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Some Inequalities on Relative Type and Relative Weak Type of Entire Functions

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Abstract

In the paper we study some relative growth properties of entire functions with respect to another entire function on the basis relative type and relative weak type.

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1 Introduction, Definitions and Notations

Let $\mathbb{C}$ be the set of all finite complex numbers. Let f be an entire function defined in the open complex plane $\mathbb{C}$ and $M_f(r) = \max\{|f(z)| : |z| = r\}$. In the sequel, we use the following notation :

$$\log^{[k]} x = \log\left(\log^{[k-1]} x\right) \text{ for } k = 1, 2, 3, ... \text{ and } \log^{[0]} x = x.$$

To start our paper we just recall the following definitions:

The order ρ_f and lower order λ_f of an entire function f are defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r}.$$

An entire function for which order and lower order are the same is said to be of regular growth. Functions which are not of regular growth are said to be of irregular growth.

The type σ_f and lower type $\overline{\sigma}_f$ of an entire function f are defined as

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\rho_f}} \text{ and } \overline{\sigma}_f = \liminf_{r\to\infty} \frac{\log M_f(r)}{r^{\rho_f}},\ 0 < \rho_f < \infty.$$

Datta and Jha [3] introduced the definition of weak type of an entire function of finite positive lower order in the following way:

[3] The weak type τ_f and the growth indicator $\overline{\tau}_f$ of an entire function f of finite positive lower order λ_f are defined by

$$\overline{\tau}_f = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\lambda_f}} \text{ and } \tau_f = \liminf_{r\to\infty} \frac{\log M_f(r)}{r^{\lambda_f}},\ 0 < \lambda\rho_f < \infty.$$

If an entire function g is non-constant then $M_g(r)$ is strictly increasing and continuous and its inverse $M_g{}^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists and is such that $\lim_{s\to\infty} M_g^{-1}(s) = \infty$.

Bernal {[1], [2]} introduced the definition of relative order of an entire function f with respect to an entire function g , denoted by $\rho_g(f)$ as follows :

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : M_f(r) < M_g(r^\mu) \text{ for all } r > r_0(\mu) > 0\} \\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.\end{aligned}$$

The definition coincides with the classical one [7] if $g(z) = \exp z$.

Similarly one can define the relative lower order of an entire function f with respect to an entire function g denoted by $\lambda_g(f)$ as follows :

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.$$

To compare the relative growth of two entire functions having same non zero finite relative order with respect to another entire function, Roy [6] recently introduced the notion of relative type of two entire functions in the following manner:

[6] Let f and g be any two entire functions such that $0 < \rho_g(f) < \infty$. Then the relative type $\sigma_g(f)$ of f with respect to g is defined as :

$$\begin{aligned} & \sigma_g(f) \\ = \; & \inf\left\{k > 0 : M_f(r) < M_g\left(kr^{\rho_g(f)}\right) \text{ for all sufficiently large values of } r\right\} \\ = \; & \limsup_{r\to\infty} \frac{M_g^{-1} M_f(r)}{r^{\rho_g(f)}} \, . \end{aligned}$$

Likewise one can define the relative lower type of an entire function f with respect to an entire function g denoted by $\overline{\sigma}_g(f)$ as follows :

$$\overline{\sigma}_g(f) = \liminf_{r\to\infty} \frac{M_g^{-1} M_f(r)}{r^{\rho_g(f)}}, \; 0 < \rho_g(f) < \infty \, .$$

Analogously to determine the relative growth of two entire functions having same non zero finite relative lower order with respect to another entire function, Datta and Biswas [4] introduced the definition of relative weak type of an entire function f with respect to another entire function g of finite positive relative lower order $\lambda_g(f)$ in the following way:

[4] The relative weak type $\tau_g(f)$ of an entire function f with respect to another entire function g having finite positive relative lower order $\lambda_g(f)$ is defined as:

$$\tau_g(f) = \liminf_{r\to\infty} \frac{M_g^{-1} M_f(r)}{r^{\lambda_g(f)}} \, .$$

Also one may define the growth indicator $\overline{\tau}_g(f)$ of an entire function f with respect to an entire function g in the following way :

$$\overline{\tau}_g(f) = \limsup_{r\to\infty} \frac{M_g^{-1} M_f(r)}{r^{\lambda_g(f)}}, \; 0 < \lambda_g(f) < \infty \, .$$

Considering $g = \exp z$, one may easily verify that Definition 1 and Definition 1 coincide with the classical type (lower type) and weak type respectively.

In the paper we study some relative growth properties of entire functions with respect to another entire function on the basis of relative type and relative weak type. We do not explain the standard definitions and notations in the theory of entire functions as those are available in [8].

2 Lemmas

In this section we present some lemmas which will be needed in the sequel.

In the line of Datta and Biswas [5] we may state the following two lemmas:

Let f be an entire function with $0 \leq \lambda_f \leq \rho_f < \infty$ and g be entire of regular growth . Then

$$\lambda_g(f) = \frac{\lambda_f}{\lambda_g} \quad \text{and} \quad \rho_g(f) = \frac{\rho_f}{\rho_g} \; .$$

Let f be an entire function with regular growth and g be entire with $0 \leq \lambda_g \leq \rho_g < \infty$. Then

$$\lambda_g(f) = \frac{\rho_f}{\rho_g} \quad \text{and} \quad \rho_g(f) = \frac{\lambda_f}{\lambda_g} \; .$$

3 Theorems

In this section we present the main results of the paper.

Let f and g be any two entire functions with finite non-zero order. Also let g is of regular growth. Then

$$\left[\frac{\overline{\sigma}_f}{\sigma_g}\right]^{\frac{1}{\rho_g}} \leq \overline{\sigma}_g(f) \leq \min\left\{\left[\frac{\overline{\sigma}_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}, \left[\frac{\sigma_f}{\sigma_g}\right]^{\frac{1}{\rho_g}}\right\}$$

$$\leq \max\left\{\left[\frac{\overline{\sigma}_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}, \left[\frac{\sigma_f}{\sigma_g}\right]^{\frac{1}{\rho_g}}\right\} \leq \sigma_g(f) \leq \left[\frac{\sigma_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}} \; .$$

From the definitions of σ_f and $\overline{\sigma}_f$, we have for all sufficiently large values of r that

$$M_f(r) \leq \exp\{(\sigma_f + \varepsilon) r^{\rho_f}\}, \tag{1}$$
$$M_f(r) \geq \exp\{(\overline{\sigma}_f - \varepsilon) r^{\rho_f}\} \tag{2}$$

and also for a sequence of values of r tending to infinity we get that

$$M_f(r) \geq \exp\{(\sigma_f - \varepsilon) r^{\rho_f}\}, \tag{3}$$
$$M_f(r) \leq \exp\{(\overline{\sigma}_f + \varepsilon) r^{\rho_f}\} \; . \tag{4}$$

Similarly from the definitions of σ_g and $\overline{\sigma}_f$, it follows for all sufficiently large values of r that

$$\begin{aligned} M_g(r) &\leq \exp\{(\sigma_g+\varepsilon)\, r^{\rho_g}\} \\ i.e.,\ r &\leq M_g^{-1}\left[\exp\{(\sigma_g+\varepsilon)\, r^{\rho_g}\}\right] \\ i.e.,\ M_g^{-1}(r) &\geq \left[\left(\frac{\log r}{(\sigma_g+\varepsilon)}\right)^{\frac{1}{\rho_g}}\right], \end{aligned} \tag{5}$$

$$\begin{aligned} M_g(r) &\geq \exp\{(\overline{\sigma}-\varepsilon)\, r^{\rho_g}\} \\ i.e.,\ r &\geq M_g^{-1}\left[\exp\{(\overline{\sigma}-\varepsilon)\, r^{\rho_g}\}\right] \\ i.e.,\ M_g^{-1}(r) &\leq \left[\left(\frac{\log r}{(\overline{\sigma}-\varepsilon)}\right)^{\frac{1}{\rho_g}}\right] \end{aligned} \tag{6}$$

and for a sequence of values of r tending to infinity we obtain that

$$\begin{aligned} M_g(r) &\geq \exp\{(\sigma_g-\varepsilon)\, r^{\rho_g}\} \\ i.e.,\ r &\geq M_g^{-1}\left[\exp\{(\sigma_g-\varepsilon)\, r^{\rho_g}\}\right] \\ i.e.,\ M_g^{-1}(r) &\leq \left[\left(\frac{\log r}{(\sigma_g-\varepsilon)}\right)^{\frac{1}{\rho_g}}\right], \end{aligned} \tag{7}$$

$$\begin{aligned} M_g(r) &\leq \exp\{(\overline{\sigma}_g+\varepsilon)\, r^{\rho_g}\} \\ i.e.,\ r &\leq M_g^{-1}\left[\exp\{(\overline{\sigma}_g+\varepsilon)\, r^{\rho_g}\}\right] \\ i.e.,\ M_g^{-1}(r) &\geq \left[\left(\frac{\log r}{(\overline{\sigma}_g-\varepsilon)}\right)^{\frac{1}{\rho_g}}\right]. \end{aligned} \tag{8}$$

Now from (3) and in view of (5), we get for a sequence of values of r tending to infinity that

$$M_g^{-1}M_f(r) \geq M_g^{-1}\left[\exp\{(\sigma_f-\varepsilon)\, r^{\rho_f}\}\right]$$

$$i.e.,\ M_g^{-1}M_f(r) \geq \left[\left(\frac{\log\exp\{(\sigma_f-\varepsilon)\, r^{\rho_f}\}}{(\sigma_g+\varepsilon)}\right)^{\frac{1}{\rho_g}}\right]$$

$$\begin{aligned} i.e.,\ M_g^{-1}M_f(r) &\geq \left[\frac{(\sigma_f-\varepsilon)}{(\sigma_g+\varepsilon)}\right]^{\frac{1}{\rho_g}} \cdot r^{\frac{\rho_f}{\rho_g}} \\ i.e.,\ \frac{M_g^{-1}M_f(r)}{r^{\frac{\rho_f}{\rho_g}}} &\geq \left[\frac{(\sigma_f-\varepsilon)}{(\sigma_g+\varepsilon)}\right]^{\frac{1}{\rho_g}}. \end{aligned}$$

As $\varepsilon\,(>0)$ is arbitrary, in view of Lemma 2 it follows that

$$\limsup_{r\to\infty}\frac{M_g^{-1}M_f(r)}{r^{\rho_g(f)}} \geq \left[\frac{\sigma_f}{\sigma_g}\right]^{\frac{1}{\rho_g}}$$
$$i.e.,\ \sigma_g(f) \geq \left[\frac{\sigma_f}{\sigma_g}\right]^{\frac{1}{\rho_g}}. \tag{9}$$

Analogously from (2) and in view of (8), it follows for a sequence of values of r tending to infinity that

$$M_g^{-1}M_f(r) \geq M_g^{-1}\left[\exp\left\{(\overline{\sigma}_f-\varepsilon)\,r^{\rho_f}\right\}\right]$$

$$i.e.,\ M_g^{-1}M_f(r) \geq \left[\left(\frac{\log\exp\{(\overline{\sigma}_f-\varepsilon)\,r^{\rho_f}\}}{(\overline{\sigma}_g+\varepsilon)}\right)^{\frac{1}{\rho_g}}\right]$$

$$i.e.,\ M_g^{-1}M_f(r) \geq \left[\frac{(\overline{\sigma}_f-\varepsilon)}{(\overline{\sigma}_g+\varepsilon)}\right]^{\frac{1}{\rho_g}}\cdot r^{\frac{\rho_f}{\rho_g}}$$
$$i.e.,\ \frac{M_g^{-1}M_f(r)}{r^{\frac{\rho_f}{\rho_g}}} \geq \left[\frac{(\overline{\sigma}_f-\varepsilon)}{(\overline{\sigma}_g+\varepsilon)}\right]^{\frac{1}{\rho_g}}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we get from above and Lemma 2 that

$$\limsup_{r\to\infty}\frac{M_g^{-1}M_f(r)}{r^{\rho_g(f)}} \geq \left[\frac{\overline{\sigma}_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}$$
$$i.e.,\ \sigma_g(f) \geq \left[\frac{\overline{\sigma}_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}. \tag{10}$$

Again in view of (6), we have from (1) for all sufficiently large values of r that

$$M_g^{-1}M_f(r) \leq M_g^{-1}\left[\exp\{(\sigma_f+\varepsilon)\,r^{\rho_f}\}\right]$$

$$i.e.,\ M_g^{-1}M_f(r) \leq \left[\left(\frac{\log\exp\{(\sigma_f+\varepsilon)\,r^{\rho_f}\}}{(\overline{\sigma}_g-\varepsilon)}\right)^{\frac{1}{\rho_g}}\right]$$

$$i.e.,\ M_g^{-1}M_f(r) \leq \left[\frac{(\sigma_f+\varepsilon)}{(\overline{\sigma}_g-\varepsilon)}\right]^{\frac{1}{\rho_g}}\cdot r^{\frac{\rho_f}{\rho_g}}$$
$$i.e.,\ \frac{M_g^{-1}M_f(r)}{r^{\frac{\rho_f}{\rho_g}}} \leq \left[\frac{(\sigma_f+\varepsilon)}{(\overline{\sigma}_g-\varepsilon)}\right]^{\frac{1}{\rho_g}}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain in view of Lemma 2 that

$$\begin{aligned}\limsup_{r\to\infty}\frac{M_g^{-1}M_f(r)}{r^{\rho_g(f)}} &\le \left[\frac{\sigma_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}\\ i.e.,\ \sigma_g(f) &\le \left[\frac{\sigma_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}\ . \qquad (11)\end{aligned}$$

Now from (2) and in view of (5), we get for all sufficiently large values of r that

$$M_g^{-1}M_f(r) \ge M_g^{-1}\left[\exp\left\{(\overline{\sigma}_f-\varepsilon)\,r^{\rho_f}\right\}\right]$$

$$i.e.,\ M_g^{-1}M_f(r) \ge \left[\left(\frac{\log\exp\left\{(\overline{\sigma}_f-\varepsilon)\,r^{\rho_f}\right\}}{(\sigma_g+\varepsilon)}\right)^{\frac{1}{\rho_g}}\right]$$

$$\begin{aligned}i.e.,\ M_g^{-1}M_f(r) &\ge \left[\frac{(\overline{\sigma}_f-\varepsilon)}{(\sigma_g+\varepsilon)}\right]^{\frac{1}{\rho_g}}\cdot r^{\frac{\rho_f}{\rho_g}}\\ i.e.,\ \frac{M_g^{-1}M_f(r)}{r^{\frac{\rho_f}{\rho_g}}} &\ge \left[\frac{(\overline{\sigma}_f-\varepsilon)}{(\sigma_g+\varepsilon)}\right]^{\frac{1}{\rho_g}}\ .\end{aligned}$$

As $\varepsilon\,(>0)$ is arbitrary, it follows from above and Lemma 2 that

$$\begin{aligned}\liminf_{r\to\infty}\frac{M_g^{-1}M_f(r)}{r^{\rho_g(f)}} &\ge \left[\frac{\overline{\sigma}_f}{\sigma_g}\right]^{\frac{1}{\rho_g}}\\ i.e.,\ \overline{\sigma}_g(f) &\ge \left[\frac{\overline{\sigma}_f}{\sigma_g}\right]^{\frac{1}{\rho_g}}\ . \qquad (12)\end{aligned}$$

Also in view of (7), we obtain from (1) for a sequence of values of r tending to infinity that

$$M_g^{-1}M_f(r) \le M_g^{-1}\left[\exp\left\{(\sigma_f+\varepsilon)\,r^{\rho_f}\right\}\right]$$

$$i.e.,\ M_g^{-1}M_f(r) \le \left[\left(\frac{\log\exp\left\{(\sigma_f+\varepsilon)\,r^{\rho_f}\right\}}{(\sigma_g-\varepsilon)}\right)^{\frac{1}{\rho_g}}\right]$$

$$\begin{aligned}i.e.,\ M_g^{-1}M_f(r) &\le \left[\frac{(\sigma_f+\varepsilon)}{(\sigma_g-\varepsilon)}\right]^{\frac{1}{\rho_g}}\cdot r^{\frac{\rho_f}{\rho_g}}\\ i.e.,\ \frac{M_g^{-1}M_f(r)}{r^{\frac{\rho_f}{\rho_g}}} &\le \left[\frac{(\sigma_f+\varepsilon)}{(\sigma_g-\varepsilon)}\right]^{\frac{1}{\rho_g}}\ .\end{aligned}$$

Since $\varepsilon\,(>0)$ is arbitrary, we get from Lemma 2 and above that

$$\begin{aligned}\liminf_{r\to\infty}\frac{M_g^{-1}M_f\left(r\right)}{r^{\rho_g(f)}} &\leq \left[\frac{\sigma_f}{\sigma_g}\right]^{\frac{1}{\rho_g}}\\ i.e.,\ \overline{\sigma}_g\left(f\right) &\leq \left[\frac{\sigma_f}{\sigma_g}\right]^{\frac{1}{\rho_g}}.\end{aligned}\tag{13}$$

Similarly from (4) and in view of (6), it follows for a sequence of values of r tending to infinity that

$$M_g^{-1}M_f\left(r\right)\leq M_g^{-1}\left[\exp\left\{\left(\overline{\sigma}_f+\varepsilon\right)r^{\rho_f}\right\}\right]$$

$$i.e.,\ M_g^{-1}M_f\left(r\right)\leq\left[\left(\frac{\log\exp\left\{\left(\overline{\sigma}_f+\varepsilon\right)r^{\rho_f}\right\}}{\left(\overline{\sigma}_g-\varepsilon\right)}\right)^{\frac{1}{\rho_g}}\right]$$

$$\begin{aligned}i.e.,\ M_g^{-1}M_f\left(r\right) &\leq \left[\frac{\left(\overline{\sigma}_f+\varepsilon\right)}{\left(\overline{\sigma}_g-\varepsilon\right)}\right]^{\frac{1}{\rho_g}}\cdot r^{\frac{\rho_f}{\rho_g}}\\ i.e.,\ \frac{M_g^{-1}M_f\left(r\right)}{r^{\frac{\rho_f}{\rho_g}}} &\leq \left[\frac{\left(\overline{\sigma}_f+\varepsilon\right)}{\left(\overline{\sigma}_g-\varepsilon\right)}\right]^{\frac{1}{\rho_g}}.\end{aligned}$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain from Lemma 2 and above that

$$\begin{aligned}\liminf_{r\to\infty}\frac{M_g^{-1}M_f\left(r\right)}{r^{\rho_g(f)}} &\leq \left[\frac{\overline{\sigma}_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}\\ i.e.,\ \overline{\sigma}_g\left(f\right) &\leq \left[\frac{\overline{\sigma}_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}.\end{aligned}\tag{14}$$

Thus the theorem follows from (9), (10), (11), (12), (13) and (14).

Although some portion of the above inequality i.e.,

$$\left[\frac{\sigma_f}{\sigma_g}\right]^{\frac{1}{\rho_g}}\leq\sigma_g\left(f\right)$$

has already been proved by Roy [6], we again give here only for carrying out the proofs of the remaining theorems of the paper.

In view of Theorem 3 one can easily deduce the following corollaries :

Let f be an entire function such that $\sigma_f=\overline{\sigma}_f$ and g be an entire function of regular growth . Then

$$\sigma_g\left(f\right)=\left[\frac{\sigma_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}\quad\text{and}\ \overline{\sigma}_g\left(f\right)=\left[\frac{\sigma_f}{\sigma_g}\right]^{\frac{1}{\rho_g}}.$$

Let f be an entire function with non zero finite order and g be entire of regular growth with $\sigma_g = \overline{\sigma}_g$. Then

$$\overline{\sigma}_g(f) = \left[\frac{\overline{\sigma}_f}{\sigma_g}\right]^{\frac{1}{\rho_g}} .$$

In addition, if $\sigma_f = \overline{\sigma}_f$ then

$$\sigma_g(f) = \overline{\sigma}_g(f) = \left[\frac{\sigma_f}{\sigma_g}\right]^{\frac{1}{\rho_g}} .$$

Let f be an entire function with non zero finite order. Then for any entire function g,

$$\begin{aligned} (i)\ \ \overline{\sigma}_g(f) &= \infty \text{ when } \sigma_g = 0 , \\ (ii)\ \ \sigma_g(f) &= \infty \text{ when } \overline{\sigma}_g = 0 , \\ (iii)\ \ \overline{\sigma}_g(f) &= 0 \text{ when } \sigma_g = \infty \end{aligned}$$

and

$$(iv)\ \ \sigma_g(f) = \infty \text{ when } \overline{\sigma}_g = \infty,$$

where g is of regular growth.

Let g be an entire function with regular growth . Then for any entire function f,

$$\begin{aligned} (i)\ \ \sigma_g(f) &= 0 \text{ when } \sigma_f = 0 , \\ (ii)\ \ \overline{\sigma}_g(f) &= 0 \text{ when } \overline{\sigma}_f = 0 , \\ (iii)\ \ \sigma_g(f) &= \infty \text{ when } \sigma_f = \infty \end{aligned}$$

and

$$(iv)\ \ \overline{\sigma}_g(f) = \infty \text{ when } \overline{\sigma}_f = \infty .$$

In the line of Theorem 3 and with the help of Lemma 2, one can prove the following theorem and therefore its proof is omitted:

Let f and g be any two entire functions with finite non-zero order. Also let f is of regular growth. Then

$$\left[\frac{\overline{\sigma}_f}{\sigma_g}\right]^{\frac{1}{\rho_g}} \le \tau_g(f) \le \min\left\{\left[\frac{\overline{\sigma}_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}, \left[\frac{\sigma_f}{\sigma_g}\right]^{\frac{1}{\rho_g}}\right\}$$

$$\le \max\left\{\left[\frac{\overline{\sigma}_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}, \left[\frac{\sigma_f}{\sigma_g}\right]^{\frac{1}{\rho_g}}\right\} \le \overline{\tau}_g(f) \le \left[\frac{\sigma_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}} .$$

In view of Theorem 3 one can easily derive the following corollaries :

Let f be an entire function with regular growth and $\sigma_f = \overline{\sigma}_f$ and g be an entire function of non zero finite order . Then

$$\overline{\tau}_g(f) = \left[\frac{\sigma_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}} \quad \text{and} \quad \tau_g(f) = \left[\frac{\sigma_f}{\sigma_g}\right]^{\frac{1}{\rho_g}} .$$

Let f be an entire function with regular growth and g be an entire function with $\sigma_g = \overline{\sigma}_g$. Then

$$\overline{\tau}_g(f) = \left[\frac{\sigma_f}{\sigma_g}\right]^{\frac{1}{\rho_g}} \quad \text{and} \quad \tau_g(f) = \left[\frac{\overline{\sigma}_f}{\sigma_g}\right]^{\frac{1}{\rho_g}} .$$

In addition, if $\sigma_f = \overline{\sigma}_f$ then

$$\overline{\tau}_g(f) = \tau_g(f) = \left[\frac{\sigma_f}{\sigma_g}\right]^{\frac{1}{\rho_g}} .$$

Let g be an entire function with non zero finite order. Then for any entire function f,

$$\begin{aligned} (i)\ \tau_g(f) &= \infty \text{ when } \sigma_g = 0 , \\ (ii)\ \overline{\tau}_g(f) &= \infty \text{ when } \overline{\sigma}_g = 0 , \\ (iii)\ \tau_g(f) &= 0 \text{ when } \sigma_g = \infty \end{aligned}$$

and

$$(iv)\ \overline{\tau}_g(f) = \infty \text{ when } \overline{\sigma}_g = \infty,$$

where f is of regular growth.

Let f be an entire function with regular growth . Then for any entire function g,

$$\begin{aligned} (i)\ \overline{\tau}_g(f) &= 0 \text{ when } \sigma_f = 0 , \\ (ii)\ \tau_g(f) &= 0 \text{ when } \overline{\sigma}_f = 0 , \\ (iii)\ \overline{\tau}_g(f) &= \infty \text{ when } \sigma_f = \infty \end{aligned}$$

and

$$(iv)\ \tau_g(f) = \infty \text{ when } \overline{\sigma}_f = \infty .$$

Similarly in the line of Theorem 3 and Theorem 3 and with the help of Lemma 2 and Lemma 2, one may easily prove the following two theorems and therefore their proofs are omitted:

Let f and g be any two entire functions with finite non-zero lower order. Also let g be of regular growth. Then

$$\left[\frac{\tau_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}} \leq \tau_g(f) \leq \min\left\{\left[\frac{\tau_f}{\tau_g}\right]^{\frac{1}{\lambda_g}}, \left[\frac{\overline{\tau}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}\right\}$$

$$\leq \max\left\{\left[\frac{\tau_f}{\tau_g}\right]^{\frac{1}{\lambda_g}}, \left[\frac{\overline{\tau}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}\right\} \leq \overline{\tau}_g(f) \leq \left[\frac{\overline{\tau}_f}{\tau_g}\right]^{\frac{1}{\lambda_g}} .$$

In view of Theorem 3, the following corollaries may also be obtained:

Let f be an entire function such that $\tau_f = \overline{\tau}_f$ and g be an entire function of regular growth . Then

$$\overline{\tau}_g(f) = \left[\frac{\tau_f}{\tau_g}\right]^{\frac{1}{\lambda_g}} \text{ and } \tau_g(f) = \left[\frac{\overline{\tau}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}} .$$

Let f be an entire function with non zero finite lower order and g be entire of regular growth with $\tau_g = \overline{\tau}_g$. Then

$$\overline{\tau}_g(f) = \left[\frac{\overline{\tau}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}} \text{ and } \tau_g(f) = \left[\frac{\tau_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}} .$$

In addition, if $\tau_f = \overline{\tau}_f$ then

$$\tau_g(f) = \overline{\tau}_g(f) = \left[\frac{\overline{\tau}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}} .$$

Let f be an entire function with non zero finite lower order. Then for any entire function g,

$$\begin{aligned} (i)\ \tau_g(f) &= \infty \text{ when } \overline{\tau}_g = 0 , \\ (ii)\ \overline{\tau}_g(f) &= \infty \text{ when } \tau_g = 0 , \\ (iii)\ \tau_g(f) &= 0 \text{ when } \overline{\tau}_g = \infty \end{aligned}$$

and

$$(iv)\ \overline{\tau}_g(f) = \infty \text{ when } \tau_g = \infty,$$

where g is of regular growth.

Let g be an entire function with regular growth . Then for any entire function f,

$$\begin{aligned} (i)\ \overline{\tau}_g(f) &= 0 \text{ when } \overline{\tau}_f = 0 , \\ (ii)\ \tau_g(f) &= 0 \text{ when } \tau_f = 0 , \\ (iii)\ \overline{\tau}_g(f) &= \infty \text{ when } \overline{\tau}_f = \infty \end{aligned}$$

and

$$(iv)\ \ \tau_g(f) = \infty \text{ when } \tau_f = \infty\ .$$

Let f and g be any two entire functions with finite non-zero order. Also let f be of regular growth. Then

$$\left[\frac{\tau_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}} \leq \overline{\sigma}_g(f) \leq \min\left\{\left[\frac{\tau_f}{\tau_g}\right]^{\frac{1}{\lambda_g}}, \left[\frac{\overline{\tau}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}\right\}$$

$$\leq \max\left\{\left[\frac{\tau_f}{\tau_g}\right]^{\frac{1}{\lambda_g}}, \left[\frac{\overline{\tau}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}\right\} \leq \sigma_g(f) \leq \left[\frac{\overline{\tau}_f}{\tau_g}\right]^{\frac{1}{\lambda_g}}\ .$$

From Theorem 3 the following corollaries are immediate:

Let f be an entire function with regular growth and $\overline{\tau}_f = \tau_f$ and g be an entire function of non zero finite lower order . Then

$$\sigma_g(f) = \left[\frac{\overline{\tau}_f}{\tau_g}\right]^{\frac{1}{\lambda_g}} \text{ and } \overline{\sigma}_g(f) = \left[\frac{\overline{\tau}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}\ .$$

Let f be an entire function with regular growth and g be an entire function with $\tau_g = \overline{\tau}_g$. Then

$$\sigma_g(f) = \left[\frac{\overline{\tau}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}} \text{ and } \overline{\sigma}_g(f) = \left[\frac{\tau_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}\ .$$

In addition, if $\tau_f = \overline{\tau}_f$ then

$$\sigma_g(f) = \overline{\sigma}_g(f) = \left[\frac{\overline{\tau}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}\ .$$

Let g be an entire function with non zero finite lower order. Then for any entire function f,

$$\begin{aligned} (i)\ \ \overline{\sigma}_g(f) &= \infty \text{ when } \overline{\tau}_g = 0\ , \\ (ii)\ \ \sigma_g(f) &= \infty \text{ when } \tau_g = 0\ , \\ (iii)\ \ \overline{\sigma}_g(f) &= 0 \text{ when } \overline{\tau}_g = \infty \end{aligned}$$

and

$$(iv)\ \ \sigma_g(f) = \infty \text{ when } \tau_g = \infty,$$

where f is of regular growth.

Let f be an entire function with regular growth . Then for any entire function g,

$$\begin{aligned} (i)\ \sigma_g(f) &= 0 \text{ when } \overline{\tau}_f = 0\ , \\ (ii)\ \overline{\sigma}_g(f) &= 0 \text{ when } \tau_f = 0\ , \\ (iii)\ \sigma_g(f) &= \infty \text{ when } \overline{\tau}_f = \infty \end{aligned}$$

and

$$(iv)\ \overline{\sigma}_g(f) = \infty \text{ when } \tau_f = \infty\ .$$

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