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Removable singularities for some degenerate non-linear elliptic equations

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Abstract

This paper is devoted to studying the removable singularities of solutions for the Dirichlet problem for degenerate non-linear elliptic equations on the boundary of domain. Method a priori energetic estimates of solutions to elliptic boundary value problems is used. The applied method differs from the way for obtaining appropriate results in linear situation.

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1 Introduction

The corresponding results for linear equations were obtained in the papers of L.Carleson [1], V.A.Kondratyev, O.A.Oleynik [2], O.A.Oleynik, G.A.Iosifyan [3], V.A.Kondratyev, E.M.Landis [4], D.Gilbarg, N.Trudinger [5], T.Gadjiev, V.Mamedova [6], J.Diederich [7], R.Harvey, J.Polking [8], for non-linear equations in the papers of T.Kilpelainen, X.Zhong [9] and others.

Let $\Omega \subset R^n$, $n \geq 2$ be a bounded domain. Consider the following equation

$$\sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, Du, ..., D^m u) = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha F_\alpha(x), \qquad (1)$$

where

$$D^{\alpha}=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}...\partial x_n^{\alpha_n}},\quad |\alpha|=\alpha_1+\alpha_2+...+\alpha_n,\quad m\geq 1.$$

Assume that the coefficients $A_\alpha(x,\xi)$ of the equation (1) are measurable with respect to $x\in\bar{\Omega}$, are continuous with respect to $\xi\in R^M$ (M is the number of different multi-indexes of lengths no more than m) and satisfy the conditions

$$\sum_{|\alpha|=m}A_\alpha(x,\xi)\xi_\alpha^m\geq\omega(x)\left|\xi^m\right|^p-c_1\omega(x)\sum_{i=1}^{m-1}|\xi_i|^p-f_1(x),$$

$$|A_\alpha(x,\xi)|\leq c_2\omega(x)\sum_{i=0}^{m}\left|\xi^i\right|^{p-1}+f_2(x), \tag{2}$$

where $\xi=(\xi^0,...,\xi^m)$, $\xi^i=(\xi^i_\alpha)$, $|\alpha|=i$, $p>1$,

$$f_1(x)\in L_{\frac{p}{p-1},loc}(\Omega),\quad f_2(x)\in L_{1,loc}(\Omega),\quad F_\alpha\in L_{\frac{p}{p-1},loc}(\Omega).$$

Suppose that $\omega(x)$, $x\in\Omega$ is a measurable non-negative function satisfying the conditions: $\omega\in L_{1,loc}(\Omega)$, and for any $p>0$ and some $\sigma>1$

$$\int\limits_{\Omega_\rho}\omega^{-1/(\sigma-1)}dx<\infty,\quad ess\sup_{x\in\Omega_\rho}\omega(x)\leq c_3\rho^{n(\sigma-1)}\left(\int\limits_{\Omega_\rho}\omega^{-\frac{1}{(\sigma-1)}}dx\right)^{1-\sigma}. \tag{3}$$

Here $\Omega_\rho=\Omega\cap B_\rho$, $B_\rho=\{x:|x|<\rho\}$, c_i are positive constants dependent only on the problem data. In particular, it follows from conditions (3) that $\omega\in A_\sigma$ ([10]), i.e.

$$\int\limits_{\Omega_\rho}\omega dx\left[\int\limits_{\Omega_\rho}\omega^{-\frac{1}{\sigma-1}}dx\right]^{\sigma-1}\leq c_4\rho^{n\sigma}. \tag{4}$$

The following estimate also follows from (3)

$$ess\sup_{x\in\Omega_\rho}\omega(x)\leq c_5\rho^{-n}\left(\int\limits_{\Omega_\rho}\omega dx\right). \tag{5}$$

Furthermore, assume that

$$\frac{\omega(\Omega_s)}{\omega(\Omega_h)}\leq c_6\left(\frac{s}{h}\right)^{n\mu}, \tag{6}$$

$\mu<1+p/n$, for any $s\geq h>0$, where $\omega(\Omega_s)=\int\limits_{\Omega_s}\omega(x)dx$.

2 Some definitions and auxiliary results

We'll describe geometry of $\partial\Omega$ by means of the non-linear basic frequency $\lambda_p^p(r)$ of the cross section S_r

$$\lambda_p^p(r) = \inf \left(\int_{S_r} |\nabla_S v|^p \, ds \right) \left(\int_{S_r} |v|^p \, ds \right)^{-1}, \tag{7}$$

where the lower bound is taken on all continuously differentiable in some vicinity of S_r functions vanishing on $\partial\Omega$; $\nabla_S v(x)$ is a projection of the vector $\nabla v(x)$ on a tangential plane to S_r at the point x. For $p = 2$ the number $\lambda_2^2(r)$ is the first eigenvalue of Beltrami-Laplace operator on S_r, for $p \neq 2$ $\lambda_p^p(r)$ was studied in various papers. Some examples of calculation, or its lower estimates for a number of specific sets are for example in [11].

By $W_{p,\omega}^m(\Omega)$ we denote a closure of the functions from $C^m(\bar{\Omega})$ with respect to the norm

$$\|u\|_{W_{p,\omega}^m(\Omega)} = \left(\int_{\Omega} \omega(x) \sum_{|\alpha| \leq m} |D^\alpha u|^p \, dx \right)^{\frac{1}{p}}$$

$\circ W_{p,\omega}^{\,m}$ is a closure of the functions from $C_0^\infty(\Omega)$ to $W_{p,\omega}^m(\Omega)$.

We say that the function $u(x) \in \circ W_{p,\omega}^{\,m}(\Omega)$ is a generalized solution of the Dirichlet problem for equation (1) if the following integral identity is fulfilled for an arbitrary function $\eta(x) \in C_0^\infty(\Omega)$

$$\int_{\Omega} \sum_{|\alpha| \leq m} A_\alpha(x, u, ..., D^m u) D^\alpha \eta dx = \int_{\Omega} \sum_{|\alpha| \leq m} F_\alpha(x) D^\alpha \eta dx. \tag{8}$$

We'll divide the considered domains into two classes. The first class is "narrow" domains whose complement in the vicinity of the point 0 is sufficiently massive, for example it contains some cone with a vertex at this point. In the terms of frequency of set this class of domains satisfies the condition
$A) r\lambda_p(r) > d_1 > 0, \quad \forall r \in (0, r_0), \quad r_0 > 0.$

The second class contains "wide domains", i.e. such that have "inwards cusp" at the point 0. In the terms of frequency of the set this class of domains is described as following
$B) r\lambda_p(r) < d_2 < \infty, \quad \forall r \in (0, r_0).$

Determine the function $\psi(r)$ on $(0, r_0)$ by the inequality

$$\inf_{r\psi(r) < |x| < r} \lambda_p(|x|)(r - r\psi(r))\omega(x) \geq \mu > 0, \tag{9}$$

where μ is such that $0 < 1 - c_0 < \psi(r) < 1$. For monotonically decreasing functions $\lambda_p(r)$ (we meet them in applications) inequality (9) accepts the following form

$$r\lambda_p(r)(1 - \psi(r))\omega(x) \geq \mu, \quad \text{for} \quad \varphi(r) \equiv 1 - \psi(r) \geq \mu\omega^{-1}(x)(r\lambda_p(r))^{-1}. \tag{10}$$

Consider the distance function from the point x to $\partial\Omega - g(x) = \rho(x, \partial\Omega)$. It is known that $\exists\delta > 0$ such that $\Gamma_\delta = \{x:\ 0 < \rho(x,\partial\Omega) < \delta\}$, $g(x) \in C^m$, $|\nabla g(x)| = 1$. Furthermore, it follows from [5] that

$$\left|\nabla^j g(x)\right| \leq h_0 (g(x))^{1-j}, \quad \forall x \in \Gamma_\delta,\ j = \overline{1,m}. \tag{11}$$

Denote $\Omega_r = \Omega \cap \{x : g(x) < r\}$

For an arbitrary $\Gamma \subset \partial\Omega$ by $\circ W^{\,m}_{p,\omega}(\Omega, \Gamma)$ we denote a closure in the norm $W^m_{p,\omega}(\Omega)$ of the set of functions from $C^\infty(\Omega)$ vanishing near $\partial\Omega\backslash\Gamma$. We'll say that $u(x) \in \circ W^{\,m}_{p,\omega,loc}(\Omega,\Gamma)$ if $u(x) \in \circ W^{\,m}_{p,\omega}(\Omega', \partial\Omega'\backslash\partial\Omega)$ for any subdomain $\Omega' \subset \Omega$ such that $\Gamma \cap \partial\Omega' = \emptyset$.

Let $u(x) \in \circ W^{\,m}_{p,\omega}(\Omega, \Gamma)$ be a generalized solution of equation (1), i.e. $u(x)$ satisfy integral identity (8) for any function $\eta(x) \in C_0^\infty(\Omega')$, $\Omega' \subset \Omega$, $\Gamma \cap \partial\Omega' = \emptyset$.

Formulate some auxiliary lemmas.

Lemma 2.1 *Let $I(r)$ be a non-negative non-increasing on the interval $(0, r_0)$, $r_0 > 0$ function satisfying the condition*

$$I(r) < \theta I(r\varepsilon(r)) + G(r\varepsilon(r)), \quad 0 < \theta < 1, \tag{12}$$

where $\varepsilon(r)$ is a measurable function, $0 < c_0 < \varepsilon(r) < 1$ is such that

$$K(r) \equiv (\varphi(r))^{-1} r\varepsilon(r) < \tau < r\inf \varphi(\tau) \geq \nu > 0,\ \varphi(r) \equiv 1 - \varepsilon(r), \tag{13}$$

and $G(r)$ is measurable and locally bounded. Then the following alternative is valid. Either $I(r_i) < c_7 G(r_i)$ for some sequence $r_i \to 0$, or $I(r)$ sufficiently rapidly grows as $r \to 0$, exactly

$$I(r) \geq c_7 \exp\left(c_8 \nu \ln(\theta + \delta)^{-1} \int_r^{r_0} \frac{d\tau}{\tau(1 - \varepsilon(\tau))}\right) I(r_0),\ 0 < \delta < 1 - \theta, \tag{14}$$

where $c_7, c_8 > 0$ are constants. In particular, the last estimate also holds in the case of boundedness of $G(r)$ for any unbounded function $I(r)$ satisfying condition (12).

Lemma 2.2 *Let $I(r)$ be a non-negative non-increasing function on the interval $(0, r_0)$ satisfying the condition*

$$I(r) \leq (1 - \varphi(r)) I(r\varepsilon) + \varphi(r) G(r\varepsilon), \quad 0 < \varepsilon < 1, \quad r \in (0, r_0), \tag{15}$$

where $\varphi(r)$ is a measurable function and $0 < \varphi(r) < c_0 < 1, \forall r \in (0, r_0)$. $G(r)$ is measurable and locally bounded. Then the following alternative is valid for $I(r)$.

1. Either for some sequence $r_i \to 0$ the estimate

$$I(r_i) < c_9 G(r_i), \quad c_9 < \infty$$

is fulfilled;

2. or $I(r)$ rapidly grows as $r \to 0$, exactly,

$$I(r) \geq c_{10} \exp\left(\frac{1-\delta}{\ln \varepsilon^{-1}} \int\limits_{r}^{r_0} \frac{\bar{\varphi}(\tau)\, d\tau}{\tau}\right) I(r_0), \ \forall \delta > 0, \ \forall r \in (0, r_0), \ r_0 = r_0(\delta), \tag{16}$$

where $\bar{\varphi}(r)$ is an arbitrary continuous, non-decreasing function satisfying the inequality $\bar{\varphi}(r) \leq \varphi(r)$, $\forall r \in (0, r_0)$.

3 The behaviour of integral energy

Now we study behaviour of $I(r)$ for small r.

Theorem 3.1 *Let $u(x) \in \circ W^{\,m}_{p,\omega,loc}(\Omega, \Gamma)$ be a generalized solution of the Dirichlet problem for equation (1). Suppose that the coefficients of the equation satisfy condition (2), the domain Ω satisfies the condition A). $\psi(t)$ be a measurable function satisfying conditions (9), and let for the function $K(r)$ estimate B) be fulfilled with respect to $\varphi(r) = 1 - \psi(r)$. Then the following alternative is valid for $I(r)$:*

1. Either $I(r_i) < c_{11}(1 + G(r_i))$ for some sequence $r_i \to 0$, where $c_{11} < \infty$ is a constant;

2. or $I(r)$ grows rapidly as $r \to 0$, exactly

$$I(r) > c_{12}(\gamma) \exp\left(c_0 \nu \ln(k_0 + \gamma)^{-1} \int\limits_{r}^{r_0} \frac{d\tau}{\tau \varphi(\tau)}\right), \ \forall r < r_0' = r_0'(\gamma), \tag{17}$$

where k_0 is some constant.

Proof. Substitute the test function $\eta(x) = u(x)\left(1 - \xi_{\psi(r)}(r^{-1} g(x))\right)$ into integral identity (17). For simplicity, we'll assume that the solution $u(x,t)$ is sufficiently smooth. Therefore we admit some formality in reasonings. Since, these reasonings may be precise by passing to a regularized problem by the known method (see [12]) and later tending the regularization parameter to zero we'll obtain the result for generalized solution. Continuing the proof of the theorem, we use condition (2) and get

$$\int\limits_{\Omega \setminus \Omega_r} \omega(x) |D^m u|^p \, dx \leq$$

$$\int\limits_{\Omega\backslash\Omega_{r\psi(r)}}\left[c_3k_1\omega(x)\sum_{|\alpha|\le m}|D^\alpha u|^{p-1}\cdot\sum_{|\alpha|\le m}\sum_{|\beta|<|\alpha|}\left|D^{\alpha-\beta}u\right|\left|D^\beta\xi\right|+\right.$$

$$\left.+\sum_{|\alpha|\le m}\sum_{|\beta|\le|\alpha|}|f_2(x)|\left|D^{\alpha-\beta}u\right|\left|D^\beta\xi\right|\right]dx+$$

$$+\int\limits_{\Omega_{r_0}\backslash\Omega_{r\psi(r)}}\left[k_2\omega(x)\sum_{|\alpha|\le m-1}|D^\alpha u|^p+k_1\omega(x)\sum_{|\alpha|\le m}|D^\alpha u|^{p-1}\times\right.$$

$$\left.\times\sum_{|\alpha|\le m-1}|D^\alpha u|+\sum_{|\alpha|<m}|f_2(x)|\cdot|D^\alpha u|+f_1(x)\right]\xi dx+$$

$$+\int\limits_{\Omega\backslash\Omega_{r_0}}\left[k_2\omega(x)\sum_{|\alpha|\le m-1}|D^\alpha u|^p+k_1\omega(x)\sum_{|\alpha|\le m}|D^\alpha u|^{p-1}\times\right.$$

$$\left.\times\sum_{|\alpha|\le m-1}|D^\alpha u|+\sum_{|\alpha|<m}|f_2(x)||D^\alpha u|+f_1(x)\right]\xi dx. \qquad (18)$$

Notice that we are in the class of domains for which $\lambda_p(r)\to\infty$ as $r\to 0$, consequently for any $\delta>0$ there exists $r_0=r_0(\delta)$ such that for any $r<r_0$ $\lambda_p(r)>\delta^{-1}$. Using the Young inequality with ε, from (18) we get

$$I(r)\le[I(r\psi(r))-I(r)]\left(k_3(1-\delta^p)^{\frac{1-p}{p}}+\frac{\varepsilon}{p}\right)+(p-1)\varepsilon^{\frac{1}{1-p}}k_4\times$$

$$\times\int\limits_{\Omega_r\backslash\Omega_{r\psi(r)}}\sum_{|\alpha|\le m}|f_2(x)|^{\frac{p}{p-1}}\cdot\lambda_p^{-\frac{m-|\alpha|}{p-1}p}(g(x))\,dx+[I(r\psi(r))-I(r_0)]\times$$

$$\times\left(k_5+\frac{\varepsilon}{p}\right)+I(r_0)\left(k_6+\frac{\varepsilon}{p}\right)+\frac{(p-1)\varepsilon^{\frac{1}{1-p}}}{p}\times$$

$$\times\int\limits_{\Omega\backslash\Omega_{r\psi(r)}}\left[\sum_{|\alpha|<m}|f_2(x)|^{\frac{p}{p-1}}\cdot\lambda_p^{-\frac{m-|\alpha|}{p-1}p}(g(x))\,dx+|f_1(x)|\right]dx. \qquad (19)$$

Denote $G(r)\equiv\int\limits_{\Omega\backslash\Omega_r}\left[\sum\limits_{|\alpha|\le m}|f_2(x)|^{\frac{p}{p-1}}\cdot\lambda_p^{-\frac{m-|\alpha|}{p-1}p}(g(x))\,dx+|f_1(x)|\right]dx.$

Then from (19) we get

$$I(r)\le\alpha(\delta,\varepsilon)\,I(r\psi(r))+A_1I(r_0)+B_1G(r\psi(r)), \qquad (20)$$

where α,A_1,B_1 are some constants that are exactly calculated and $\alpha_0=\alpha(0,0)<1$. Thus, from (20) and Lemma 2.1 we get the proof of the theorem.

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