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On the n-Normed Space of p-Integrable Functions

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Abstract

The space $L^p(X)$, where X is a measure space with at least n disjoint subsets of positive measure, can be equipped with an n-norm, which makes $L^p(X)$ an n-normed space. The purpose of this paper is to study some properties of this n-normed space. In particular, we examine the completeness of the n-normed space and prove a contractive mapping theorem on this space.

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1 Introduction

Let X be a (real) vector space (of dimension at least n, where n is a fixed number in $\mathbb{N}$). A mapping $\|\cdot, \ldots, \cdot\| : X^n \to \mathbb{R}$ satisfying the following properties:

(1.1) $\|x_1, \ldots, x_n\| = 0$ if and only if $x_1, \ldots, x_n$ are linearly dependent,

(1.2) $\|x_1, \ldots, x_n\|$ is invariant under permutation,

(1.3) $\|\alpha x_1, \ldots, x_n\| = |\alpha| \, \|x_1, \ldots, x_n\|$ for every $x_1, \ldots, x_n \in X$ and $\alpha \in \mathbb{R}$,

(1.4) $\|x + y, x_2, \ldots, x_n\| \leq \|x, x_2, \ldots, x_n\| + \|y, x_2, \ldots, x_n\|$ for every $x, y, x_2, \ldots, x_n \in X$,

is called an *n-norm* on X, and the pair $(X, \|\cdot, \ldots, \cdot\|)$ is called an *n-normed space.*

Geometrically, the value of $\|x_1, \ldots, x_n\|$ may be interpreted as the volume of the n-dimensional parallelepiped spanned by $x_1, \ldots, x_n$ in X. The concept of n-normed spaces was developed by Gähler in the period of 1964-1970 [3, 4, 5, 6, 7]. More recent works may be found in [1, 8, 10, 11, 12, 14, 15, 16, 17].

Let $(X, \|\cdot, \ldots, \cdot\|)$ be an n-normed space. A sequence (x_k) in X is said to *converge* to an $x \in X$ (in the n-norm) if

$$\lim_{k \to \infty} \|x_k - x, y_2, \ldots, y_n\| = 0,$$

for every $y_2, \ldots, y_n \in X$. Also, a sequence (x_k) in X is called a *Cauchy* sequence if

$$\lim_{k,l \to \infty} \|x_k - x_l, y_2, \ldots, y_n\| = 0,$$

for every $y_2, \ldots, y_n \in X$.

If every Cauchy sequence (x_k) in X converges to some $x \in X$, then X is said to be *complete*. A complete n-normed space is called an *n-Banach space.*

On the space $L^p(X)$ $(1 \leq p < \infty)$, the following an n-norm was defined by Gunawan in [9],

$$\|f_1, \ldots, f_n\|_p := \left(\frac{1}{n!} \int_X \cdots \int_X \left| \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ f_1(x_2) & \cdots & f_n(x_2) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix} \right|^p dx_1 \ldots dx_n \right)^{\frac{1}{p}}.$$

The aim of this note is to study $L^p(X)$, $1 \leq p < \infty$, as an n-normed space with the above n-norm. We investigate the completeness of this n-normed space, and use the result to prove a contractive mapping theorem on this space.

2 Main Results

Let X be a measure space with at least n disjoint subsets of positive measure. Recall that $L^p(X)$, $1 \leq p < \infty$, is the space of equivalence classes (modulo equivalence almost everywhere) of functions such that $\int_X |f(x)|^p dx < \infty$ and the function $\|f\|_p := \left(\int_X |f(x)|^p dx\right)^{\frac{1}{p}}$ defines a norm on $L^p(X)$.

Before we reveal our main results, we present some lemmas and proposition.

Lemma 2.1. *For every $f_1, \ldots, f_n \in L^p(X)$, we have*

$$\|f_1, \ldots, f_n\|_p \le (n!)^{1-\frac{1}{p}} \|f_1\|_p \cdots \|f_n\|_p.$$

Proof. Let Φ be a set of all permutations of $\{1, \ldots, n\}$. By the triangle inequality for real numbers and Minkowski's inequality, we have

$$\begin{aligned}
&\|f_1, \ldots, f_n\|_p \\
&= \left(\frac{1}{n!}\int_X \cdots \int_X |\det(f_i(x_j))|^p dx_1 \cdots dx_n\right)^{\frac{1}{p}} \\
&= \left(\frac{1}{n!}\int_X \cdots \int_X \left|\sum_{\phi=(j_1,\ldots,j_n)\in\Phi} \operatorname{sgn}(\phi) f_1(x_{j_1}) f_2(x_{j_2}) \cdots f_n(x_{j_n})\right|^p dx_{j_1} \cdots dx_{j_n}\right)^{\frac{1}{p}} \\
&\le \left(\frac{1}{n!}\int_X \cdots \int_X \left(\sum_{(j_1,\ldots,j_n)\in\Phi} |f_1(x_{j_1}) f_2(x_{j_2}) \cdots f_n(x_{j_n})|\right)^p dx_{j_1} \cdots dx_{j_n}\right)^{\frac{1}{p}} \\
&\le (n!)^{-\frac{1}{p}} \sum_{(j_1,\ldots,j_n)\in\Phi} \left(\left[\int_X |f_1(x_{j_1})|^p dx_{j_1}\right]^{\frac{1}{p}} \cdots \left[\int_X |f_n(x_{j_n})|^p dx_{j_n}\right]^{\frac{1}{p}}\right) \\
&= (n!)^{-\frac{1}{p}} \sum_{(j_1,\ldots,j_n)\in\Phi} \|f_1\|_p \cdots \|f_n\|_p \\
&= (n!)^{1-\frac{1}{p}} \|f_1\|_p \cdots \|f_n\|_p,
\end{aligned}$$

for every $f_1, \ldots, f_n \in L^p(X)$, as claimed. □

Now, as shown in [9], we can derive a norm from the n-norm in a certain way. Indeed, if $\{a_1, \ldots, a_n\}$ is a linearly independent set in $L^p(X)$, then one may observe that

$$\|f\|_p^* := \left(\sum_{\{i_2,\ldots,i_n\}\subset\{1,\ldots,n\}} \|f, a_{i_2}, \ldots, a_{i_n}\|_p^p\right)^{\frac{1}{p}} \tag{2.1}$$

defines a norm on $L^p(X)$. The mapping $\|\cdot\|_p^*$ in (2.1) can be easily seen to satisfy the properties of a norm. In particular, we may check that if $\|f\|_p^* = 0$, then $f = 0$ almost everywhere. Indeed, if $\|f\|_p^* = 0$, then we obtain $\|f, a_{i_2}, \ldots, a_{i_n}\|_p = 0$ for every $\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}$. This means that f is in the linear span of $\{a_{i_2}, \ldots, a_{i_n}\}$ almost everywhere, for every

$\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}$. This forces us to conclude that $f = 0$ almost everywhere.

We know that $L^p(X)$ equipped with $\|\cdot\|_p$ is complete. Now, we will show that $L^p(X)$ as an n-normed space is complete with respect to the n-norm, through the following proposition.

Proposition 2.2. *Let $\{a_1, \ldots, a_n\}$ be a linearly independent set in $L^p(X)$, and the norm $\|\cdot\|_p^*$ be defined by (2.1). Then $\|\cdot\|_p^*$ is equivalent to the usual norm $\|\cdot\|_p$. Precisely, we have*

$$\frac{n\|a_1, \ldots, a_n\|_p}{(2n-1)\left(\|a_1\|_p + \cdots + \|a_n\|_p\right)}\|f\|_p \le \|f\|_p^*$$

and

$$\|f\|_p^* \le (n!)^{1-\frac{1}{p}} \left(\sum_{\{i_2,\ldots,i_n\}\subset\{1,\ldots,n\}} \|a_{i_2}\|_p^p \cdots \|a_{i_n}\|_p^p \right)^{\frac{1}{p}} \|f\|_p,$$

for every $f \in L^p(X)$.

Proof. For every $f \in L^p(X)$ and any subset $\{i_2, \ldots, i_n\}$ of $\{1, 2, \ldots, n\}$, we have

$$\|f, a_{i_2}, \ldots, a_{i_n}\|_p \le (n!)^{1-\frac{1}{p}}\|f\|_p\|a_{i_2}\|_p \cdots \|a_{i_n}\|_p,$$

by Lemma 2.1. Hence we obtain

$$\begin{aligned} \|f\|_p^* &= \left(\sum_{\{i_2,\ldots,i_n\}\subset\{1,\ldots,n\}} \|f, a_{i_2}, \ldots, a_{i_n}\|_p^p \right)^{\frac{1}{p}} \\ &\le (n!)^{1-\frac{1}{p}} \left(\sum_{\{i_2,\ldots,i_n\}\subset\{1,\ldots,n\}} \|a_{i_2}\|_p^p \cdots \|a_{i_n}\|_p^p \right)^{\frac{1}{p}} \|f\|_p. \end{aligned}$$

To prove the reverse inequality, we observe that

$$\|f\|_p^p \, \|a_1, \ldots, a_n\|_p^p = \frac{1}{n!} \int_X \cdots \int_X \left| f(x) \begin{vmatrix} a_1(x_1) & \cdots & a_n(x_1) \\ \vdots & \ddots & \vdots \\ a_1(x_n) & \cdots & a_n(x_n) \end{vmatrix} \right|^p dx dx_1 \cdots dx_n.$$

By Minkowski's inequality, we have

$$\left(\frac{1}{n!}\int_X\cdots\int_X\left|f(x)\begin{vmatrix} a_1(x_1) & \cdots & a_n(x_1)\\ \vdots & \ddots & \vdots\\ a_1(x_n) & \cdots & a_n(x_n)\end{vmatrix}\right|^p dxdx_1\cdots dx_n\right)^{\frac{1}{p}}$$

$$\leq\left(\frac{1}{n!}\int_X\cdots\int_X\left|a_1(x_1)\begin{vmatrix} f(x) & \cdots & a_n(x)\\ \vdots & \ddots & \vdots\\ f(x_n) & \cdots & a_n(x_n)\end{vmatrix}\right|^p dxdx_1\cdots dx_n\right)^{\frac{1}{p}}+$$

$$\cdots+\left(\frac{1}{n!}\int_X\cdots\int_X\left|a_1(x_n)\begin{vmatrix} f(x_1) & \cdots & a_n(x_1)\\ \vdots & \ddots & \vdots\\ f(x) & \cdots & a_n(x)\end{vmatrix}\right|^p dxdx_1\cdots dx_n\right)^{\frac{1}{p}}+$$

$$+\left(\frac{1}{n!}\int_X\cdots\int_X\left|a_2(x)\begin{vmatrix} a_1(x_1) & f(x_1) & \cdots & a_n(x_1)\\ \vdots & \vdots & \ddots & \vdots\\ a_1(x_n) & f(x_n) & \cdots & a_n(x_n)\end{vmatrix}\right|^p dxdx_1\cdots dx_n\right)^{\frac{1}{p}}+$$

$$\cdots+\left(\frac{1}{n!}\int_X\cdots\int_X\left|a_n(x)\begin{vmatrix} a_1(x_1) & \cdots & f(x_1)\\ \vdots & \ddots & \vdots\\ a_1(x_n) & \cdots & f(x_n)\end{vmatrix}\right|^p dxdx_1\cdots dx_n\right)^{\frac{1}{p}}$$

$$= n\,\|a_1\|_p\|f,a_2,\ldots,a_n\|_p+\|a_2\|_p\|f,a_1,a_3,\ldots,a_n\|_p+ \cdots+\|a_n\|_p\|f,a_1,\ldots,a_{n-1}\|_p.$$

Hence we obtain

$$\begin{aligned}\|f\|_p\|a_1,a_2,\ldots,a_n\|_p &\leq n\|a_1\|_p\|f,a_2,a_3,\ldots,a_n\|_p+\\ &\quad+\|a_2\|_p\|f,a_1,a_3,\ldots,a_n\|_p+\\ &\quad\cdots+\|a_n\|_p\|f,a_1,a_2,\ldots,a_{n-1}\|_p\end{aligned}$$

$$\begin{aligned}\|f\|_p\|a_2,a_1,\ldots,a_n\|_p &\leq n\|a_2\|_p\|f,a_1,a_3,\ldots,a_n\|_p+\\ &\quad+\|a_1\|_p\|f,a_2,a_3,\ldots,a_n\|_p+\\ &\quad\cdots+\|a_n\|_p\|f,a_1,a_2,\ldots,a_{n-1}\|_p\\ &\vdots\end{aligned}$$

$$\begin{aligned}\|f\|_p\|a_n,a_1,\ldots,a_{n-1}\|_p &\leq n\|a_n\|_p\|f,a_1,a_2,\ldots,a_{n-1}\|_p+\\ &\quad+\|a_1\|_p\|f,a_n,a_2,\ldots,a_{n-1}\|_p+\\ &\quad\cdots+\|a_{n-1}\|_p\|f,a_n,a_1,\ldots,a_{n-2}\|_p,\end{aligned}$$

whence

$$\begin{aligned}n\|f\|_p\|a_1,a_2,\ldots,a_n\|_p &\leq (2n-1)\|a_1\|_p\|f,a_2,a_3,\ldots,a_n\|_p+\\ &\quad\cdots+(2n-1)\|a_n\|_p\|f,a_1,a_2,\ldots,a_{n-1}\|_p.\end{aligned}$$

Next, we observe that

$$\begin{aligned} \|f, a_2, a_3, \dots, a_n\|_p &\le \left(\sum_{\{i_2,\dots,i_n\} \subset \{1,\dots,n\}} \|f, a_{i_2}, \dots, a_{i_n}\|_p^p \right)^{\frac{1}{p}} = \|f\|_p^* \\ &\vdots \\ \|f, a_1, a_2, \dots, a_{n-1}\|_p &\le \left(\sum_{\{i_2,\dots,i_n\} \subset \{1,\dots,n\}} \|f, a_{i_2}, \dots, a_{i_n}\|_p^p \right)^{\frac{1}{p}} = \|f\|_p^*. \end{aligned}$$

Hence, we obtain

$$n\|f\|_p\|a_1, a_2, \dots, a_n\|_p \le (2n-1)\left(\|a_1\|_p + \cdots + \|a_n\|_p\right)\|f\|_p^*.$$

This completes the proof. □

Corollary 2.3. *If $A := \{a_1, \dots, a_n\}$ and $B := \{b_1, \dots, b_n\}$ are two linearly independent sets in $L^p(X)$, then the norm defined by (2.1) using A is equivalent to that using B.*

Corollary 2.4. *The space $(L^p(X), \|\cdot\|_p^*)$ is complete. In other words, it is a Banach space.*

By Lemma 2.1, if a sequence (f_n) converges to $f \in L^p(X)$ with respect to the usual norm $\|\cdot\|_p$, then it also converges to f with respect to the n-norm $\|\cdot,\dots,\cdot\|_p$. Similarly, if (f_n) is a Cauchy sequence in $L^p(X)$ with respect to $\|\cdot\|_p$, then it is also a Cauchy sequence with respect to $\|\cdot,\dots,\cdot\|_p$. Another consequence of Proposition 2.2 is the following theorem.

Theorem 2.5. *If a sequence $(f_n) \in L^p(X)$ converges to some $f \in L^p(X)$ with respect to $\|\cdot,\dots,\cdot\|_p$, then it also converges to f with respect to $\|\cdot\|_p$. Also, if (f_n) is a Cauchy sequence with respect to $\|\cdot,\dots,\cdot\|_p$, then it is a Cauchy sequence with respect to $\|\cdot\|_p$.*

Proof. Let $\{a_1, \dots, a_n\}$ be a linearly independent set in $L^p(X)$, and $\|\cdot\|_p^*$ be defined by (2.1). Now, if (f_n) converges to some $f \in L^p(X)$ with respect to $\|\cdot,\dots,\cdot\|_p$, then for every $\{i_2, \dots, i_n\} \subset \{1, \dots, n\}$ we have

$$\|f_n - f, a_{i_2}, \dots, a_{i_n}\| \to 0, \quad \text{as } n \to \infty.$$

It follows that

$$\|f(n) - f\|_p^* \to 0, \quad \text{as } n \to \infty,$$

that is, (f_n) converges to f with respect to $\|\cdot\|_p^*$. By Proposition 2.2, we conclude that (f_n) also converges to f with respect to $\|\cdot\|_p$. The second statement of the theorem is proved in a similar way. □

Corollary 2.6. $(L^p(X), \|\cdot, \dots, \cdot\|_p)$ *is an n-Banach space.*

Proof. Let (f_n) be a Cauchy sequence in $L^p(X)$ with respect to $\|\cdot, \dots, \cdot\|_p$. Then, by Theorem 2.5, (f_n) is a Cauchy sequence with respect to $\|\cdot\|_p$. We know that $(L^p(X), \|\cdot\|_p)$ is a Banach space, and so (f_n) must converge to an element $f \in L^p(X)$ with respect to $\|\cdot\|_p$. By Lemma 2.1, (f_n) must also converge to f with respect to $\|\cdot, \dots, \cdot\|_p$. Therefore, $(L^p(X), \|\cdot, \dots, \cdot\|_p)$ is an n-Banach space. □

Remark 2.7. Up to this point, one may ask what then is the purpose of having an n-norm on $L^p(X)$? There are two answers to this question. First, we can use the n-norm to define "volumes" of n-dimensional parallelepiped spanned by n elements in $L^p(X)$. Second, we did not know the relation between the topology generated by the n-norm and that by the usual norm on $L^p(X)$, until we proved Proposition 2.2. The result enriches our knowledge on particular n-normed spaces such as $L^p(X)$ and $\ell^p(\mathbb{N})$ spaces, as part of an effort in understanding the notion of n-normed spaces in general.

3 An Application

A contractive mapping theorem on standard and finite dimensional n-normed spaces was formulated by Gunawan and Mashadi [11, 12] in 2001. What distinguishes their work from others' many years earlier is that they proved the theorem by involving a derived norm from the n-norm, rather than doing the same steps in n-normed spaces as in the proof of the analogous theorem in normed spaces. In 2013, Idris, Ekariani and Gunawan [13] formulated a contractive mapping theorem on the infinite dimensional vector space ℓ^p as a 2-normed space. Its generalization for ℓ^p as an n-normed space, where $n > 2$, can be found in [2].

With our previous result, we can now prove the following contractive mapping theorem on $(L^p(X), \|\cdot, \dots, \cdot\|_p)$.

Theorem 3.1. (Contractive Mapping Theorem) *Let T be a self-mapping of $L^p(X)$ which is contractive with respect to a linearly independent set $\{a_1, \dots, a_n\}$ in $L^p(X)$, that is, there exists a constant $C \in (0, 1)$ such that the inequality*

$$\|Tf - Tg, a_{i_2}, \dots, a_{i_n}\|_p \le C\, \|f - g', a_{i_2}, \dots, a_{i_n}\|_p$$

holds for all $f, g \in L^p(X)$ and $\{i_2, \dots, i_n\} \subset \{1, \dots, n\}$. Then T has a unique fixed point in $L^p(X)$.

Proof. For every $f, g \in L^p(X)$, we observe that

$$\begin{aligned}\|Tf - Tg\|_p^* &= \left(\sum_{\{i_2,\dots,i_n\}\subset\{1,\dots,n\}} \|Tf - Tg, a_{i_2}, \dots, a_{i_n}\|_p^p \right)^{\frac{1}{p}} \\ &\le C \left(\sum_{\{i_2,\dots,i_n\}\subset\{1,\dots,n\}} \|f - g, a_{i_2}, \dots, a_{i_n}\|_p^p \right)^{\frac{1}{p}} \\ &= C\,\|f - g\|_p^*.\end{aligned}$$

This result tells us that T is a contractive mapping on $(L^p(X), \|\cdot\|_p^*)$, which is a Banach space (by Corollary 2.4). Thus T must have a unique fixed point in $L^p(X)$. $\square$

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