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# Optimal Bounds for Neuman-Sándor Mean in Terms of the Convex Combination of Geometric and Quadratic Means

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**Abstract**

In this paper, we present the least value $\alpha$ and the greatest value $\beta$ such that the double inequality

$$\alpha G(a,b) + (1-\alpha)Q(a,b) < M(a,b) < \beta G(a,b) + (1-\beta)Q(a,b)$$

holds for all $a, b > 0$ with $a \neq b$, where G(a,b), M(a,b) and Q(a,b) are respectively the geometric, Neuman-Sándor and quadratic means of a and b.

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# 1 Introduction

For $a, b > 0$ with $a \neq b$ the Neuman-Sándor mean $M(a,b)$[1] was defined by

$$M(a,b) = \frac{a-b}{2\sinh^{-1}(\frac{a-b}{a+b})}, \tag{1.1}$$

where $\sinh^{-1}(x) = \log(x + \sqrt{1+x^2})$ is the inverse hyperbolic sine function.

Recently, the Neuman-Sándor mean has been the subject of intensive research. In particular, many remarkable inequalities for the Neuman-Sándor mean $M(a,b)$ can be found in the literature [1,2].

Let $H(a,b) = (2ab)/(a+b)$, $G(a,b) = \sqrt{ab}$, $L(a,b) = (a-b)/(\log a - \log b)$, $P(a,b) = (a-b)/(4\arctan\sqrt{a/b} - \pi)$, $A(a,b) = (a+b)/2$, $T(a,b) = (a-b)/[2\arctan(a-b)/(a+b)]$, $Q(a,b) = \sqrt{(a^2+b^2)/2}$, and $C(a,b) = (a^2+b^2)/(a+b)$ be the harmonic, geometric, logarithmic, first Seiffert, arithmetic, second Seiffert, quadratic, and contra-harmonic mean of $a$ and $b$, respectively. Then

$$\begin{aligned} &\min\{a,b\} < H(a,b) < G(a,b) < L(a,b) < P(a,b) < A(a,b) \\ &< M(a,b) < T(a,b) < Q(a,b) < C(a,b) < \max\{a,b\} \end{aligned} \tag{1.2}$$

hold for all $a, b > 0$ with $a \neq b$.

In [3], Neuman proved that the double inequalities

$$\alpha Q(a,b) + (1-\alpha)A(a,b) < M(a,b) < \beta Q(a,b) + (1-\beta)A(a,b) \tag{1.3}$$

and

$$\lambda C(a,b) + (1-\lambda)A(a,b) < M(a,b) < \mu C(a,b) + (1-\mu)A(a,b) \tag{1.4}$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq [1-\log(1+\sqrt{2})]/[(\sqrt{2}-1)\log(1+\sqrt{2})] = 0.3249\cdots, \beta \geq 1/3, \lambda \leq [1-\log(1+\sqrt{2})]/\log(1+\sqrt{2}) = 0.1345\cdots$ and $\mu \geq 1/6$.

In [4], Li etc showed that the double inequality

$$L_{p_0}(a,b) < M(a,b) < L_2(a,b) \tag{1.5}$$

holds for all $a, b > 0$ with $a \neq b$, where $L_p(a,b) = [(a^{p+1} - b^{p+1})/((p+1)(a-b))]^{1/p} (p \neq -1, 0)$, $L_0(a,b) = 1/e(a^a/b^b)^{1/(a-b)}$ and $L_{-1}(a,b) = (a-b)/(\log a - \log b)$ is the p-th generalized logarithmic mean of $a$ and $b$, and $p_0 = 1.843\cdots$ is the unique solution of the equation $(p+1)^{1/p} = 2\log(1+\sqrt{2})$.

In[5], Chu etc proved that the double inequalities

$$\alpha_1 L(a,b) + (1-\alpha_1)Q(a,b) < M(a,b) < \beta_1 L(a,b) + (1-\beta_1)Q(a,b) \tag{1.6}$$

and

$$\alpha_2 L(a,b) + (1-\alpha_2)C(a,b) < M(a,b) < \beta_2 L(a,b) + (1-\beta_2)C(a,b) \tag{1.7}$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \geq 2/5, \beta_1 \leq 1 - 1/[\sqrt{2}\log(1+\sqrt{2})] = 0.1977\cdots, \alpha_2 \geq 5/8$ and $\beta_2 \leq 1 - 1/[2\log(1+\sqrt{2})] = 0.4327\cdots$.

The main purpose of this paper is to find the least value $\alpha$ and the greatest value $\beta$ such that the double inequality

$$\alpha G(a,b) + (1-\alpha)Q(a,b) < M(a,b) < \beta G(a,b) + (1-\beta)Q(a,b)$$

holds for all $a, b > 0$ with $a \neq b$.

# 2 Lemmas

In order to establish our main result we need several lemmas, which we present in this section.

**Lemma 2.1** *Let* $f(x) = 1/\sqrt{1+x^2}$, $g(x) = 1/\sqrt{1-x^2}$, $h(x) = \sqrt{1-x^4}$, *and* $k(x) = 1/\sqrt{1-x^4}$. *Then the inequalities*

$$1 - \frac{x^2}{2} < f(x) < 1 - \frac{x^2}{2} + \frac{3}{8}x^4, \tag{2.1}$$

$$g(x) > 1 + \frac{x^2}{2}, \tag{2.2}$$

$$h(x) < 1 - \frac{x^4}{2}, \tag{2.3}$$

*and*

$$k(x) > 1 + \frac{x^4}{2} \tag{2.4}$$

*hold for all* $x \in (0, 1)$.

**Proof.** The first inequality in (2.1) is known (see [5, lemma 2.1]). The second inequality in (2.1) and the inequalities (2.2), (2.3) follow in turn from the inequalities

$$\left(1 - \frac{x^2}{2} + \frac{3}{8}x^4\right)^2 - f^2(x) = \frac{x^6}{64(1+x^2)}[9x^2(x^2+1) +4(10-6x^2)] > 0, \tag{2.5}$$

$$g^2(x) - \left(1 + \frac{x^2}{2}\right)^2 = \frac{x^4}{4(1-x^2)}(x^2+3) > 0, \tag{2.6}$$

and

$$\left(1 - \frac{x^4}{2}\right)^2 - h^2(x) = \frac{x^8}{4} > 0 \tag{2.7}$$

for all $x \in (0, 1)$. Making use of (2.2) with $x$ replaced by $x^2$ the inequality (2.4) is obtained.

**Lemma 2.2** *(see* [5, *lemma* 2.4]) *Let* $\varphi(x) = x/[\sqrt{1+x^2}\left(\sinh^{-1}(x)\right)^2] - 1/\sinh^{-1}(x)$. *Then the inequality*

$$\varphi(x) < -\frac{x}{3} + \frac{17}{90}x^3 \tag{2.8}$$

*holds for all* $x \in (0, 1)$.

**Lemma 2.3** *Let $\psi(x) = \log(x + \sqrt{1+x^2})$. Then the double inequality*

$$x - \frac{x^3}{6} < \psi(x) < x \tag{2.9}$$

*holds for all $x \in (0,1)$.*

**Proof.** Let

$$\psi_1(x) = \psi(x) - (x - \frac{x^3}{6}) \tag{2.10}$$

Then simple computations lead to

$$\lim_{x \to 0^+} \psi_1(x) = 0, \tag{2.11}$$

and

$$\psi_1'(x) = \frac{1}{\sqrt{1+x^2}} - (1 - \frac{x^2}{2}). \tag{2.12}$$

Making use of the first inequality in (2.1) for (2.12) cause the conclusion that

$$\psi_1'(x) > \frac{1}{\sqrt{1+x^2}} - \frac{1}{\sqrt{1+x^2}} = 0. \tag{2.13}$$

for $x \in (0,1)$. Therefore, the first inequality in (2.9) follows from (2.10) and (2.11) together with (2.13).

Let $\psi_2(x) = x - \psi(x)$. Then from $\lim_{x \to 0^+} \psi_2(x) = 0$ and $\psi_2'(x) = 1 - 1/\sqrt{1+x^2} > 0$ the second inequality in (2.9) is obtained.

**Lemma 2.4** *Let $\lambda = 1 - 1/\left[\sqrt{2}\log(1+\sqrt{2})\right] = 0.1977\cdots$ and*

$$\begin{aligned} F(x) = \ & 4\lambda x^{16} + 2(37\lambda - 2)x^{14} + (41 - 36\lambda)x^{12} + 2(129\lambda - 65)x^{10} \\ & +4(46 - 125\lambda)x^8 - 2(170\lambda + 29)x^6 - (120\lambda + 161)x^4 \\ & +16(12 - 37\lambda)x^2 + 16(5\lambda - 4). \end{aligned} \tag{2.14}$$

*Then the inequality*

$$F(x) < 0 \tag{2.15}$$

*holds for all $x \in (0, 1/2]$.*

**Proof.** Making use of the transform $x^2 = 1/t$ $(t \in [4, +\infty))$ for $F(x)$ we get

$$F(x) = t^{-8} F_1(t), \tag{2.16}$$

where

$$\begin{aligned} F_1(t) = \ & 16(5\lambda - 4)t^8 + 16(12 - 37\lambda)t^7 - (120\lambda + 161)t^6 \\ & -2(170\lambda + 29)t^5 + 4(46 - 125\lambda)t^4 + 2(129\lambda - 65)t^3 \\ & +(41 - 36\lambda)t^2 + 2(37\lambda - 2)t + 4\lambda. \end{aligned} \tag{2.17}$$

Simple calculations of derivative yield

$$\begin{aligned} F_1'(t) = \; & 2[64(5\lambda-4)t^7+56(12-37\lambda)t^6-3(120\lambda+161)t^5 \\ & -5(170\lambda+29)t^4+8(46-125\lambda)t^3+3(129\lambda-65)t^2 \\ & +(41-36\lambda)t+(37\lambda-2)], \end{aligned} \tag{2.18}$$

$$\begin{aligned} F_1''(t) = \; & 2[448(5\lambda-4)t^6+336(12-37\lambda)t^5-15(120\lambda+161)t^4 \\ & -20(170\lambda+29)t^3+24(46-125\lambda)t^2+6(129\lambda-65)t \\ & +(41-36\lambda)], \end{aligned} \tag{2.19}$$

$$\begin{aligned} F_1'''(t) = \; & 12[448(5\lambda-4)t^5+280(12-37\lambda)t^4-10(120\lambda+161)t^3 \\ & -10(170\lambda+29)t^2+8(46-125\lambda)t+(129\lambda-65)], \end{aligned} \tag{2.20}$$

$$\begin{aligned} F_1^{(4)}(t) = \; & 24[1120(5\lambda-4)t^4+560(12-37\lambda)t^3-15(120\lambda \\ & +161)t^2-10(170\lambda+29)t+4(46-125\lambda)], \end{aligned} \tag{2.21}$$

$$\begin{aligned} F_1^{(5)}(t) = \; & 240[448(5\lambda-4)t^3+168(12-37\lambda)t^2 \\ & -3(120\lambda+161)t-(170\lambda+29)], \end{aligned} \tag{2.22}$$

$$F_1^{(6)}(t)=720[448(5\lambda-4)t^2+112(12-37\lambda)t-(120\lambda+161)] \tag{2.23}$$

and

$$F_1^{(7)}(t)=80640[8(5\lambda-4)t+(12-37\lambda)]. \tag{2.24}$$

Noticing that $0<\lambda<1/5$, from (2.17)-(2.24) we have

$$F_1(4)=-4[(1351973\lambda+432000)]<0, \tag{2.25}$$

$$F_1'(4)=-2[(3888187\lambda+1952910)]<0, \tag{2.26}$$

$$F_1''(4)=-2[(4278668\lambda+3850479)]<0, \tag{2.27}$$

$$F_1'''(4)=-12[(466271\lambda+1081121)]<0, \tag{2.28}$$

$$F_1^{(4)}(4)=96[17855\lambda-189104]<0, \tag{2.29}$$

$$F_1^{(5)}(4)=720[14098\lambda-28131]<0, \tag{2.30}$$

$$F_1^{(6)}(4)=720[19144\lambda-23457]<0, \tag{2.31}$$

and

$$F_1^{(7)}(t)<-967680(2t-1)<0 \tag{2.32}$$

for all $t\in[4,+\infty)$.

From(2.32) we clearly see that $F_1^{(6)}(t)$ is strictly decreasing in $[4,+\infty)$. Therefore, the conclusion of lemma 2.4 follows easily from (2.25)-(2.31) and (2.16) together with the monotonicity of $F_1^{(6)}(t)$.

**Lemma 2.5** *Let* $\lambda = 1 - 1/\left[\sqrt{2}\log(1+\sqrt{2})\right] = 0.1977\cdots$ *and*

$$\begin{aligned} H(x) = \ & 8\lambda x^{14} + 12(14\lambda - 3)x^{12} + 15(11 - 21\lambda)x^{10} + (809\lambda \\ & -403)x^8 + (707 - 1773\lambda)x^6 + 3(361\lambda - 271)x^4 \\ & +4(127 - 335\lambda)x^2 + 32(5\lambda - 4). \end{aligned} \tag{2.33}$$

*Then the inequality*

$$H(x) < 0 \tag{2.34}$$

*holds for all* $x \in (1/2, 1)$.

**Proof.** Let $x^2 = t$ ($t \in (1/4, 1)$). Then

$$\begin{aligned} H(x) &= 8\lambda t^7 + 12(14\lambda - 3)t^6 + 15(11 - 21\lambda)t^5 + (809\lambda - 403)t^4 \\ & \quad +(707 - 1773\lambda)t^3 + 3(361\lambda - 271)t^2 + 4(127 - 335\lambda)t \\ & \quad +32(5\lambda - 4) \\ &= H_1(t). \end{aligned} \tag{2.35}$$

Simple calculations of derivative yield

$$\begin{aligned} H_1'(t) = \ & 56\lambda t^6 + 72(14\lambda - 3)t^5 + 75(11 - 21\lambda)t^4 + 4(809\lambda - 403)t^3 \\ & +3(707 - 1773\lambda)t^2 + 6(361\lambda - 271)t + 4(127 - 335\lambda), \end{aligned} \tag{2.36}$$

$$\begin{aligned} H_1''(t) = \ & 6[56\lambda t^5 + 60(14\lambda - 3)t^4 + 50(11 - 21\lambda)t^3 + 2(809\lambda \\ & -403)t^2 + (707 - 1173\lambda)t + (361\lambda - 271)], \end{aligned} \tag{2.37}$$

and

$$\begin{aligned} H_1'''(t) = \ & 6[280\lambda t^4 + 240(14\lambda - 3)t^3 + 150(11 - 21\lambda)t^2 \\ & +4(809\lambda - 403)t + (707 - 1173\lambda)]. \end{aligned} \tag{2.38}$$

Whereafter, making use of the transform $t = 1/u$ ($u \in (1, 4)$) for $H_1'''(t)$ one has

$$H_1'''(t) = 6u^{-4}H_2(u), \tag{2.39}$$

where

$$\begin{aligned} H_2(u) = \ & (707 - 1173\lambda)u^4 + 4(809\lambda - 403)u^3 + 150(11 \\ & -21\lambda)u^2 + 240(14\lambda - 3)u + 280\lambda. \end{aligned} \tag{2.40}$$

Again calculations of derivative result in

$$\begin{aligned} H_2'(u) = \ & 4[(707 - 1173\lambda)u^3 + 3(809\lambda - 403)u^2 \\ & +75(11 - 21\lambda)u + 60(14\lambda - 3)], \end{aligned} \tag{2.41}$$

$$H_2''(u) = 12[(707 - 1173\lambda)u^2 + 2(809\lambda - 403)u + 25(11 - 21\lambda)], \tag{2.42}$$

and

$$H_2'''(u) = 24[(707 - 1173\lambda)u + (809\lambda - 403)]. \tag{2.43}$$

Noticing that $49/250 < \lambda < 1/5$, from (2.35) – (2.37) and (2.40) – (2.43) one has

$$\lim_{t\to\frac{1}{4}^+} H_1(t) = -\frac{45}{2048}(6013\lambda + 1920) < 0, \ \lim_{t\to 1^-} H_1(t) = -1200\lambda < 0, \quad (2.44)$$

$$\begin{aligned} &\lim_{t\to\frac{1}{4}^+} H_1'(t) = \frac{1}{512}(108486 - 555791\lambda) < 0, \\ &\lim_{t\to 1^-} H_1'(t) = -1768\lambda < 0, \end{aligned} \quad (2.45)$$

$$\lim_{t\to\frac{1}{4}^+} H_1''(t) = \frac{3}{64}(743\lambda - 17502) < 0, \ \lim_{t\to 1^-} H_1''(t) = 312\lambda > 0, \quad (2.46)$$

$$\lim_{u\to 1^+} H_2(u) = 1953\lambda + 25 > 0, \quad (2.47)$$

$$\lim_{u\to 1^+} H_2'(u) = 4(143 - 81\lambda) > 0, \quad (2.48)$$

$$\lim_{u\to 1^+} H_2''(u) = 96(22 - 85\lambda) > 0, \quad (2.49)$$

and

$$H_2'''(u) > 24[351u - 247] > 0 \quad (2.50)$$

for all $u \in (1, 4)$.

From (2.50) we clearly see that $H_2''(u)$ is strictly increasing in $(1, 4)$. Thus $H_2(u) > 0$ for $u \in (1, 4)$ follows from (2.47)-(2.49) and the monotonicity of $H_2''(u)$. From (2.39) and $H_2(u) > 0$ we know that $H_1'''(t) > 0$ for $t \in (1/4, 1)$, hence $H_1''(t)$ is strictly increasing in $(1/4, 1)$. It follows from (2.46) together with the monotonicity of $H_1''(t)$ that there exists $t_0 \in (1/4, 1)$ such that $H_1''(t) < 0$ for $t \in (1/4, t_0)$ and $H_1''(t) > 0$ for $t \in (t_0, 1)$, thus $H_1'(t)$ is strictly decreasing in $(1/4, t_0)$ and strictly increasing in $[t_0, 1)$. From (2.45) and the monotonicity of $H_1'(t)$ we affirm $H_1'(t) < 0$ for $t \in (1/4, 1)$, so that $H_1(t)$ is strictly decreasing in $(1/4, 1)$. Therefore, the inequality $H(x) < 0$ follows from (2.44) and (2.35) together with the monotonicity of $H_1(t)$.

# 3 Main Results

**Theorem 3.1** *The double inequality*

$$\alpha G(a,b) + (1-\alpha)Q(a,b) < M(a,b) < \beta G(a,b) + (1-\beta)Q(a,b) \quad (1)$$

*holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \geq 1/3$ and $\beta \leq 1 - 1/\left[\sqrt{2}\log(1+\sqrt{2})\right] = 0.1977\cdots$.*

**Proof.** Without loss of generality, we assume that $a > b > 0$. Let $x = (a-b)/(a+b) \in (0,1)$ and $\lambda = 1 - 1/\left[\sqrt{2}\log(1+\sqrt{2})\right] = 0.1977\cdots$. Then

$$\frac{G(a,b)}{A(a,b)} = \sqrt{1-x^2}, \frac{M(a,b)}{A(a,b)} = \frac{x}{\sinh^{-1}(x)}, \frac{Q(a,b)}{A(a,b)} = \sqrt{1+x^2}. \tag{3.1}$$

Firstly, we prove that

$$\frac{1}{3}G(a,b) + \frac{2}{3}Q(a,b) < M(a,b). \tag{3.2}$$

Equations (3.1) lead to

$$\begin{aligned} & \frac{G(a,b)}{3A(a,b)} + \frac{2Q(a,b)}{3A(a,b)} - \frac{M(a,b)}{A(a,b)} \\ = & \frac{1}{3}\sqrt{1-x^2} + \frac{2}{3}\sqrt{1+x^2} - \frac{x}{\sinh^{-1}(x)} \\ = & d(x) \end{aligned} \tag{3.3}$$

Simple computations yield

$$\lim_{x\to 0^+} d(x) = 0, \tag{3.4}$$

and

$$\begin{aligned} d'(x) = & -\frac{x}{3\sqrt{1-x^2}} + \frac{2x}{3\sqrt{1+x^2}} - \frac{1}{\sinh^{-1}(x)} \\ & + \frac{x}{\sqrt{1+x^2}\left[\sinh^{-1}(x)\right]^2} \\ = & \frac{2}{3}xf(x) - \frac{1}{3}xg(x) + \varphi(x), \end{aligned} \tag{3.5}$$

where $f(x), g(x)$ and $\varphi(x)$ are defined as in Lemma 2.1 and Lemma 2.2, respectively.

From (3.5), lemma 2.1 and lemma 2.2 one has

$$\begin{aligned} d'(x) < & \frac{2}{3}x(1 - \frac{x^2}{2} + \frac{3}{8}x^4) - \frac{1}{3}x(1 + \frac{x^2}{2}) + (-\frac{x}{3} + \frac{17}{90}x^3) \\ = & -\frac{x^3}{2}(1 - \frac{x^2}{60}) \\ < & 0 \end{aligned} \tag{3.6}$$

for all $x \in (0,1)$. Therefore, inequality (3.2) follows from (3.3) and (3.4) together with (3.6).

Secondly, we prove that

$$\lambda G(a,b) + (1-\lambda)Q(a,b) > M(a,b). \tag{3.7}$$

Equations (3.1) lead to

$$\begin{aligned}&\frac{\lambda G(a,b)}{A(a,b)}+\frac{(1-\lambda)Q(a,b)}{A(a,b)}-\frac{M(a,b)}{A(a,b)}\\&=\lambda\sqrt{1-x^2}+(1-\lambda)\sqrt{1+x^2}-\frac{x}{\log(x+\sqrt{1+x^2})}\\&=\frac{D(x)}{\log(x+\sqrt{1+x^2})},\end{aligned}\tag{3.8}$$

where

$$D(x)=\left[\lambda\sqrt{1-x^2}+(1-\lambda)\sqrt{1+x^2}\right]\log(x+\sqrt{1+x^2})-x.\tag{3.9}$$

Some tedious, but not difficult, calculations lead to

$$\lim_{x\to 0^+}D(x)=0,\ \lim_{x\to 1^-}D(x)=0,\tag{3.10}$$

$$\begin{aligned}D'(x)=\ & x\left(\frac{1-\lambda}{\sqrt{1+x^2}}-\frac{\lambda}{\sqrt{1-x^2}}\right)\log(x+\sqrt{1+x^2})\\&+\frac{\lambda(1-x^2)}{\sqrt{1-x^4}}-\lambda,\end{aligned}\tag{3.11}$$

$$\lim_{x\to 0^+}D'(x)=0,\ \lim_{x\to 1^-}D'(x)=-\infty,\tag{3.12}$$

$$\begin{aligned}D''(x)=\ &\left[\frac{1-\lambda}{(1+x^2)^{3/2}}-\frac{\lambda}{(1-x^2)^{3/2}}\right]\log(x+\sqrt{1+x^2})\\&-\frac{\lambda x(3+x^2)}{(1+x^2)\sqrt{1-x^4}}+\frac{(1-\lambda)x}{1+x^2},\end{aligned}\tag{3.13}$$

$$\lim_{x\to 0^+}D''(x)=0,\ \lim_{x\to 1^-}D''(x)=-\infty,\tag{3.14}$$

$$\begin{aligned}D''(\frac{1}{3})\ &=\ \frac{27}{2}\left(\frac{1-\lambda}{5\sqrt{10}}-\frac{\lambda}{8\sqrt{2}}\right)\log(\frac{\sqrt{10}+1}{3})+\frac{3}{10}-\frac{3\lambda}{10}-\frac{21\lambda}{10\sqrt{5}}\\&>\ \frac{27}{2}\left(\frac{1-\frac{1}{5}}{5\sqrt{10}}-\frac{\frac{1}{5}}{8\sqrt{2}}\right)\log(\frac{\sqrt{10}+1}{3})+\frac{3}{10}-\frac{3}{50}-\frac{21}{50\sqrt{5}}\\&=\ 0.1976\cdots>0,\end{aligned}\tag{3.15}$$

$$\begin{aligned}D'''(x)=\ &-3x\left[\frac{1-\lambda}{(1+x^2)^{5/2}}-\frac{\lambda}{(1-x^2)^{5/2}}\right]\log(x+\sqrt{1+x^2})\\&-\frac{\lambda(x^6+8x^4-x^2+4)}{(1+x^2)(1-x^4)^{3/2}}+\frac{(1-\lambda)(2-x^2)}{(1+x^2)^2},\end{aligned}\tag{3.16}$$

and

$$\begin{aligned} D^{(4)}(x) = & \ 3\left[\frac{(1-\lambda)(4x^2-1)}{(1+x^2)^{7/2}} - \frac{\lambda(4x^2+1)}{(1-x^2)^{7/2}}\right]\log(x+\sqrt{1+x^2}) \\ & -\frac{\lambda x(2x^8+33x^6-7x^4+75x^2-7)}{(1+x^2)(1-x^4)^{5/2}} \\ & +\frac{(1-\lambda)x(2x^2-13)}{(1+x^2)^3}. \end{aligned} \tag{3.17}$$

In order to discuss $D^{(4)}(x)$ is positive or negative, we divide the range of variable $x$ into two intervals (0,1/2] and (1/2,1).

For $x \in (0, 1/2]$, (3.17) is rewritten into

$$\begin{aligned} D^{(4)}(x) = & \ 3\left[\frac{(1-\lambda)(4x^2-1)}{(1+x^2)^3}f(x) - \frac{\lambda(4x^2+1)}{(1-x^2)^3}g(x)\right]\psi(x) \\ & +\frac{7\lambda x(1+x^4)}{(1+x^2)(1-x^4)^3}h(x) - \frac{\lambda x^3(2x^6+33x^4+75)}{(1+x^2)(1-x^4)^2}k(x) \\ & +\frac{(1-\lambda)x(2x^2-13)}{(1+x^2)^3}, \end{aligned} \tag{3.18}$$

where $f(x), g(x), \psi(x), h(x)$ and $k(x)$ are defined as in Lemma 2.1 and 2.3, respectively. From (3.18), lemma 2.1 and lemma 2.3 one has

$$\begin{aligned} D^{(4)}(x) \ < & \ 3\left[\frac{(1-\lambda)(4x^2-1)}{(1+x^2)^3}(1-\frac{x^2}{2}) - \frac{\lambda(4x^2+1)}{(1-x^2)^3}\right](x-\frac{x^3}{6}) \\ & -\frac{\lambda x^3(2x^6+33x^4+75)}{(1+x^2)(1-x^4)^2}(1+\frac{x^4}{2}) + \frac{7\lambda x(1+x^4)}{(1+x^2)(1-x^4)^3}\cdot \\ & (1-\frac{x^4}{2}) + \frac{(1-\lambda)x(2x^2-13)}{(1+x^2)^3} \\ = & \ \frac{x}{4(1+x^2)^4(1-x^2)^3}F(x), \end{aligned} \tag{3.19}$$

where $F(x)$ is defined as in lemma 2.4. It fllows from (3.19) and lemma 2.4 that

$$D^{(4)}(x) < 0. \tag{3.20}$$

For $x \in (1/2, 1)$, (3.17) is rewritten into

$$\begin{aligned} D^{(4)}(x) = & \ 3\left[\frac{(1-\lambda)(4x^2-1)}{(1+x^2)^3}f(x)\psi(x) - \frac{\lambda(4x^2+1)}{(1-x^2)^3}g(x)\psi(x)\right] \\ & -\frac{\lambda x(2x^8+33x^6-7x^4+75x^2-7)}{(1+x^2)(1-x^4)^2}k(x) \\ & +\frac{(1-\lambda)x(2x^2-13)}{(1+x^2)^3}, \end{aligned} \tag{3.21}$$

where $f(x), g(x), h(x)$ and $\psi(x)$ are defined as in lemma 2.1 and 2.3, respectively. From (3.21), lemma 2.1 and lemma 2.3 together with the fact that

$$2x^8+33x^6-7x^4+75x^2-7 > 2(0)^8+33(0)^6-7\cdot 1^4+75(1/2)^2-7 = 19/4 > 0,$$

one has

$$\begin{aligned} D^{(4)}(x) &< 3\Big[\frac{(1-\lambda)(4x^2-1)}{(1+x^2)^3}(1-\frac{x^2}{2}+\frac{3x^4}{8})x-\frac{\lambda(4x^2+1)}{(1-x^2)^3}(1+ \\ &\quad \frac{x^2}{2})(x-\frac{x^3}{6})\Big]-\frac{\lambda x(2x^8+33x^6-7x^4+75x^2-7)}{(1+x^2)(1-x^4)^2}\cdot \\ &\quad (1+\frac{x^4}{2})+\frac{(1-\lambda)x(2x^2-13)}{(1+x^2)^3} \\ &= \frac{x}{8(1-x^4)^3}H(x), \end{aligned} \tag{3.22}$$

where $H(x)$ is defined as in Lemma 2.5. It follows from (3.22) and Lemma 2.5 that

$$D^{(4)}(x) < 0. \tag{3.23}$$

Synthesizing the above two cases we affirm that $D^{(4)}(x) < 0$ for all $x \in (0,1)$, hence the function $D''(x)$ is concave in (0,1). It follows from (3.14) and (3.15) together with the concavity of $D''(x)$ that there exists $x_0 \in (0,1)$ such that $D''(x) > 0$ for $x \in (0, x_0)$ and $D''(x) < 0$ for $x \in (x_0, 1)$, hence $D'(x)$ is strictly increasing in $(0, x_0)$ and strictly decreasing in $[x_0, 1)$. From (3.12) together with the monotonicity of $D'(x)$ we know that there exists $x_1 \in (x_0, 1)$ such that $D'(x) > 0$ for $x \in (0, x_1)$ and $D'(x) < 0$ for $x \in (x_1, 1)$, so that $D(x)$ is strictly increasing in $(0, x_1)$ and strictly decreasing in $[x_1, 1)$. It follows from (3.10) together with the monotonicity of $D(x)$ that

$$D(x) > 0 \tag{3.24}$$

for all $x \in (0,1)$. Therefore, the inequality (3.7) follows from (3.8) and (3.24).

At least, we prove that $1/3G(a,b) + 2/3Q(a,b)$ is the best possible lower convex combination bound and $\lambda G(a,b) + (1-\lambda)Q(a,b)$ is the best possible upper convex combination bound of the geometric and quadratic means for the Neuman-Sándor mean.

From equations (3.1) one has

$$\begin{aligned} \frac{Q(a,b)-M(a,b)}{Q(a,b)-G(a,b)} &= \frac{\sqrt{1+x^2}\log(x+\sqrt{1+x^2})-x}{(\sqrt{1+x^2}-\sqrt{1-x^2})\log(x+\sqrt{1+x^2})} \\ &= B(x). \end{aligned} \tag{3.25}$$

It is easy to calculate that

$$\lim_{x\to 0^+} B(x) = \frac{1}{3}, \tag{3.26}$$

and

$$\lim_{x\to 1^-} B(x) = \lambda. \tag{3.27}$$

If $\alpha < 1/3$, then equations (3.25) and (3.26) lead to conclusion that there exists $\delta_1 = \delta_1(\alpha) \in (0,1)$ such that $M(a,b) < \alpha G(a,b) + (1-\alpha)Q(a,b)$ for $(a-b)/(a+b) \in (0,\delta_1)$.

If $\beta > \lambda$, then equations (3.25) and (3.27) imply the conclusion that there exists $\delta_2 = \delta_2(\beta) \in (0,1)$ such that $M(a,b) > \beta G(a,b) + (1-\beta)Q(a,b)$ for $(a-b)/(a+b) \in (1-\delta_2, 1)$.

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