K. Kamthan, M. Gupta, Sequence Spaces and Series, Marcel Dekker Inc., New York and Basel, 1981.

[36] A.M. Jarrah, E. Malkowsky, BK spaces, bases and linear operators, Rendiconti Circ. Mat. Palermo II **52**(1990), 177–191.

[37] B. Altay, F. Başar, Certain topological properties and duals of the matrix domain of a triangle matrix in a sequence space, J. Math. Anal. Appl. **336** (1) (2007) 632-645.

[38] G.G. Lorentz, A contribution to the theory of divergent sequences, Acta Math. **80**(1948), 167–190.

[39] J.A. Sıddıqi, Infinite matrices summing every almost periodic sequences, Pac. J. Math. **39**(1)(1971), 235–251.

[40] J.P. King, Almost summable sequences, Proc. Amer. Math. Soc. **17**(1966), 1219–1225.

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