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Comorphisms of Lie algebroids and groupoids: a short introduction

Krzysztof Drachal

Faculty of Mathematics and Information Science
at the Warsaw University of Technology
ul. Koszykowa 75, 00 – 662 Warszawa, Poland

Abstract

The aim of this paper is to present a short (and a bit informal) exposition to comorphisms of Lie algebroids and comorphisms of Lie groupoids. We briefly present the motivation to study such a concept. We also review some recent works done in this direction.

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1 Introduction

The concept of *comorphism* in case of Lie algebroids and Lie groupoids was first fully described by Higgins and Mackenzie [16]. However, such a concept was tentatively discussed in some earlier works (see, for example, references in [16]). Recently, this notion was described with a help of suitable *graphs* and *Lie pseudo–algebras* by Chen and Liu [6].

One can be interested in the question why we do introduce comorphisms. First of all, notice that it is well–known that a linear map of Lie algebras, $\mathfrak{g} \to \mathfrak{h}$, is a morphism if and only if its dual, i.e., $\mathfrak{h}^* \to \mathfrak{g}^*$, is a Poisson morphism. The comorphism concept allows us to extend this result to Poisson bundles and Lie algebroids. Secondly, the module of sections of a Lie algebroid is a Lie pseudo–algebra, but the concepts of a Lie algebroid morphism and a Lie pseudo–algebra morphism do not correspond. And thirdly, notice that for a Lie algebroid the dual vector bundle possesses the structure of a Poisson bundle and the dual of a Poisson bundle has a Lie algebroid structure. However, these dualities are on objects only. Whereas, with a help of the comorphism concept, they can be expanded on the whole categories.

Moreover, according to Weinstein, comorphisms can be regarded as a reformulation of canonical relations of the symplectic geometry (see [16]).

We do not give detailed proofs in this papers. Indeed, most of the content of this paper is heavily based on [16, 6] and [10], from which we extracted the most important (in our opinion) theorems. The Reader interested in deeper understanding should refer to the cited literature and references therein. On the other hand, this paper is a try to present some recently developed ideas in a reasonably simple and short draft.

2 Fundamental concepts

Now, we will remind the fundamental concepts from the theory of Lie algebroids and Lie groupoids. In case of any doubts, [19] is a very well–known (and good) reference.

Definition 2.1. A morphism of Lie algebras*,* $\mathfrak{g}$ *and* $\mathfrak{h}$*, is a linear map* $\varphi : \mathfrak{g} \to \mathfrak{h}$*, such that* $\varphi([a,b]) = [\varphi(a), \varphi(b)]$ *for every* $a, b \in \mathfrak{g}$*, where* $[\cdot,\cdot]$ *denotes the Lie bracket.*

Definition 2.2. A dual vector bundle *is denoted in the following way* $E^* \stackrel{q^*}{\to} M$*, where* $E \stackrel{q}{\to} M$ *is the initial bundle. Fibers of the dual vector bundle are, by the definition, the dual spaces of the fibers of* E *(i.e., all linear functionals on fibers of* E*).*

Definition 2.3. *Let* $E \stackrel{q}{\to} M$ *be a vector bundle and let* $f : N \to M$ *be a mapping. Then,* $f^!E \stackrel{q^!}{\to} N$ *is called* an inverse image bundle*, if* $f^!E := \{(x,p) \mid f(x) = q(p)\}$ *and* $q^!((x,p)) := p$*.*

Definition 2.4. A canonical morphism *is the following mapping* $f_! : f^!E \to E$*.*

$$\begin{array}{ccc} E & & \\ {\scriptstyle q}\downarrow & & \\ M & \xleftarrow{\;f\;} & N \end{array}$$

Sections of $f^!E$ can be regarded as sums $\sum_i \alpha_i \otimes X_i$, where $\alpha_i \in C^\infty(N)$ and $X_i \in \Gamma E$. Notice also, that there exists an isomorphism of $C^\infty(N)$–modules: $C^\infty(N) \otimes \Gamma E \cong \Gamma(f^!E)$, $\alpha \otimes X \mapsto (\alpha \circ f)X^!$, where $X^!$ is a pullback of X.

Definition 2.5. A groupoid *is a small category, in which every morphism is an isomorphism.*

Unfortunately, except aesthetic elegance, this definition has few advantages. In practical usage it gives no help. More illustrative is the reformulation in terms of *arrows*. Indeed, we can also see a groupoid, denoted by $G \rightrightarrows M$, as the pair consisting of a collection of arrows G, and a base M (consisting of objects). Then, for every $x \in M$ there exists $\mathbb{1}_x \in G$. Of course, arrows are composable, associative and invertible. We strongly refer the Reader to the alternative definition of a groupoid in terms of relations, which was proposed by Zakrzewski [33, 34]. This formulation is also discussed in [26] and [3].

Definition 2.6. A Lie groupoid *is a groupoid, such that:*

- *its source and target maps, $s, t : G \to M$, are surjective submersions (i.e., differentiable surjections, which differentials are also surjections),*
- *its inclusion map, $M \ni x \mapsto \mathbb{1}_x \in G$ is smooth,*
- *the multiplication in G is smooth.*

Now, we will present some examples of Lie groupoids.

Example 2.7. *Let M be an arbitrary manifold. Let $s = \mathrm{id}_M = t$ and let every element be a unity.*

Example 2.8. *Let $G \times M \to M$ be a smooth action of a Lie group on a manifold M. Let $s := \mathrm{pr}_M$ and $t : G \times M \to M$. Let $i : x \mapsto \mathbb{1}_x = (1, x)$ and $(g_2, y)(g_1, x) := (g_2 g_1, x)$, where $y = g_1 x$. Let $(g, x)^{-1} := (g^{-1}, gx)$. Then, $G \times M$ is so called* an action groupoid.

Example 2.9. *Let $E \xrightarrow{q} M$ be a vector bundle. Consider $\Phi(E)$, the set of all vector space isomorphisms $\xi : E_x \to E_y$, where $x, y \in M$. Let $s(\xi) := x$, $t(\xi) := y$ and $i : x \mapsto \mathrm{id}_{E_x}$. Let the multiplication be defined as a composition of mappings and let the inverse be defined as an inverse of mappings. Such a Lie groupoid is called* a frame groupoid.

Example 2.10. *Consider $U \subset M$ and $V \subset M$, where M is a manifold. Consider also a diffeomorphism $\varphi : U \to V$. Denote by $j^1_x\varphi$ the first order jet of φ at x and denote by $J^1(M, M)$ the set of all first order jets. Then, $J^1(M, M)$ can be given a Lie groupoid structure. Indeed, let $s(j^1_x\varphi) := x$, $t(j^1_x\varphi) = \varphi(x)$ and $j^1_{\varphi(x)}\psi \cdot j^1_x\varphi = j^1_x(\psi \circ \varphi)$. (It is isomorphic to $\Phi(TM)$, i.e., consider the following mapping $j^1_x\varphi \mapsto T_x\varphi$.)*

Definition 2.11. A morphism of Lie groupoids, $(\varphi, f) : G \rightrightarrows M \to H \rightrightarrows N$*, is a pair consisting of smooth mappings $\varphi : G \to H$ and $f : M \to N$, such that*

- *$s_H \circ \varphi = f \circ s_G$ and $t_H \circ \varphi = f \circ t_G$, where s, t are source and target maps respectively,*

- *$\varphi(hg) = \varphi(h)\varphi(g)$ for every $h, g \in G$ that can be composed (multiplied).*

Definition 2.12. *If the assumption of a smoothness in the above definition is dropped, one obtains the definition of* a morphism of groupoids.

Definition 2.13. A Lie algebroid *is defined as triple consisting of*

- *a vector bundle $E \xrightarrow{q} M$,*
- *an anchor mapping $a : E \to TM$,*
- *and a Lie bracket $[\cdot, \cdot] : \Gamma E \times \Gamma E \to \Gamma E$, such that*
  - $[X, fY] = f[X, Y] + (a(X)f)Y$ *for every* $f \in C^\infty(M)$ *and* $X, Y \in \Gamma E$,
  - $a([X, Y]) = [a(X), a(Y)]$.

We remind that a Lie bracket means an operator, which is $\mathbb{R}$–bilinear, alternating and satisfying the Jacobi identity. The Jacobi identity is usually written in the following way: $[a, [b, c]] + [c, [a, b]] + [b, [c, a]] = 0$. However, requiring $\mathrm{ad}_a b := [a, b]$ to be a derivation, i.e., to fulfil the following rule $\mathrm{ad}_a[b, c] = [\mathrm{ad}_a b, c] + [b, \mathrm{ad}_a c]$, gives a conceptually better definition [13].

Now, we will present two examples of Lie algebroids.

Example 2.14. *Every Lie algebra over a point.*

Example 2.15. *A tangent bundle, TM, with the identity as the anchor.*

Definition 2.16. A morphism of vector bundles*, $E_1 \xrightarrow{q_1} M$ and $E_2 \xrightarrow{q_2} N$, is defined as a pair (f, g), such that*

- *$g \circ q_1 = q_2 \circ f$,*
- *for every $p \in M$ the mapping $q_1^{-1}(\{p\}) \to q_2^{-1}(\{g(x)\})$ induced by f is a linear mapping of vector spaces.*

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle q_2} \\ M & \xrightarrow[g]{} & N \end{array}$$

Notice, that a morphism of vector bundles induces a morphism of modules of sections only in base–preserving case (i.e., if $g = \mathrm{id}_M$).

Definition 2.17. A morphism of Lie algebroids *is defined as a pair* (φ, f) *satisfying the following conditions. In particular, let* E_1 *be a Lie algebroid on a base* M *and let* E_2 *be a Lie algebroid on a base* N*. Now, consider a vector bundles morphism, such that*

- $a_1 \circ \varphi = T(f) \circ a_2$*, i.e., the anchor is preserved, where* $T(f) : TN \to TM$*,*
- *for* $X, Y \in \Gamma E_2$*, if* $\varphi^!(X) = \sum_i u_i \otimes X_i$ *and* $\varphi^!(Y) = \sum_j v_j \otimes Y_j$*, then* $\varphi^!([X,Y]) = \sum_{i,j} u_i v_j \otimes [X_i, Y_j] + \sum_j a_2(X)(v_j) \otimes Y_j - \sum_i a_2(Y)(u_i) \otimes X_i$*, i.e., the bracket is preserved.*

$$\begin{array}{ccc} E_2 & \xrightarrow{\ \varphi\ } & E_1 \\ {\scriptstyle q_2}\downarrow & & \downarrow{\scriptstyle q_1} \\ N & \xrightarrow[\ f\ ]{} & M \end{array}$$

As a result, we obtain a category of Lie algebroids, which we will denote by LA.

Definition 2.18. A Lie pseudo–algebra *is defined as an* A*–module* $\mathcal{C}$*, such that*

- A *is a commutative and unitary* $\Bbbk$*–algebra,*
- *there exists a Lie bracket* $[\cdot,\cdot]$ *on* $\mathcal{C}$*,*
- *there exists an anchor* $a : \mathcal{C} \to \mathrm{Der}A$*,*
- $[X, fY] = f[X,Y] + (a(X)f)Y$*,*
- $a([X,Y]) = [a(X), a(Y)]$*,*

for every $X, Y \in \mathcal{C}$ *and every* $f \in A$*.*

Sometimes, (if $\Bbbk = \mathbb{R}$) a Lie pseudo–algebra is called a Lie–Rinehart algebra (especially in cohomology theories). Such algebras were introduced by Herz in 1953 [15].

The notion of a Lie pseudo–algebra is a generalization of a Lie algebroid concept. In particular, a vector bundle is replaced by a module $\mathcal{C}$. Notice, that due to the well–known Serre–Swan theorem [24, 27, 21] the category of smooth vector bundles on M is equivalent to the category of finitely generated projective modules over $C^\infty(M)$.

We remind that a module is called *finitely generated*, if it has a finite generating set. In other words, suppose that $\mathcal{C}$ is a left A–module. Then, there exist $c_1, \ldots, c_n \in \mathcal{C}$, such that for an arbitrary $c \in \mathcal{C}$ there exist $a_1, \ldots, a_n \in A$, such that $c = a_1c_1 + \cdots + a_nc_n$.

The projectivity of a module can be defined in various ways (see, for example, [21]). For the purpose of this paper, we would call an A–module $\mathcal{C}$ *projective*, if there exist $\{c_i \in \mathcal{C} \mid i \in I\}$ and $\{h_i \in \mathrm{Hom}(\mathcal{C}, A) \mid i \in I\}$, such that for an arbitrary $c \in \mathcal{C}$ we can write $c = \sum_{i \in I} h_i(c)a_i$ and $h_i(c) \neq 0$ for at least finite number of i–s. In other words, a projective module posses a dual basis.

Notice also, that an arbitrary Lie algebroid (E, M) gives rise to a Lie pseudo–algebra $(\Gamma(E), C^\infty(M))$.

Definition 2.19. A morphism of modules*, A–module $\mathcal{C}$ and B–module $\mathcal{D}$, where A and B are unitary, commutative, $\Bbbk$–algebras, is defined as a pair (φ, f), such that*

- *$f : A \to B$ is a morphism of corresponding algebras,*
- *$\varphi : \mathcal{C} \to \mathcal{D}$ is an additive mapping (i.e., $\varphi(p_1 + p_2) = \varphi(p_1) + \varphi(p_2)$ for every $p_1, p_2 \in \mathcal{C}$) and $\varphi(ap) = f(a)\varphi(p)$ for every $a \in A$ and $p \in \mathcal{C}$.*

As a result, we obtained the category, further denoted by Mod.

Now, let $(\mathcal{C}, A)$ and $(\mathcal{D}, B)$ be two Lie pseudo–algebras.

Definition 2.20. A morphism of Lie pseudo–algebras*, $(\varphi, f) : (\mathcal{C}, A) \to (\mathcal{D}, B)$, is defined as a morphism of modules $\mathcal{C}$ and $\mathcal{D}$, such that*

- *$f(a_\mathcal{C}(X)(\alpha)) = a_\mathcal{D}(\varphi(X))(f(\alpha))$ for every $\alpha \in A$ and every $X \in \mathcal{C}$, where $a_\mathcal{C}$ and $a_\mathcal{D}$ are anchors,*
- *$\varphi([X_1, X_2]) = [\varphi(X_1), \varphi(X_2)]$ for every $X_1, X_2 \in \mathcal{C}$.*

As a result, the category LPA is obtained.

Definition 2.21. A Poisson bracket *is a Lie bracket, which also acts as a derivation.*

Definition 2.22. A Poisson bundle *is a pair consisting of a vector bundle (E, M) and a Poisson bracket $\{\cdot, \cdot\} : C^\infty(E) \times C^\infty(E) \to C^\infty(E)$, such that fiberwise linear functions from $C^\infty(E)$ are a subalgebra with respect to this bracket.*

By fiberwise linearity of $f \in C^\infty(E)$ we understand that $f(ap) = af(p)$. Notice also, that the collection of fiberwise functions is isomorphic to $\Gamma(E^*)$.

Definition 2.23. A morphism of Poisson bundles*, denoted by $(\varphi, f) : (E_1, M) \to (E_2, N)$, is defined as a morphism of vector bundles, such that $\varphi : E_1 \to E_2$ is a Poisson mapping (i.e., a mapping $\alpha \mapsto \alpha \circ \varphi$ preserves the bracket for every $\alpha \in C^\infty(E_2)$).*

3 Comorphisms

Now, we will sketch the idea of comorphisms. Comorphisms will consist of certain pairs. Similarly as in [16], we will denote such pairs with semicolons, i.e., by $(\cdot;\cdot)$, in order to distinguish them from morphisms, which will be denoted with colons, i.e., $(\cdot,\cdot)$.

Definition 3.1. A comorphism of vector bundles, $E_1 \overset{q_1}{\to} M$ *and* $E_2 \overset{q_2}{\to} N$, *is defined as a pair* $(\varphi; f)$, *such that*

- $f : N \to M$ *is a smooth mapping,*
- $\varphi : f^! E_1 \to E_2$ *is a vector bundle morphism,*

i.e., the below diagram commutes.

$$\begin{array}{ccccc} E_1 & \xrightarrow{f_!} & f^! E_1 & \xrightarrow{\varphi} & E_2 \\ \downarrow & & \downarrow & & \downarrow \\ M & \xleftarrow[f]{} & N & = & N \end{array}$$

This comorphism will be denoted by $(\varphi; f) : (E_1, M) \overset{c}{\to} (E_2, N)$. We will also emphasise by the symbol $\overset{c}{\to}$ that the mapping is a comorphism.

Of course, such defined comorphisms can be composed, i.e., consider $(\varphi; f) : (E_1, M) \overset{c}{\to} (E_2, N)$ and $(\psi; g) : (E_2, N) \overset{c}{\to} (E_3, P)$. Then, $(\psi; g) \bullet (\varphi; f) : (E_1, M) \overset{c}{\to} (E_3, P)$, where $(\psi; g) \bullet (\varphi; f) := (\psi \circ g^!(\varphi); f \circ g)$ is the composition.

As a result, the category $\overleftarrow{VB}$ is obtained.

If $f = \mathrm{id}_M$, then $(\varphi; f)$ is called *a base–preserving comorphism over* M. (Notice that in such a case, it is just a morphism of vector bundles over M.)

Base–preserving comorphisms over M, denoted by $\overleftarrow{VB}_M$, form a subcategory of $\overleftarrow{VB}$.

Let E_1^* be a vector bundle dual to a vector bundle E_1. Let E_2^* be a vector bundle dual to a vector bundle E_2. Let $(\varphi, f) : (E_1, M) \to (E_2, N)$ be a morphism of vector bundles. Consider $\varphi^! : E_1 \to f^! E_2$ and $(\varphi^!)^* : (f^! E_2)^* \to E_1^*$. Of course, $(f^! E_2)^*$ can be identified with $f^!(E_2^*)$. Then, a comorphism $(\varphi, f)^* := ((\varphi^!)^*; f) : (E_2^*, N) \overset{c}{\to} (E_1^*, M)$ is obtained. Such a construction gives a contravariant functor $*$ from the category of vector bundles with morphisms to $\overleftarrow{VB}$.

Let $(\varphi; f)^* := (f_! \circ \varphi^*, f) : (E_2^*, N) \to (E_1^*, M)$ be such that the below diagram commutes.

$$\begin{array}{ccccccc}
E_2^* & \xrightarrow{\varphi^*} & (f^! E_1)^* & \overset{\cong}{\longleftrightarrow} & f^!(E_1)^* & \xrightarrow{f_!} & E_1^* \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
N & = & N & = & N & \xrightarrow[f]{} & M
\end{array}$$

$*$ is a contravariant functor from $\overleftarrow{VB}$ to the category of vector bundles with morphisms. As a result, mutually inverse functors are obtained. In other words, the considered categories are opposite to each other. This is also true, even if we restrict to base–preserving subcategories.

Let us denote by $B \otimes_A \mathcal{C}$ the B–module obtained from an A–module $\mathcal{C}$ by changing the base algebra.

Definition 3.2. A comorphism of modules *consists of a module morphism* $\varphi : \mathcal{C} \to A \otimes_B \mathcal{D}$ *and an algebra morphism* $f : B \to A$*. It will be denoted by* $(\varphi; f) : (\mathcal{C}, A) \overset{c}{\to} (\mathcal{D}, B)$*.*

Of course, comorphisms of modules can be composed, i.e., $(\psi; g) \bullet (\varphi; f) := (A \otimes_B \psi \circ \varphi; f \circ g)$, where $A \otimes_B \psi$ denotes the induced morphism. In other words, consider $\psi : \mathcal{C} \to \mathcal{D}$, and then consider $A \otimes_B \psi : A \otimes_B \mathcal{C} \to A \otimes_B \mathcal{D}$. As a result, the category $\overleftarrow{Mod}$ is obtained.

However, in order to obtain a duality, we have to restrict to finitely generated projective modules, $\overleftarrow{FMod}$. There is a well–known duality functor, i.e., $* : \mathcal{C}^* = \mathrm{Hom}_A(\mathcal{C}, A)$. $\mathcal{C}^*$ is a dual A–module. The dual morphism for $\varphi : \mathcal{C} \to \mathcal{D}$ is given by $\varphi^* : \mathcal{D}^* \to \mathcal{C}^*$.

Let (φ, f) be a morphism of modules $(\mathcal{C}, A)$ and $(\mathcal{D}, B)$. Then, $\varphi^! : B \otimes_A \mathcal{C} \to \mathcal{D}$, $\sum_i b_i \otimes p_i \mapsto \sum_i b_i \varphi(p_i)$ is a morphism of B–modules. Of course, there is an isomorphism $I : (B \otimes_A \mathcal{C})^* \to B \otimes_A \mathcal{C}^*$. Therefore, there exists a mapping $\varphi^* := I \circ (\varphi^!)^*$, $\varphi^* : \mathcal{D}^* \to B \otimes_A \mathcal{C}^*$. φ^* and $f : A \to B$ constitute a comorphism $(\varphi, f)^* : (\mathcal{D}^*, B) \overset{c}{\to} (\mathcal{C}^*, A)$. This construction gives a contravariant functor from the category of finitely generated projective modules to the category $\overleftarrow{FMod}$.

Now, let $(\varphi; f) : (\mathcal{C}, A) \overset{c}{\to} (\mathcal{D}^*, B)$ be a comorphism in the category $\overleftarrow{FMod}$. Consider $\varphi^\sharp : \mathcal{D}^* \overset{\alpha}{\to} A \otimes_B \mathcal{D}^* \overset{\beta}{\to} (A \otimes_B \mathcal{D})^* \overset{\varphi^*}{\to} \mathcal{C}^*$, where $\alpha(\omega) := 1 \otimes \omega$ and β is a canonical isomorphism. Then, $(\varphi^\sharp, f) : (\mathcal{D}^*, B) \to (\mathcal{C}^*, A)$ is a morphism. This construction gives a contravariant functor $* : \overleftarrow{FMod} \to FMod$. Moreover, $\overleftarrow{FMod}$ and $FMod$ are the opposite categories.

Now, let $(\varphi, f) : (E_1, M) \to (E_2, N)$ be a morphism of vector bundles. Then, $\varphi^! : E_1 \to f^! E_2$ induces $\Gamma(\varphi^!) : \Gamma E_1 \to \Gamma(f^! E_2)$. Then, $(\Gamma(\varphi^!); C(f))$ is a comorphism $(\Gamma E_1, C^\infty(M)) \overset{c}{\to} (\Gamma E_2, C^\infty(N))$, where $C(f) : C^\infty(M) \to C^\infty(N)$, $F \mapsto F \circ f$, $f : N \to M$. Notice that $E \mapsto \Gamma E$ and $(\varphi, f) \mapsto$

$(\Gamma(\varphi^!), C(f))$ constitute a covariant functor from the category VB to the category $\overleftarrow{FMod}$.

Let $(\varphi; f) : (E_1, M) \overset{c}{\to} (E_2, N)$ be a comorphism. Let $\varphi^\sharp(X) := \Gamma(\varphi)(1 \otimes X)$, where $X \in \Gamma E_1$. Then, $\varphi^\sharp : \Gamma E_1 \to \Gamma E_2$ and $(\varphi^\sharp, C(f))$ is a morphism from $(\Gamma E_1, C^\infty(M))$ to $(\Gamma E_2, C^\infty(N))$. Therefore, a covariant functor Γ from the category $\overleftarrow{VB}$ to the category $FMod$ is obtained, i.e., $\Gamma(\varphi; f) = (\varphi^\sharp, C(f))$.

Notice that, if $(\varphi; f) : (E_1, M) \overset{c}{\to} (E_2, N)$ is a comorphism, then the mapping $\Gamma(\varphi) : C^\infty(N) \otimes_{C^\infty(M)} \Gamma E_1 \to \Gamma E_2$ is determined by $\varphi^\sharp$. In other words, for every $\alpha \in C^\infty(N)$ and for every $X \in \Gamma E_1$ it is true that $\Gamma(\varphi)(\alpha \otimes X) = \alpha\varphi^\sharp(X)$.

Finally, the below diagram of functors commutes. Horizontal functors are covariant, whereas vertical functors are contravariant.

$$\begin{array}{ccc} VB & \xrightarrow{\Gamma} & \overleftarrow{FMod} \\ {\scriptstyle *}\updownarrow & & \updownarrow{\scriptstyle *} \\ \overleftarrow{VB} & \xrightarrow[\Gamma]{} & FMod \end{array}$$

Definition 3.3. A comorphism of Lie algebroids, $(\varphi; f) : (E_1, M) \overset{c}{\to} (E_2, N)$, *is defined as a comorphism of vector bundles* E_1 *and* E_2*, such that* $\varphi^\sharp([X_1, X_2]) = [\varphi^\sharp(X_1), \varphi^\sharp(X_2)]$ *for every* $X_1, X_2 \in \Gamma E_1$ *and the following diagram commutes.*

$$\begin{array}{ccc} f^!E_1 & \xrightarrow{\varphi} & E_2 \\ {\scriptstyle f^!(a_1)}\downarrow & & \downarrow{\scriptstyle a_2} \\ f^!TM & \xleftarrow[T(f)^!]{} & TN \end{array}$$

Of course, such comorphisms can be composed. As a result, the category $\overleftarrow{LA}$ is obtained.

Now, let $(\mathcal{C}, A)$ and $(\mathcal{D}, B)$ be two Lie pseudo–algebras.

Definition 3.4. A comorphism of Lie pseudo–algebras, $(\varphi; f) : (\mathcal{C}, A) \overset{c}{\to} (\mathcal{D}, B)$, *is defined as a comorphism of modules* $\mathcal{C}$ *and* $\mathcal{D}$*, such that*

- *if* $X \in \mathcal{C}$ *and* $\varphi(X) = \sum_i \alpha_i \otimes Y_i$*, then for every* $b \in B$ *it is true that* $a_{\mathcal{C}}(X)(f(b)) = \sum_i \alpha_i f(a_{\mathcal{D}}(Y_i)(b))$*, where* $a_{\mathcal{C}}$ *and* $a_{\mathcal{D}}$ *are anchors,*

- *if* $X_1, X_2 \in \mathcal{C}$*,* $\varphi(X_1) = \sum_i \alpha_i \otimes Y_{1i}$*,* $\varphi(X_2) = \sum_j \beta_j \otimes Y_{2j}$*, then* $\varphi([X_1, X_2]) = \sum_i \alpha_i\beta_j \otimes [Y_{1i}, Y_{2j}] + \sum_j a_{\mathcal{C}}(X_1)(\beta_j) \otimes Y_{2j} - \sum_i a_{\mathcal{C}}(X_2)(\alpha_i) \otimes Y_{1i}$.

Of course, also such comorphisms can be composed. As a result, the category $\overleftarrow{LPA}$ is obtained.

Let $(\mathcal{C}, A)$ be a Lie pseudo–algebra. Consider $d : \bigwedge^k \mathcal{C}^* \to \bigwedge^{k+1} \mathcal{C}^*$ $(k \geq 0)$ such that

- $d \circ d = 0$,
- $(df)(X) = (a(X))(f)$ for every $f \in A$ and $X \in \mathcal{C}$, where a is an anchor,
- $(dp)(X \wedge Y) = (a(X))(p(Y)) - (a(Y))(p(X)) - p([X, Y])$ for every $p \in \mathcal{C}^*$ and every $X, Y \in \mathcal{C}$,
- $d(p_1 \wedge \cdots \wedge p_n) = dp_1 \wedge p_2 \wedge \cdots \wedge p_n \cdots + (-1)^{n-1} p_1 \wedge \cdots \wedge p_{n-1} \wedge dp_n$.

Definition 3.5. *Such a mapping d will be called* an exterior differential operator.

Theorem 3.6. *$(\varphi; f) : (\mathcal{C}, A) \overset{c}{\to} (\mathcal{D}, B)$ is a comorphism of Lie pseudo–algebras, if and only if $\varphi^* : \mathcal{D}^* \to \mathcal{C}^*$ satisfies the following condition $d_{\mathcal{D}} \circ \varphi^* = \varphi^* \circ d_{\mathcal{C}}$, where φ^* is considered as a mapping $\bigwedge^k \mathcal{D}^* \to \bigwedge^{k+1} \mathcal{C}^*$. In other words, $d_{\mathcal{C}}(f(b)) = \varphi^*(d_{\mathcal{D}}(b))$ for every $b \in B$ and $d_{\mathcal{C}}(\varphi^*(p)) = \varphi^*(d_{\mathcal{D}}(p))$ for every $p \in \mathcal{D}^*$.*

Theorem 3.7. *$(\varphi, f) : (E_1, M) \to (E_2, N)$ is a morphism of Lie algebroids, if and only if it is a vector bundle morphism and $\widetilde{\varphi^*} : \Gamma(\bigwedge^k E_2^*) \to \Gamma(\bigwedge^k E_1^*)$ satisfies the following condition $d_{E_1} \circ \widetilde{\varphi^*} = \widetilde{\varphi^*} \circ d_{E_2}$, i.e., $\widetilde{\varphi^*} = f^* : C^\infty(N) \cong \Gamma(\bigwedge^0 E_2^*) \to \Gamma(\bigwedge^0 E_1^*) \cong C^\infty(M)$.*

Definition 3.8. An (infinitesimal) action of E on f, *where E is a Lie algebroid on M and $f : N \to M$ is a smooth mapping, is understood as a mapping $\Gamma E \ni X \mapsto X^\dagger \in \mathfrak{X}(N)$, such that*

- $(X + Y)^\dagger = X^\dagger + Y^\dagger$,
- $(\alpha X)^\dagger = (\alpha \circ f) X^\dagger$,
- $[X, Y]^\dagger = [X^\dagger, Y^\dagger]$,
- $X^\dagger(\alpha \circ f) = a(X) \circ f$,

for every $\alpha \in C^\infty(M)$ and every $X, Y \in \Gamma E$.

Definition 3.9. An action Lie algebroid (or: transformation Lie algebroid), *corresponding to $X \mapsto X^\dagger$ is, by the definition, a Lie algebroid structure defined on the pullback vector bundle $f^! E$, such that*

- *the anchor is defined in the following way $f^! E \to TN$, $(p_N, X) \mapsto X^\dagger(p_N)$,*

- *the bracket on* $\Gamma(f^! E) \cong C^\infty(N) \otimes_{C^\infty(M)} \Gamma E$ *is defined in the following way* $[\sum_i \alpha_i \otimes X_i, \sum_j \beta_j \otimes Y_j] = \sum_{i,j} \alpha_i \beta_j \otimes [X_i, Y_j] + \sum_{i,j} \alpha_i X_i^\dagger(\beta_j) \otimes Y_j - \sum_{i,j} \beta_j Y_j^\dagger(\alpha_i) \otimes X_i$.

In such a case, $f^! E$ is denoted by $E \ltimes f$.

Definition 3.10. An action morphism *is a morphism of Lie algebroids,* $(\varphi, f) : (E_2, N) \to (E_1, M)$*, such that* $\varphi^! : E_2 \to f^! E_1$ *is an isomorphism of vector bundles.*

Now, we will show another look on a comorphism of Lie algebroids, which can serve as a definition (of course, an equivalent to the previous one). For an action morphism, $(\varphi^!)^{-1} : f^! E_1 \to E_2$ and $f : N \to M$ constitute a comorphism of Lie algebroids.

Theorem 3.11. *Let* $(\varphi; f) : (E_1, M) \overset{c}{\to} (E_2, N)$ *be a comorphism of Lie algebroids. Then,* $\Gamma(a_2) \circ \varphi^\sharp : \Gamma E_1 \to \mathfrak{X}(N)$ *is an infinitesimal action of* E_1 *on* f*. Moreover,* $\varphi : f^! E_1 \to E_2$ *is a Lie algebroid morphism from* $E_1 \ltimes f$ *to* E_2*.*

Definition 3.12. A comorphism of Lie groupoids, $(\varphi; f) : G \rightrightarrows M \overset{c}{\to} H \rightrightarrows N$*, is defined as a triple consisting of*

- *a smooth map* $f : M \to N$*,*
- *an infinitesimal action of* G *on* f*,*
- *a base–preserving morphism* $\varphi : G \ltimes f \to H$*.*

Of course, such comorphisms can be composed. Therefore, the category $\overleftarrow{LG}$ is obtained.

It should be mentioned that every comorphism of Lie groupoids can be also understood as a morphism of certain C^*–algebras (see, for example, [25]).

Definition 3.13. $(\varphi; f) : G \rightrightarrows M \overset{c}{\to} H \rightrightarrows N$ *is called* a comorphism of groupoids*, if*

- $f : N \to M$*,*
- $\varphi : N \times_f G \to H$*, where* $N \times_f G := \{(p, g) \mid p \in N, f(p) = s_G(g)\}$*,*

and the following diagram commutes

$$\begin{array}{ccc} N \times_f G & \xrightarrow{\ \varphi\ } & H \\ {\scriptstyle \mathrm{pr}_N}\downarrow & & \downarrow{\scriptstyle s_H} \\ N & \xrightarrow[\mathrm{id}_N]{} & N \end{array}$$

and, moreover, the following conditions hold:

- $\varphi(p, f(p)) = p$ *for every* $p \in N$,
- $f \circ t_H \circ \varphi(p, g) = t_G(p)$ *for every* $(p, g) \in N \times_f G$,
- $\varphi(p, gh) = \varphi(p, g)\varphi(t_H \circ \varphi(p, g), h)$ *for every* $(p, g) \in N \times_f G$ *and* $(t_H \circ \varphi(p, g), h) \in N \times_f G$.

Now, let $(\mathcal{C}, A)$ be a Lie pseudo–algebra. Let $I \subseteq A$ be an ideal of A. Let $\mathcal{C}^I := \{p \in \mathcal{C} \mid \forall_{a\in I}\ [p, a] \in I\}$. Let $I\mathcal{C} := \{\sum_i a_i p_i \mid a_i \in I, p_i \in \mathcal{C}\}$. Of course, $I\mathcal{C} \subset \mathcal{C}^I$. Let $\mathcal{C}_I := \mathcal{C}^I/I\mathcal{C}$. Then, $(\mathcal{C}_I, A/I)$ is a Lie pseudo–algebra with structures inherited from $(\mathcal{C}, A)$. It will be called *I–restriction of* $(\mathcal{C}, A)$.

Example 3.14. *Let N be a submanifold of M. Let $I = \{f \in C^\infty(M) \mid \forall_{p\in N}\ f(p) = 0\} \subset C^\infty(M)$. Then, $C^\infty(M)/I \cong C^\infty(N)$ and $(\mathfrak{X}(M))_I = \mathfrak{X}(N)$. So it is a restriction of tangent vector fields on a submanifold.*

Let $\psi : A \to B$ be an algebra morphism. Then, $\widetilde{\psi} : A \otimes B \to B$ is understood as a mapping $a \otimes b \mapsto \psi(a)b$.

Let $(\mathcal{C}, A)$ and $(\mathcal{C}, A)$ be two Lie pseudo–algebras and let $\psi : A \to B$ be a morphism. Consider $\mathcal{P} := (\mathcal{C} \otimes B) \oplus (A \otimes \mathcal{D})$. Consider also $I = \ker \widetilde{\psi}$.

Definition 3.15. *An I–restriction of $(\mathcal{P}, A \otimes B)$ will be called ψ*–sum of $(\mathcal{C}, A)$ and $(\mathcal{D}, B)$. *It will be denoted by $(\mathcal{C} \oplus_\psi \mathcal{D}, B)$.*

$\mathcal{C} \oplus_\psi \mathcal{D}$ is a B–submodule of $(\mathcal{C} \otimes_A B) \oplus \mathcal{D}$. Moreover, $\sum_i X_i \otimes_A b_i + Y \in (\mathcal{C} \oplus_\psi \mathcal{D}, B) \Leftrightarrow \forall_{a\in A}\ \sum_i \psi([X_i, a])b_i = [Y, \psi(a)]$.

Consider $\widetilde{\varphi} : \mathcal{C} \otimes_A B \to \mathcal{D}$, $X \otimes_A b \mapsto \varphi(X)b$.

Theorem 3.16. $(\varphi, f) : (\mathcal{C}, A) \to (\mathcal{D}, B)$ *is a morphism of Lie pseudo–algebras, if and only if* $\{p + \widetilde{\varphi}(p) \mid p \in \mathcal{C} \otimes_A B\} \subset (\mathcal{C} \otimes_A B) \oplus \mathcal{D}$ *is a Lie pseudo–subalgebra of* $\mathcal{C} \oplus_f \mathcal{D}$.

Theorem 3.17. $(\varphi; f) : (\mathcal{C}, A) \overset{c}{\to} (\mathcal{D}, B)$ *is a comorphism of Lie pseudo–algebras, if and only if* $\{\varphi(Y) + Y \mid Y \in \mathcal{C}\} \subset (\mathcal{D} \otimes_B A) \oplus \mathcal{C}$ *is a Lie pseudo–subalgebra of* $\mathcal{D} \oplus_f \mathcal{C}$.

Consider two Lie algebroids (E_1, M) and (E_2, N). Let $f : M \to N$.

Definition 3.18. $E_1 \oplus_f E_2 := \{(p_1, p_2) \in E_1 \oplus f^! E_2 \mid p_1 \in q_1^{-1}(\{x\}), p_2 \in q_2^{-1}(\{f(x)\}), T(f) \circ a_1(p_1) = a_2(p_2), x \in M\}$ *will be called f*–sum of Lie algebroids (E_1, M) and (E_2, N). *(By a_1 and a_2 we denoted anchors.)*

Theorem 3.19. $(\varphi, f) : (E_1, M) \to (E_2, N)$ *is a morphism of Lie algebroids, if and only if* $\mathcal{G} := \{(p, \varphi(p)) \mid x \in M, p \in q_1^{-1}(\{x\})\} \subset E_1 \oplus f^! E_2$ *is contained in* $E_1 \oplus_f E_2$ *and* $\Gamma(\mathcal{G})$ *is a Lie subalgebra of* $\Gamma(E_1 \oplus_f E_2)$.

Theorem 3.20. $(\varphi; f) : (E_1, M) \overset{c}{\to} (E_2, N)$ *is a comorphism of Lie algebroids, if and only if* $\mathcal{G} := \{(\varphi(p), p) \mid x \in N, p \in q_1^{-1}(\{f(x)\})\} \subset E_2 \oplus f^! E_1$ *is contained in* $E_2 \oplus_f E_1$ *and* $\Gamma(\mathcal{G})$ *is a Lie subalgebra of* $\Gamma(E_2 \oplus_f E_1)$.

Definition 3.21. *Let* $G \rightrightarrows M$ *and* $H \rightrightarrows N$ *be two groupoids. Consider a groupoid structure* $G \times H \rightrightarrows M \times N$ *defined by the following conditions*

- $s_{G\times H}(p, q) := (s_G(p), s_H(q))$,
- $t_{G\times H}(p, q) := (t_G(p), t_H(q))$,
- $(p, q)(r, s) := (pr, qs)$, *if* p *and* r *are composable and* q *and* s *are composable, where* $p, r \in G$ *and* $q, s \in H$.

Then, $G \times H \rightrightarrows M \times N$ *will be called* a direct product of groupoids $G \rightrightarrows M$ *and* $H \rightrightarrows N$.

Let $f : M \to N$. Let $G \times_f H := \{(p, q) \in G \times H \mid s_H(q) = f \circ s_G(p), t_H(q) = f \circ t_G(p)\}$. Then, $G \times_f H$ will be called *f–product* of groupoids $G \rightrightarrows M$ and $H \rightrightarrows N$.

Theorem 3.22. $(\varphi, f) : G \rightrightarrows M \to H \rightrightarrows N$ *is a morphism of groupoids, if and only if* $\{(p, \varphi(p)) \mid p \in G\}$ *is a subgroupoid of* $G \times_f H$.

We remind that $H \rightrightarrows N$ is called *a subgroupoid* of the groupoid $G \rightrightarrows M$, if $H \rightrightarrows N$ is a groupoid and there exists an injective groupoid morphism $i : H \rightrightarrows N \to G \rightrightarrows M$.

Now, let $f : N \to M$. Let $\varphi : N \times_f G \to H$.

Theorem 3.23. $(\varphi; f) : G \rightrightarrows M \overset{c}{\to} H \rightrightarrows N$ *is a comorphism of groupoids, if and only if* $\{(\varphi(p, q), q) \mid p \in N, q \in s_G^{-1}(f(p))\}$ *is a subgroupoid of* $H \times_f G$.

Now, consider a Poisson bundle (E, M). Let $\overline{\alpha}$ denote a fiberwise linear function from $C^\infty(E)$, corresponding to $\alpha \in \Gamma E^*$. (Notice that the collection of fiberwise functions is isomorphic to ΓE^*.) Let $\alpha, \beta \in \Gamma E^*$. Then, consider $\Gamma E^* \ni [\alpha, \beta] := \overline{[\alpha, \beta]} = \{\overline{\alpha}, \overline{\beta}\}$. The Hamiltonian vector field $H_{\overline{\alpha}} = \{\overline{\alpha}, \cdot\} \in \Gamma TE$ projects under $q : E \to M$ to the vector field $a(\alpha)$ on M. As a result, $a : E^* \to TM$ finishes making E^* a Lie algebroid on M.

Conversely, if E is a Lie algebroid on M, then (E^*, M) can be made a Poisson bundle. Indeed, let $X, Y \in \Gamma E$ and let $\alpha, \beta \in C^\infty(M)$. Then, consider $\{\overline{X}, \overline{Y}\} := \overline{[X, Y]}$, $\{\overline{X}, \alpha \circ q\} := a(X)(\alpha) \circ q$ and $\{\alpha \circ q, \beta \circ q\} := 0$. Next, extend to all functions $E^* \to \mathbb{R}$ by differentiation.

Details of the above constructions can be found, for example, in [7].

Theorem 3.24. *Let* $(\varphi, f) : (E_1, M) \to (E_2, N)$ *be a morphism of Poisson bundles. Then the dual vector bundle comorphism* $(\varphi, f)^* : (E_2^*, N) \overset{c}{\to} (E_1^*, M)$ *is a comorphism of Lie algebroids.*

Theorem 3.25. *Let* $(\varphi; f) : (E_1, M) \stackrel{c}{\to} (E_2, N)$ *be a comorphism of Lie algebroids. Then the dual vector bundle morphism* $(\varphi; f)^* : (E_2^*, N) \to (E_1^*, M)$ *is a morphism of Poisson bundles.*

Definition 3.26. A comorphism of Poisson bundles, $(\varphi; f) : (E_1, M) \stackrel{c}{\to} (E_2, N)$, *is defined as a comorphism of the underlying vector bundles, such that if* α *and* β *are fiberwise linear functions from* $C^\infty(E_2)$*, such that* $\alpha \circ \varphi = \sum_i u_i \otimes F_i$ *and* $\beta \circ \varphi = \sum_j v_j \otimes G_j$*, where* $u_i, v_j \in C^\infty(N)$ *and* F_i *and* G_j *are fiberwise linear functions from* $C^\infty(E_1)$*, then* $\{\alpha, \beta\} \circ \varphi = \sum_{i,j} u_i v_j \otimes \{F_i, G_j\} + \sum_j b(\beta)(v_j) \otimes G_j - \sum_i b(\beta)(u_i) \otimes F_i$*, where* $b : E_2^* \to TN$ *is an anchor.*

Theorem 3.27. *Let* $(\varphi; f) : (E_1, M) \stackrel{c}{\to} (E_2, N)$ *be a comorphism of Poisson bundles. Then, the dual vector bundle morphism* $(\varphi; f)^* : (E_2^*, N) \to (E_1^*, M)$ *is a morphism of Lie algebroids.*

Theorem 3.28. *Let* $(\varphi, f) : (E_1, M) \to (E_2, N)$ *be a morphism of Lie algebroids. Then, the dual vector bundle comorphism* $(\varphi, f)^* : (E_2^*, N) \stackrel{c}{\to} (E_1^*, M)$ *is a comorphism of Poisson bundles.*

Finally, notice that every comorphism can be factored into an infinitesimal action and a base–preserving morphism.

4 Integration of a Lie algebroid

Now, we will show the connection of comorphisms and the problem of "integrating" a Lie algebroid. Indeed, when a groupoid is given, we can construct its infinitesimal analog – an algebroid. It is interesting to discuss the opposite construction.

First, we remind the well–known Lie theorems, which are motivation for the above question.

Theorem 4.1 (The first Lie theorem)**.** *Let* G *be a Lie group and let* $\mathfrak{g}$ *be its Lie algebra. Let* $\mathfrak{h}$ *be a Lie subalgebra of* $\mathfrak{g}$*. Then, there exists a unique connected Lie subgroup* H *of* G*, which Lie algebra is* $\mathfrak{h}$*.*

Theorem 4.2 (The second Lie theorem)**.** *Let* G *and* H *be two Lie groups and let* G *be simply connected. Let* $\mathfrak{g}$ *and* $\mathfrak{h}$ *be corresponding Lie algebras and let* $\varphi : \mathfrak{g} \to \mathfrak{h}$ *be a morphism. Then, there is a unique morphism* $F : G \to H$*, which induces* φ*.*

Theorem 4.3 (The third Lie theorem)**.** *Let* $\mathfrak{g}$ *be a Lie algebra. Then, there exists a Lie group, which Lie algebra is isomorphic to* $\mathfrak{g}$*.*

Definition 4.4. s–fiber at x *is understood as the collection of all arrows beginning in* x*.*

One can associate to every Lie groupoid some Lie algebroid. However, it is not always possible to associate a Lie groupoid to the given Lie algebroid [10].

In case of a Lie group G, one has the vector space T_eG, where e is a unit in G. Then, right invariant vector fields can be considered, on which a Lie bracket $[\cdot,\cdot]$ can be defined. As a result, one obtains a Lie algebra $\mathfrak{g}$. However, in the case of a Lie groupoid $G \rightrightarrows M$, a unity might be not unique. Instead of a vector space, one considers rather a vector bundle A, defined in the following way. Its fiber at $x \in M$ consists, by the definition, of a tangent space at a unit $\mathbb{1}_x$ of a s–fiber at x.

Now, let $t : G \to M$ be a target map. Then, $Tt : TG \to TM$. An anchor a is constructed by restricting Tt to A. (It is obvious that A is a subbundle of TG.)

Further, a bracket can be defined on ΓA. Indeed, it can be shown that ΓA and right invariant vector fields on G are isomorphic.

Definition 4.5. *A Lie algebroid constructed in the above way, i.e.,* $(A, a, [\cdot,\cdot])$, *is called* a Lie algebroid of the Lie groupoid $G \rightrightarrows M$.

Definition 4.6. An integrable Lie algebroid *is such a Lie algebroid that is isomorphic to the Lie algebroid of a certain Lie groupoid.*

Now, consider a Lie algebroid, for which A is a vector bundle over M and a in an anchor.

Definition 4.7. A–path *is defined as a pair of paths* (ρ, γ)*, where* $\rho : I \to A$ *and* $\gamma : I \to M$*, such that*

- $\rho(\tau) \in A_{\gamma(\tau)}$ *for every* $\tau \in I$,
- $a(\rho(\tau)) = \frac{d\gamma}{d\tau}(\tau)$ *for every* $\tau \in I$.

Consider an equivalence relation $\sim$ on M, such that $x \sim y$, if and only if there exists a path ρ with a base path γ, where γ connects x and y.

Definition 4.8. *Let* $G \rightrightarrows M$ *be a Lie groupoid. Let* $g : [0,1] \to G$ *be a path in* G*. Then,* g *is called* a G–path, *if*

- $g(0) = \mathbb{1}_x$,
- $s(g(\tau)) = x$ *for every* $\tau \in [0,1]$.

In other words, a G*–path starts in a unit and lays in a* s*–fiber.*

The space of A–paths and the space of G–paths can be both made topological spaces equipped with C^2 topologies.

Theorem 4.9. *If $G \rightrightarrows M$ integrates the given Lie algebroid, then G–paths and A–paths are homeomorphic.*

Notice that a certain relation of homotopy for A–paths can be defined. With a help of this relation a groupoid of "homotopic" A–paths can be defined. Such a groupoid is called in literature *a Weinstein groupoid*.

Theorem 4.10. *A Lie algebroid is integrable, if and only if its Weinstein groupoid admits a smooth structure.*

Definition 4.11. *A section of a Lie algebroid is called* complete*, if the anchor maps such a section to a complete vector field.*

Definition 4.12. *A comorphism of integrable Lie algebroids is called* complete*, if the pullback on sections takes complete sections to complete sections.*

Definition 4.13. *A comorphism of integrable Lie algebroids is called* integrable*, if it is an image of the Lie functor $\overleftarrow{LG} \to \overleftarrow{LA}$.*

Definition 4.14. A source simply connected *Lie groupoid is, by the definition, a Lie groupoid, which source fibers (i.e., $s^{-1}(x)$, $x \in M$) are simply connected.*

Theorem 4.15. *The path construction (i.e., the construction of a Weinstein groupoid) is a functor from integrable Lie algebroids with complete comorphisms to source simply connected Lie groupoids with comorphisms. Moreover, it is an inverse of the Lie functor, so, as a result, these two categories are equivalent.*

Finally, let A be an integrable Lie algebroid over X and let B be an integrable Lie algebroid over Y. Let G and H be their integrating source simply connected Lie groupoids respectively. Then, as a corollary from the above theorem, we obtain that a comorphism from A to B integrates to a unique comorphism from G to H, if and only if the comorphism from A to B is complete.

5 Final remarks

Concepts of Lie groupoids and Lie algebroids are widely explored amongst mathematicians and physicists. Therefore, the corresponding literature is really vast. The most important references in the context of this paper have already been mentioned. However, the interested Reader should additionally consult some other papers and books.

Doubtlessly, in the case of an integration of a Lie algebroid [5, 9] and [28] are very important papers. Also, the paper [11] is strictly connected with topics presented here. As a historical source, [22] contains the first introduction of the concept of a Lie algebroid. We also refer the Reader to [8].

An interesting introduction can also be found in [20]. Whereas in a short paper [32] much emphasis is put on the idea of the symmetry in the concept of a groupoid. Very nice and short papers, which also discuss Poisson structures are [31] and [23]. Of course, the Reader interested more in Poisson structures should consult the classical book [30].

Finally, we would like to mention that concepts, which we have discussed here, have also some strong physical motivations. We refer the Reader to the following papers [18, 12, 17]. Moreover, for example, in [1, 4, 2, 29, 14] the Reader would find astonishing applications of groupoids in cosmology.

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